Interval estimation for Gumbel Distribution using climate records

A. Asgharzadeh[a*], **M. Abdi**[b], **S. Nadarajah**[c]

[a] *Department of Statistics, University of Mazandaran, Babolsar, Iran*

[b] *Department of Statistics, Higher Education complex of Bam, Bam, Iran*

[c] *School of Mathematics, University of Manchester, Manchester M13 9PL, UK*

Abstract

The Gumbel distribution is one of the most popular widely used distributions in climate modeling. In this paper, we present exact confidence intervals and joint confidence regions for the parameters of Gumbel distribution based on record data. Exact confidence intervals and joint confidence regions for the parameters of inverse Weibull distribution are also discussed. Three numerical examples with climate data are presented to illustrate the proposed methods. A simulation study is conducted to study the performance of the proposed confidence interval and region.

Keywords: Interval estimation, Joint confidence region, Record data, Gumbel distribution.

2000 Mathematics Subject Classification: 62F25; 62E15.

1 Introduction

There are several situations pertaining to meteorology, hydrology, largest insurance claims, seismology and athletic events in which only observations that exceed or only those that fall below the current extreme value are recorded and the complete data are not available. For example, an electronic component ceases to function in an environment of too high temperature, a battery dies under the stress of time and a wooden beam breaks when sufficient perpendicular force is applied to it. Moreover, in some reliability experiments, the units that are experimented on are destroyed. If units are expensive the cost of the experiment is mainly the cost of the destroyed units. In such cases it is possible to set up the experiment in such a way that only units whose life length are (lower) record values are destroyed (see Ahmadi and Arghami, 2003). Hence, in such experiments, measurements may be made sequentially and only the record values are observed. Data of this type are called Record

*E-mail addresses: *a.asgharzadeh@umz.ac.ir* (A. Asgharzadeh), *me.abdi@bam.ac.ir* (M. Abdi), *mbbsssn2@manchester.ac.uk* (S. Nadarajah)

Data or Records. The statistical study of record values started with Chandler (1952) and has now spread in different directions. For more details on records and its applications, see Nevzorov (1988), Ahsanullah (1995) and Arnold et. al. (1998).

A random variable X is said to have Gumbel distribution, if its cumulative distribution function (cdf) is

$$F(x;\mu,\sigma) = e^{-e^{-\frac{x-\mu}{\sigma}}}, \quad x \in R, \quad \mu \in R, \quad \sigma > 0, \tag{1.1}$$

where μ and σ are the location and scale parameters, respectively. The Gumbel probability density function (pdf) has the form

$$f(x;\mu,\sigma) = \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma} - e^{-\frac{x-\mu}{\sigma}}}. \tag{1.2}$$

The Gumbel distribution was introduced by Gumbel (1958) and since then it received a considerable attention in the literature. It is frequently used in engineering and climate modeling. The book by Kotz and Nadarajah (2000) which describes the Gumbel distribution, presents some of its application areas in engineering include flood frequency analysis, network engineering, nuclear engineering, offshore engineering, space engineering, software reliability engineering, structural engineering, and wind engineering. See also Nadarajah (2006), Persson and Rydén (2010) and Cooray (2010) for some generalizations of the Gumbel distribution.

Many authors have studied statistical inference based on record data from the Gumbel distribution. Balakrishnan et al. (1992) derived some recurrence relations for the single and product moments of record values from Gumbel distribution. Nagaraja (1988) proved that some inference procedures based on asymptotic theory of extreme order statistics are equivalent to those based on record values from the Gumbel distribution. Based on record data, Ahsanullah (1990, 1991) derived the maximum likelihood, best linear invariant and minimum variance unbiased estimators of the Gumbel location and scale parameters μ and σ. Also, he presented two types of predictors of the s^{th} record value based on the first m ($m < s$) record values. Ali Mousa et al. (2002) considered Bayesian estimation, prediction and characterization for Gumbel distribution based on record data. Recently, Ahsanullah and Shakil (2013) have studied the characterizations of Rayleigh distribution based on order statistics and record values.

