Characterization of finite groups by the number of non-cyclic non-TI-subgroups *

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Abstract

Let G be a finite group and M a subgroup of G. Then M is said to be a TI-subgroup of G if $M^g \cap M = 1$ or M for any $g \in G$. In this paper, we characterize the solvability of finite groups only by the number of their non-cyclic non-TI-subgroups, we prove that any finite group G having at most 26 non-cyclic non-TI-subgroups is always solvable except for $G \cong A_5$ or $\mathrm{SL}(2,5)$.

Keywords: non-cyclic subgroup; non-TI-subgroup; solvable group
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1 Introduction

In this paper, all groups are considered to be finite. In many cases, the structure of a group can be studied by its local properties. In [1], H. Bao and L. Miao studied groups with some $\mathcal{M}$-permutable primary subgroups. In [16], L. Zhang, W. Shi, D. Yu et al determined all simple groups by their first prime graph components. And in [17], L. Zhang, W. Nie and D. Yu characterized the simple group $D_n(3)$ by its prime graph.

Let G be a group and M a subgroup of G. Then M is said to be a TI-subgroup of G if $M^g \cap M = 1$ or M for any $g \in G$.

For TI-subgroups, one of the most important problems is to study the structure of groups in which some particular subgroups are TI-subgroups. In [14], G. Walls classified groups in which every subgroup is a TI-subgroup. In [4], X. Guo, S. Li and P. Flavell described groups in which every abelian subgroup is a TI-subgroup. Moreover, M.R. Salarian [8] classified groups G in which every cyclic subgroup, elementary abelian 2-subgroup and abelian subgroup of order at most $4p$ are TI-subgroups, where p is a prime divisor of $|G|$. Recently, H. Mousavi, T. Rastgoo and V. Zenkov [5] determined non-nilpotent groups in which every cyclic subgroup is a TI-subgroup.

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In [11, 12], we gave some equivalent characterizations of groups in which every non-abelian subgroup is a TI-subgroup and groups in which every non-nilpotent subgroup is a TI-subgroup respectively.

Let G be a group having exactly n non-cyclic subgroups. Let $\delta(G)$ be the number of non-cyclic non-TI-subgroups of G and $\tau(G)$ the number of non-cyclic TI-subgroups of G. Then $n = \delta(G) + \tau(G)$. By [10, Theorem 3.1], any group G with $\delta(G) = 0$ is solvable. Motivated by this, the main goal of this paper is to study the influence of $\delta(G)$ on the solvability of G.

We have the following result, the proof of which is given in Section 3.

Theorem 1.1 *Any group G having at most 26 non-cyclic non-TI-subgroups is always solvable except for $G \cong A_5$ or $\mathrm{SL}(2,5)$.*

Observe that A_5 has 27 non-cyclic subgroups in all, including 21 non-cyclic non-TI-subgroups and 6 non-cyclic TI-subgroups. And $\mathrm{SL}(2,5)$ has 27 non-cyclic subgroups in all, including 26 non-cyclic non-TI-subgroups and 1 non-cyclic TI-subgroup.

In [13], we determined groups in which every non-cyclic proper subgroup is non-normal.

Theorem 1.2 [13, Theorem 1.1] *Suppose that G is a solvable group in which every non-cyclic proper subgroup is non-normal, then one of the following statements holds:*

(1) *G is cyclic or $G \cong Q_8$ or $\mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime;*

(2) *$G \cong \langle a, b \mid a^m = b^{q^n} = 1, b^{-1}ab = a^r\rangle$, where m, n are positive integers and q is the smallest prime divisor of $|G|$ such that $((r-1)q, m) = 1$ and $r^q \equiv 1 \pmod{m}$.*

As a direct consequence of Theorem 1.2, we can easily obtain the following result.

Theorem 1.3 *Let G be a solvable group having at least one non-cyclic proper subgroup. Suppose that every non-cyclic proper subgroup of G is not a TI-subgroup, then $G \cong \langle a, b \mid a^m = b^{q^n} = 1, b^{-1}ab = a^r\rangle$, where m, n are positive integers and q is the smallest prime divisor of $|G|$ such that $((r-1)q, m) = 1$ and $r^q \equiv 1 \pmod{m}$.*

The special linear group $\mathrm{SL}(2,5)$ shows that a group in which every non-cyclic proper subgroup is not a TI-subgroup might be non-solvable.