The purpose of this paper is to construct the interval estimation for the parameters of the Gumbel distribution based on lower record values. The rest of this paper is organized as follows. Section 2 provides some preliminaries. In Section 3, we present an exact confidence interval (CI) for parameter σ and an exact joint confidence region (JCR) for the parameters μ and σ. In Section 4, the exact confidence intervals and joint confidence regions for the parameters of inverse Weibull distribution are discussed. Section 5 discusses three numerical examples with climate data for illustration. In Section 6, a Monte Carlo simulation is conducted to study the performance of the proposed confidence interval and region.

2 Preliminaries

Let $X_1,\ X_2, \ldots$ be a sequence of independent and identically distributed (iid) continuous random variables with cdf $F(x)$ and pdf $f(x)$. An observation X_j is called an upper (lower) record value of this sequence if its value exceeds (is lower than) that of all previous observations. Generally, let us define $T_1 = 1,\ U_1 = X_1$, and for $n \geq 2$

$$T_n = \min\{j > T_{n-1} : X_j > X_{T_{n-1}}\}, \quad U_n = X_{T_n}.$$

Then the sequence $\{U_n\}(\{T_n\})$ is known as upper record statistics (upper record times). Similarly, the lower record times S_n and the lower record values L_n are defined as follows: $S_1 = 1,\ L_1 = X_1$, and for $n \geq 2,\ S_n = \min\{j > S_{n-1} : X_j < X_{S_{n-1}}\},\ L_n = X_{S_n}$. The following lemmas are useful in this paper.

Lemma 2.1. *Let $L_1 > L_2 > \cdots > L_m$ be the first m observed lower record values from a population with cdf $F(.)$. Define*

$$U_i = -\ln[F(L_i)], \quad i = 1, 2, \ldots, m.$$

Then $U_1 < U_2 < \cdots < U_m$ are the first m upper record values from a standard exponential distribution.

Proof: From the joint pdf of $L_1, L_2, ..., L_m$ and using a simple Jacobian argument, we can easily obtain the joint pdf of $U_1, U_2, ..., U_m$ as

$$f_{U_1,U_2,\ldots,U_m}(u_1, u_2, ..., u_m) = e^{-u_m}, \quad 0 < u_1 < u_2 < ... < u_m,$$

which is the joint pdf of the first m upper record values from a standard exponential distribution (see Arnold et al. (1998)). The proof is thus obtained.

Lemma 2.2. *If $U_1 < U_2 < \cdots < U_m$ are the first m upper record values from a standard exponential distribution. Then the spacings $U_1, U_2 - U_1, \ldots, U_m - U_{m-1}$ are iid random variables from a standard exponential distribution.*

Proof: The proof can be found in Arnold et al. (1998).

3 Confidence Interval and Joint Confidence Region

Let $L_1 > L_2 > \cdots > L_m$ be the first m observed lower record values from the Gumbel distribution. In this section, a $100(1-\alpha)\%$ confidence interval for scale parameter σ and a $100(1-\alpha)\%$ joint confidence region for (μ, σ) are constructed based on the observed lower records $\underline{L} = (L_1, L_2, \ldots, L_m)$. A $100(1-\alpha)\%$ joint confidence region (or set) for (μ, σ) is a random set $C(\underline{L})$ such that

$$P\left[(\mu, \sigma) \in C(\underline{L})\right] = 1 - \alpha.$$

One of the applications of the joint confidence regions of the parameters is to find confidence bounds for the functions of the two parameters μ and σ.

The most popular approach in obtaining confidence intervals and joint confidence regions is to use pivotal quantities (functions of data and of unknown parameters whose distributions are known and parameter free). Another useful approach is that use the approximate normality of maximum likelihood estimates (MLEs) and obtain the large sample confidence intervals and joint confidence regions. Since the asymptotic theory of the MLEs of the parameters are not practical in case of records data (see Arnold et al. (1998), P. 24)), we shall use here the first approach to obtain confidence intervals and joint confidence regions.

Let us define

$$Y_i = \exp\left(-\frac{L_i - \mu}{\sigma}\right), \qquad i = 1, 2, \ldots, m. \tag{3.1}$$

Then, by Lemma 2.1, $Y_1 < Y_2 < \cdots < Y_m$ are the first m upper record values from a standard exponential distribution. Moreover, by Lemma 2.2, we can observe that

$$\begin{cases} Z_1 = Y_1 \\ Z_2 = Y_2 - Y_1 \\ \vdots \\ Z_m = Y_m - Y_{m-1} \end{cases} \tag{3.2}$$

are iid random variables from a standard exponential distribution. Hence

$$V = 2Z_1 = 2\,Y_1$$

has a chi-square distribution with 2 degrees of freedom and

$$U = 2\sum_{i=2}^{m} Z_i = 2\,(Y_m - Y_1)$$

has a chi-square distribution with $2m-2$ degrees of freedom. We can also find that U and V are independent random variables. Let

$$P_1 = \frac{U/2(m-1)}{V/2} = \frac{U}{(m-1)V} = \frac{1}{m-1}\left(\frac{Y_m - Y_1}{Y_1}\right), \tag{3.3}$$

and

$$P_2 = U + V = 2Y_m. \tag{3.4}$$

It is easy to show that P_1 has an F distribution with $2m-2$ and 2 degrees of freedom and P_2 has a chi-square distribution with $2m$ degrees of freedom. Furthermore, P_1 and P_2 are independent, see Johnson et al. (1994, P. 350).

Let $F_\alpha(\upsilon_1, \upsilon_2)$ be the percentile of F distribution with right-tail probability α and υ_1 and υ_2 degrees of freedom. Next theorem gives an exact confidence interval for the scale parameter σ base on lower record values.

Theorem 3.1. *Suppose that $L_1 > L_2 > \cdots > L_m$ be the first m observed lower record values from the Gumbel distribution in* (1.1)*. Then, for any $0 < \alpha < 1$,*

$$\frac{L_1 - L_m}{\ln[1 + (m-1)F_{\frac{\alpha}{2}}(2m-2, 2)]} < \sigma < \frac{L_1 - L_m}{\ln[1 + (m-1)F_{1-\frac{\alpha}{2}}(2m-2, 2)]}$$

is a $100(1-\alpha)\%$ confidence interval for σ.

Proof. From (3.3), we know that the pivot

$$P_1(\sigma) = \frac{1}{m-1}\left[\frac{e^{-\frac{L_m-\mu}{\sigma}} - e^{-\frac{L_1-\mu}{\sigma}}}{e^{-\frac{L_1-\mu}{\sigma}}}\right] = \frac{1}{m-1}\left[e^{\frac{L_1-L_m}{\sigma}} - 1\right],$$

has an F distribution with $2m-2$ and 2 degrees of freedom. Hence for $0 < \alpha < 1$, we obtain

$$P\left(F_{1-\frac{\alpha}{2}}(2m-2, 2) < P_1 < F_{\frac{\alpha}{2}}(2m-2, 2)\right) = 1 - \alpha,$$

is equivalent to

$$P\left(\frac{L_1 - L_m}{\ln[1 + (m-1)F_{\frac{\alpha}{2}}(2m-2, 2)]} < \sigma < \frac{L_1 - L_m}{\ln[1 + (m-1)F_{1-\frac{\alpha}{2}}(2m-2, 2)]}\right) = 1 - \alpha$$

This completes the proof.

It should be mentioned here that we can also use $P_1(\sigma)$ to test null hypothesis $H_0 : \sigma = \sigma_0$.

Let $\chi^2_\alpha(\upsilon)$ denote the percentile of χ^2 distribution with right-tail probability α and υ degrees of freedom. Next theorem gives an exact joint confidence region for the parameters μ and σ.