All notations in this paper are standard, please see [7].

2 Preliminaries

In this section, we collect some essential lemmas needed in the sequel.

Lemma 2.1 ([7, Theorem 10.5.6 (i) and Exercise 9.1.1]) *Let G be a Frobenius group with kernel N and complement M. Then*

(1) *N is nilpotent;*

(2) *All Frobenius complements of G are conjugate in G.*

Lemma 2.2 ([7, Exercise 10.5.7]) *Suppose that G is a group having an abelian maximal subgroup, then G is solvable.*

Lemma 2.3 ([15, Theorems 3.1 and 3.2] and [9, Lemma 2.7]) *Let G be a group having exactly m maximal subgroups.*

(1) *Suppose that $m \leq 20$, then G is solvable.*

(2) *Suppose that G is a non-solvable group with $m = 21$, then $G/\Phi(G) \cong A_5$.*

(3) *Suppose that G is a non-solvable group with $m = 22$, then $G/\Phi(G) \cong \mathrm{PSL}(2,7)$ or $A_5 \times \mathbb{Z}_p$ or S_5, where p is a prime.*

(4) *Suppose that G is a non-solvable group with $m = 23$, then $G/\Phi(G) \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_p$ or $A_5 \times \mathbb{Z}_{pq}$ or $S_5 \times \mathbb{Z}_p$, where p and q are distinct primes.*

Lemma 2.4 *Suppose that G is a group in which every maximal subgroup is a TI-subgroup, then G is solvable.*

Proof. If every maximal subgroup of G is normal, then G is obviously solvable. Next assume that G has at least one non-normal maximal subgroup. Let M be a non-normal maximal subgroup of G. By the hypothesis, G is a Frobenius group with complement M. Let N be Frobenius kernel of G. One has $G = N \rtimes M$, the semidirect product of N and M, where $N \trianglelefteq G$. Let M_1 be any maximal subgroup of M. Then $N \rtimes M_1$ is a maximal subgroup of G. Since all Frobenius complements of G are conjugate by Lemma 2.1 (2) and $N \rtimes M_1$ is not a conjugate of M, one has $N \rtimes M_1 \trianglelefteq G$ by the hypothesis. Then $M_1 = M_1(N \cap M) = (N \rtimes M_1) \cap M \trianglelefteq M$. It follows that M is nilpotent. Therefore, G is solvable since N is also nilpotent by Lemma 2.1 (1). □

Lemma 2.5 *There does not exist a non-abelian simple group G such that G has exactly m maximal subgroups, where $24 \leq m \leq 26$.*

Proof. Suppose not. Assume that G is a non-abelian simple group having exactly m maximal subgroups for $24 \leq m \leq 26$. Let M be a maximal subgroup of G with the smallest index in G. Assume that $|G : M| = n$. Then G is isomorphic to a subgroup of A_n. By [6], G has at least three conjugacy classes of maximal subgroups. It follows that $n \leq [\frac{26}{3}] = 8$. By [3, page 22], A_8 has the following subgroups being non-abelian simple groups: A_5, A_6, $\mathrm{PSL}(2,7)$, A_7 and A_8. Obviously, all those non-abelian simple groups do not have exactly m maximal subgroups for $24 \leq m \leq 26$, a contradiction. □

3 Proof of Theorem 1.1

The proof of Theorem 1.1 follows from the following five lemmas.

Lemma 3.1 *Let G be a group having at most 20 non-cyclic non-TI-subgroups, then G is solvable.*

Proof. Let G be a counterexample of minimal order. It follows that G is a minimal non-solvable group, and then $G/\Phi(G)$ is a minimal non-abelian simple group. By Lemma 2.2, every maximal subgroup of G is non-cyclic. Since $G/\Phi(G)$ is a non-abelian simple group, every maximal subgroup of G is obviously not a TI-subgroup. It follows that G has at most 20 maximal subgroups by the hypothesis. Then G is solvable by Lemma 2.3 (1), a contradiction. So G is solvable. □

Lemma 3.2 *Let G be a non-solvable group having exactly 21 non-cyclic non-TI-subgroups, then $G \cong A_5$.*

Proof. Obviously, every maximal subgroup of G is non-cyclic. We claim that every maximal subgroup of G is solvable.