Theorem 3.2. *Suppose that $L_1 > L_2 > \cdots > L_m$ be the first m observed lower record values from Gumbel distribution. Then, the following inequalities determine $100(1-\alpha)\%$ joint confidence region for μ and σ:*

$$\begin{cases} \frac{L_1-L_m}{\ln[1+(m-1)F_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2,2)]} < \sigma < \frac{L_1-L_m}{\ln[1+(m-1)F_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2,2)]}, \\ L_m + \sigma \ln\left(\frac{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m)}{2}\right) < \mu < L_m + \sigma \ln\left(\frac{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m)}{2}\right). \end{cases}$$

Proof. From (3.4), we know that

$$P_2(\mu,\sigma) = 2\ e^{-\frac{L_m-\mu}{\sigma}}$$

has a χ^2 distribution with $2m$ degrees of freedom, and it is independent of P_1. Hence, for $0 < \alpha < 1$, we have

$$P\left[F_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2,2) < P_1 < F_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2,2)\right] = \sqrt{1-\alpha},$$

and

$$P\left[\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m) < P_2 < \chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m)\right] = \sqrt{1-\alpha}.$$

From these relationships, we conclude that

$$\begin{aligned} P\Bigg[&F_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2,2) < \frac{e^{\frac{L_1-L_m}{\sigma}}-1}{m-1} < F_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2,2) \\ &, \chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m) < 2\ e^{-\frac{L_m-\mu}{\sigma}} < \chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m)\Bigg] \\ &= 1-\alpha, \end{aligned}$$

or equivalently

$$\begin{cases} \frac{L_1-L_m}{\ln[1+(m-1)F_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2,2)]} < \sigma < \frac{L_1-L_m}{\ln[1+(m-1)F_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2,2)]}, \\ L_m + \sigma \ln\left(\frac{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m)}{2}\right) < \mu < L_m + \sigma \ln\left(\frac{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m)}{2}\right). \end{cases}$$

As an application, Theorem 3.2 can be used to obtain a lower confidence bound for the expected value and reliability function of a Gumbel distribution. Here we only obtain a lower confidence bound for the expected value. The expected value of the Gumbel distribution is $E(X) = \mu + \gamma\sigma$, where $\gamma = 0.57722$ is the Euler's constant. The following corollary will be used to obtain the lower confidence bound for $E(X)$. The proof is easy and omitted.

Corollary 3.3. *Suppose that* $L_1 > L_2 > \cdots > L_m$ *be the first* m *observed lower record values from Gumbel distribution. Then, the following inequalities determine* $100(1-\alpha)\%$ *joint confidence region for the parameters* μ *and* σ*:*

$$\begin{cases} \frac{L_1-L_m}{\ln[1+(m-1)F_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2,2)]} < \sigma < \frac{L_1-L_m}{\ln[1+(m-1)F_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2,2)]}, \\ \\ \mu > L_m + \sigma \ln\left(\frac{\chi^2_{\sqrt{1-\alpha}}(2m)}{2}\right) \end{cases}$$

where $0 < \alpha < 1$.

Theorem 3.4. *Suppose that* $L_1 > L_2 > \cdots > L_m$ *be the first* m *observed lower record values from Gumbel distribution. Then for any* $0 < \alpha < 1$,

$$\inf_{\sigma}\left\{L_m + \sigma\left[\ln\left(\frac{\chi^2_{\sqrt{1-\alpha}}(2m)}{2}\right) + \gamma\right]\right\},$$

is a $(1-\alpha)100\%$ *lower confidence bound for the expected value* $E(X)$*, where the infimum is taken over the interval*

$$\left\{\sigma : \frac{L_1 - L_m}{\ln[1+(m-1)F_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2,2)]} < \sigma < \frac{L_1 - L_m}{\ln[1+(m-1)F_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2,2)]}\right\}.$$