Otherwise, assume that M is a non-solvable maximal subgroup of G. By Lemma 3.1 and the hypothesis, M has exactly 21 non-cyclic non-TI-subgroups. It follows that every maximal subgroup of G is a TI-subgroup. By Lemma 2.4, G is solvable, a contradiction.

So every maximal subgroup of G is solvable. It follows that $G/\Phi(G)$ is a minimal non-abelian simple group. Therefore, every maximal subgroup of G is a non-cyclic non-TI-subgroup. By Lemma 2.3 (1) and the hypothesis, G has exactly 21 maximal subgroups. It follows that $G/\Phi(G) \cong A_5$ by Lemma 2.3 (2).

We claim that $\Phi(G) = 1$.

Otherwise, assume that $\Phi(G) \neq 1$. Let P be a Sylow 2-subgroup of G. Then $P\Phi(G)/\Phi(G)$ is a Sylow 2-subgroup of $G/\Phi(G)$. Obviously, $P\Phi(G)$ is non-cyclic and $P\Phi(G)$ is not a TI-subgroup of G. Moreover, $P\Phi(G)$ is not a maximal subgroup of G. It follows that G has more than 21 non-cyclic non-TI-subgroups, a contradiction.

So $\Phi(G) = 1$, and then $G \cong A_5$. □

Lemma 3.3 *Let G be a group having exactly 22 or 23 non-cyclic non-TI-subgroups, then G is solvable.*

Proof. Assume that G is non-solvable. Then every maximal subgroup of G is non-cyclic. We claim that every maximal subgroup of G is solvable.

Otherwise, assume that M is a non-solvable maximal subgroup of G. By Lemma 3.1 and the hypothesis, M has exactly 21 or 22 or 23 non-cyclic non-TI-subgroups. It follows that G has at most two maximal subgroups that are not TI-subgroups. Since the number of conjugates of any non-normal maximal subgroup of G must be greater than 2, it follows that every maximal subgroup of G is a TI-subgroup. Then G is solvable by Lemma 2.4, a contradiction.

Therefore, G is a minimal non-solvable group. It follows that $G/\Phi(G)$ is a minimal non-abelian simple group. Obviously, every maximal subgroup of G is not a TI-subgroup. By Lemma 2.3 (1) and the hypothesis, G might have exactly 21 or 22 or 23 maximal subgroups. It follows that $G/\Phi(G)$ might be isomorphic to A_5 or $\mathrm{PSL}(2,7)$ by Lemma 2.3. Obviously, $\mathrm{PSL}(2,7)$ has more than 23 non-cyclic non-TI-subgroups. Therefore, $G/\Phi(G)$ cannot be isomorphic to $\mathrm{PSL}(2,7)$.

Next assume that $G/\Phi(G) \cong A_5$. Obviously, by Lemma 3.2, $\Phi(G) \neq 1$. Let Q be a Sylow 2-subgroup of G. It is easy to see that $Q\Phi(G)$ is a non-cyclic non-TI-subgroup of G. Since $|G : N_G(Q\Phi(G))| = |G/\Phi(G) : N_{G/\Phi(G)}(Q\Phi(G)/\Phi(G))| = 5$ and $Q\Phi(G)$ is not a maximal subgroup of G, it follows that G has at least 26 non-cyclic non-TI-subgroups, a contradiction. Hence, G is solvable. $\square$

Lemma 3.4 *Let G be a group having exactly 24 or 25 non-cyclic non-TI-subgroups, then G is solvable.*

Proof. Suppose not. Assume that G is non-solvable. Then every maximal subgroup of G is non-cyclic. If G has a non-solvable maximal subgroup, say M. By Lemmas 3.1, 3.2, 3.3 and the hypothesis, M might have exactly 21 or 24 or 25 non-cyclic non-TI-subgroups. Arguing as in proof of Lemma 3.3, M cannot have exactly 24 or 25 non-cyclic non-TI-subgroups. If M has exactly 21 non-cyclic non-TI-subgroups, then $M \cong A_5$ by Lemma 3.2. It follows that G has at most four maximal subgroups that are not TI-subgroups. Let H be a maximal subgroup of G that is not a TI-subgroup. Obviously, $H \ntrianglelefteq G$. One has $|G : N_G(H)| = |G : H| = n$, where $3 \leq n \leq 4$. It follows that $G/\mathrm{Core}_G(H) \lesssim S_n$ for $3 \leq n \leq 4$, where $\mathrm{Core}_G(H)$ is the largest normal subgroup of G that is contained in H. Since S_n is solvable for $3 \leq n \leq 4$, $G/\mathrm{Core}_G(H)$ has a normal maximal subgroup, say $K/\mathrm{Core}_G(H)$. Then K is a normal maximal subgroup of G. Obviously, $K \neq M$. Then $K \cap M = 1$. One has $G = K \rtimes M$. It follows that $M \cong G/K$ is a cyclic group of prime order, a contradiction.

Therefore, every maximal subgroup of G is solvable. It follows that $G/\Phi(G)$ is a minimal non-abelian simple group. It is clear that every maximal subgroup of G is not a TI-subgroup. By Lemma 2.3 (1) and the hypothesis, G might have exactly 21 or 22 or 23 or 24 or 25 maximal subgroups. If $\Phi(G) = 1$, by Lemmas 2.3 and 2.5, G only may be isomorphic to A_5 or $\mathrm{PSL}(2,7)$. Obviously, both A_5 and $\mathrm{PSL}(2,7)$ do not have exactly 24 or 25 non-cyclic non-TI-subgroups. It follows that $\Phi(G) \neq 1$. By Lemmas 2.3 and 2.5, $G/\Phi(G)$ only may be isomorphic to A_5. Arguing as in proof of Lemma 3.3, we have that G has at least 26 non-cyclic non-TI-subgroups, a contradiction. So G is solvable. $\square$

Lemma 3.5 *Let G be a non-solvable group having exactly 26 non-cyclic non-TI-subgroups, then $G \cong \mathrm{SL}(2,5)$.*

Proof. Since G is non-solvable, every maximal subgroup of G is non-cyclic.

We claim that every maximal subgroup of G is solvable.

Otherwise, assume that M is a non-solvable maximal subgroup of G. Arguing as in proof of Lemma 3.4, M only may have exactly 21 non-cyclic non-TI-subgroups and G has exactly five conjugate maximal subgroups that are not TI-subgroups. By Lemma 3.2, $M \cong A_5$. Let H be a maximal subgroup of G that is not a TI-subgroup. Since $|G : H| = |G : N_G(H)| = 5$, one has $G/\mathrm{Core}_G(H) \lesssim S_5$. Since $G/\mathrm{Core}_G(H)$ has a non-normal maximal subgroup $H/\mathrm{Core}_G(H)$ with index 5 in $G/\mathrm{Core}_G(H)$, one has $G/\mathrm{Core}_G(H) \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ or A_5 or S_5.

(i) Suppose that $G/\mathrm{Core}_G(H) \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ or S_5. We have that $G/\mathrm{Core}_G(H)$ has a normal maximal subgroup $L/\mathrm{Core}_G(H)$. Then L is a normal maximal subgroup of G. Obviously, $L \neq M$. Then $L \cap M = 1$. One has $G = L \rtimes M$. It follows that $M \cong G/L$ is a cyclic group of prime order, a contradiction.

(ii) Suppose that $G/\mathrm{Core}_G(H) \cong A_5$, one has $H/\mathrm{Core}_G(H) \cong A_4$. If $\mathrm{Core}_G(H) \leq M$, one has $\mathrm{Core}_G(H) = 1$. Then $G \cong A_5$, this contradicts the hypothesis. If $\mathrm{Core}_G(H) \nleq M$, one has $\mathrm{Core}_G(H) \cap M = 1$. It follows that $G = \mathrm{Core}_G(H) \rtimes M$. Let M_1 be any non-cyclic non-TI-subgroup of G contained in M. It is easy to see that $\mathrm{Core}_G(H) \rtimes M_1$ is also a non-cyclic non-TI-subgroup of G. It follows that G has at least 42 non-cyclic non-TI-subgroups, a contradiction.

So every maximal subgroup of G is solvable. It follows that $G/\Phi(G)$ is a minimal non-abelian simple group. Therefore, every maximal subgroup of G is not a TI-subgroup. By the hypothesis, G has at most 26 maximal subgroups. Moreover, by Lemmas 2.3, 2.5 and the hypothesis, we can get that $G/\Phi(G) \cong A_5$. Obviously, $\Phi(G) \neq 1$.