Proof. For fixed t and σ, $E(X)=\mu+\gamma\sigma$ is increasing in μ, then

$$\begin{aligned}
P & \left[E(X) > \inf_\sigma \left[L_m + \sigma\left[\ln\left(\frac{\chi^2_{\sqrt{1-\alpha}}(2m)}{2}\right)+\gamma\right]\right] : \right. \\
& \left. \frac{L_1-L_m}{\ln[1+(m-1)F_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2,2)]} < \sigma < \frac{L_1-L_m}{\ln[1+(m-1)F_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2,2)]}\right] \\
= & \ P\left[E(X) > \inf_\sigma[\mu+\gamma\sigma] : \mu = L_m + \sigma\ln\left(\frac{\chi^2_{\sqrt{1-\alpha}}(2m)}{2}\right), \right. \\
& \left. \frac{L_1-L_m}{\ln[1+(m-1)F_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2,2)]} < \sigma < \frac{L_1-L_m}{\ln[1+(m-1)F_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2,2)]}\right] \\
= & \ P\left[E(X) > \inf_\sigma[\mu+\gamma\sigma] : \mu > L_m + \sigma\ln\left(\frac{\chi^2_{\sqrt{1-\alpha}}(2m)}{2}\right), \right. \\
& \left. \frac{L_1-L_m}{\ln[1+(m-1)F_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2,2)]} < \sigma < \frac{L_1-L_m}{\ln[1+(m-1)F_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2,2)]}\right] \\
= & \ P\left[\mu > L_m + \sigma\ln\left(\frac{\chi^2_{\sqrt{1-\alpha}}(2m)}{2}\right), \right. \\
& \left. \frac{L_1-L_m}{\ln[1+(m-1)F_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2,2)]} < \sigma < \frac{L_1-L_m}{\ln[1+(m-1)F_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2,2)]}\right] \\
= & \ 1-\alpha
\end{aligned}$$

The proof is thus completed.

4 Results for Inverse Weibull distribution

The results in Theorems 3.1 and 3.2 can be used for constructing exact confidence interval and joint confidence region for the parameters of inverse Weibull distribution. It is known that, if the random variable X has a Gumbel distribution in (1.1), then $Z=\exp(X)$ has the inverse Weibull distribution with cdf as

$$G(z) = e^{-\lambda z^{-\beta}},$$

where $\beta = 1/\sigma$ and $\lambda = \exp(\beta\mu)$.

In the next theorem, an exact confidence interval for β and an exact joint confidence regions for the parameters β and λ are given. The proof is easy and omitted.

Theorem 4.1. *Suppose that $Z_1 > Z_2 > \cdots > Z_m$ be the first m observed lower record values from inverse Weibull distribution. Then, a $100(1-\alpha)\%$ confidence interval for β is*

$$\frac{\ln[1+(m-1)F_{1-\frac{\alpha}{2}}(2m-2,2)]}{\ln Z_1 - \ln Z_m} < \beta < \frac{\ln[1+(m-1)F_{\frac{\alpha}{2}}(2m-2,2)]}{\ln Z_1 - \ln Z_m},$$

and a $100(1-\alpha)\%$ joint confidence region for parameters β and λ is determined by the following inequalities

$$\begin{cases} \frac{\ln[1+(m-1)F_{\frac{1+\sqrt{1-\alpha}}{2}}(2m-2,2)]}{\ln Z_1 - \ln Z_m} < \beta < \frac{\ln[1+(m-1)F_{\frac{1-\sqrt{1-\alpha}}{2}}(2m-2,2)]}{\ln Z_1 - \ln Z_m}, \\ \frac{\chi^2_{\frac{1+\sqrt{1-\alpha}}{2}}(2m)}{2} e^{\beta \ln Z_m} < \lambda < \frac{\chi^2_{\frac{1-\sqrt{1-\alpha}}{2}}(2m)}{2} e^{\beta \ln Z_m}. \end{cases}$$

5 Applications

In this section, three examples with climate record data are given to illustrate the proposed confidence intervals and joint confidence regions.

5.1 Example 1. (Air temperature data)

The following data represents the first six lower records of the average annual mean monthly air temperatures (in degrees centigrade) at Babolsar city in north of Iran from 1951 to 2000 (see the website: http://www.iranhydrology.com/meteo.asp).

17.4, 16.7, 16.2, 15.9, 15.5, 14.9.

A simple plot of these six lower records against the expected values of the first Gumbel lower record values in Table 1 , we observe a very strong correlation (as high as 0.9693). This indicates that the Gumbel model provides a good fit to these record values.

Table 1. Expected value of the first Gumbel lower record values.

i	1	2	3	4	5	6
$E(L_i)$	15.2206	8.9549	5.8221	3.7335	2.1671	0.9140.