We claim that $\Phi(G)$ is cyclic. Otherwise, assume that $\Phi(G)$ is non-cyclic. Let E be any subgroup of G satisfying $\Phi(G) < E < G$. Since $G/\Phi(G) \cong A_5$, one has that E is a non-cyclic non-TI-subgroup of G. Observe that A_5 has more than 26 non-trivial subgroups. It follows that G has more than 26 non-cyclic non-TI-subgroups, a contradiction. Thus $\Phi(G)$ is cyclic.

If $|\Phi(G)|$ has an odd prime divisor, say p. Let F be a maximal subgroup of $\Phi(G)$ such that $|\Phi(G) : F| = p$. Since $\Phi(G)$ is cyclic, one has $F \trianglelefteq G$. Let $\bar{G} = G/F$. Then $\bar{G}/\Phi(\bar{G}) = (G/F)/\Phi(G/F) = (G/F)/(\Phi(G)/F) \cong G/\Phi(G) \cong A_5$. It follows that $|\bar{G}| = 60p$. We denote by $\pi(A)$ the set of all prime divisors of $|A|$, the order of group A. Since $\pi(G) = \pi(G/\Phi(G))$, we have $\pi(G) = \{2, 3, 5\}$. Then $p = 3$ or 5. It follows that $\bar{G}$ is a non-solvable group of order 180 or 300. Using the small groups library in GAP (see [2]), one has that $A_5 \times \mathbb{Z}_3$ is a unique non-solvable group of order 180 and $A_5 \times \mathbb{Z}_5$ is a unique non-solvable group of order 300. It follows that $\bar{G}$ might be isomorphic to $A_5 \times \mathbb{Z}_3$ or $A_5 \times \mathbb{Z}_5$. However, $\Phi(A_5 \times \mathbb{Z}_3) = 1 \ncong \mathbb{Z}_3$ and $\Phi(A_5 \times \mathbb{Z}_5) = 1 \ncong \mathbb{Z}_5$, this contradicts that $\Phi(\bar{G}) = \Phi(G)/F \cong \mathbb{Z}_p$. Thus $|\Phi(G)|$ cannot have any odd prime divisor. It follows that $\Phi(G)$ is a cyclic 2-group.

Let P be a Sylow 2-subgroup of G. Obviously, P is a non-cyclic non-TI-subgroup of G. Moreover, $|G : N_G(P)| = |G/\Phi(G) : N_{G/\Phi(G)}(P/\Phi(G))| = 5$ and P is not a maximal subgroup of G. Therefore, 26 non-cyclic non-TI-subgroups of G consist of 21 maximal subgroups of G and 5 Sylow 2-subgroups of G.

We claim that P is a minimal non-cyclic group.

Otherwise, assume that P_1 is a maximal subgroup of P such that P_1 is non-cyclic. By above arguments, we have that P_1 is a TI-subgroup of G. Since $P_1 \nleq \Phi(G)$ and $G/\Phi(G)$ is a non-abelian simple group, one has $P_1 \ntrianglelefteq G$. Obviously, $P_1 \cap \Phi(G) \trianglelefteq G$ since $\Phi(G)$ is cyclic. Since P_1 is a TI-subgroup of G, we must have that $P_1 \cap \Phi(G) = 1$. Then $\Phi(G) \cong \Phi(G)/(P_1 \cap \Phi(G)) \cong P_1\Phi(G)/P_1 = P/P_1 \cong \mathbb{Z}_2$. It follows that $G \cong \mathrm{SL}(2, 5)$. However, the Sylow 2-subgroup of $\mathrm{SL}(2, 5)$ is isomorphic to Q_8 so $\Phi(G) \leq P_1$, this is a contradiction.

So P is a minimal non-cyclic group. Since $|P| = 4|\Phi(G)| \geq 8$, we have that $P \cong Q_8$, the quaternion group of order 8. Then $\Phi(G) \cong \mathbb{Z}_2$. It follows that $G \cong \mathrm{SL}(2, 5)$. □

Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5 combined together give Theorem 1.1. □

4 Remark

From above arguments, a natural question arises:

Question 4.1 *Let G be a group having exactly $\delta(G)$ non-cyclic non-TI-subgroups and $\tau(G)$ non-cyclic TI-subgroups. Suppose that G is non-solvable, is it always true that $\delta(G) > \tau(G)$?*

Obviously, Question 4.1 is true for A_5 and $\mathrm{SL}(2, 5)$.

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