To find a 95% confidence interval for σ, and a joint confidence region for μ and σ, we need the following percentiles:

$$F_{0.025(10,2)} = 39.39797, \quad F_{0.975(10,2)} = 0.1832712,$$

$$F_{0.0127(10,2)} = 78.13913, \quad F_{0.9873(10,2)} = 0.1434067,$$

and

$$\chi^2_{0.0127(12)} = 25.4812\ ,\ \chi^2_{0.9873(12)} = 3.765882.$$

Using the methods described in Section 3, we can obtain CI and JCR for the parameters.

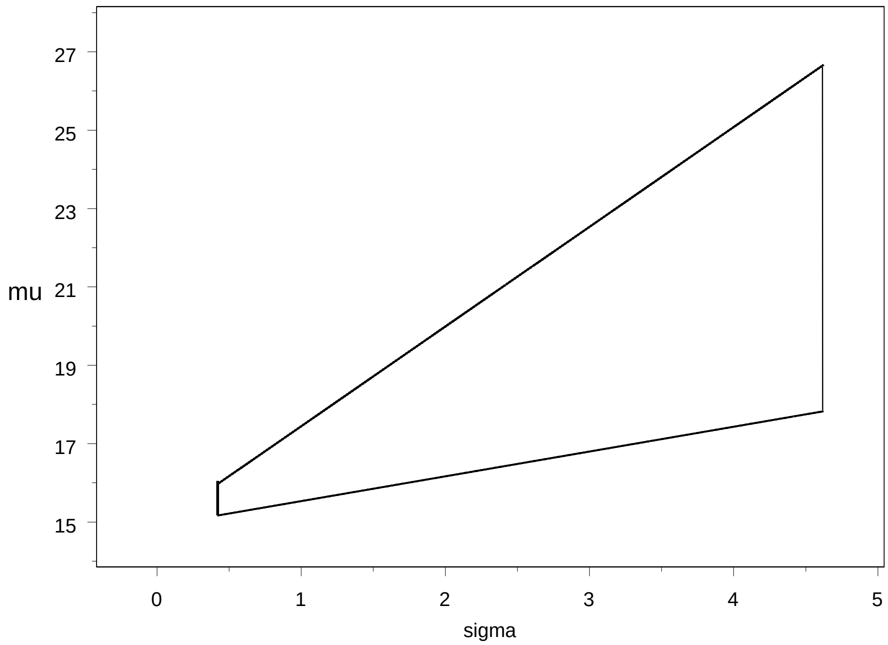

Figure 1: Joint confidence region for parameters in Example 1.

By Theorem 3.1, the 95% CI for σ is (0.4728, 3.8436), with confidence length 3.3808. By Theorem 3.2, the 95% JCR for μ and σ is determined by the following inequalities:

$$0.4187 < \sigma < 4.6245$$

$$14.9 + \sigma \ln\left(\frac{3.7659}{2}\right) < \mu < 14.9 + \sigma \ln\left(\frac{25.4812}{2}\right),$$

with area

$$\int_{0.4187}^{4.6245} \int_{14.9+\sigma\ln\left(\frac{3.7659}{2}\right)}^{14.9+\sigma\ln\left(\frac{25.4812}{2}\right)} d\mu\ d\sigma = 20.27702.$$

Figure 1 shows the above joint confidence region. Also a 95% lower confidence bound for the expected value $E(X)$ is:

$$\inf_{\sigma}[14.9 + \sigma(\ln(4.420471/2) + 0.57722)] = \inf_{\sigma}[14.9 + 1.370319\sigma]$$

where the infimum is taken over $0.4187 < \sigma < 4.6245$. Thus, the 95% lower confidence bound for $E(X)$ is $14.9 + (1.370319)(0.4187) = 15.47378$.

5.2 Example 2. (Rainfall data)

In this example we present a data analysis and illustrate application of the results in Section 3 to the seasonal (July 1-June 30) rainfall in inches recorded at Los Angeles Civic Center from 1962 to 2012 (see the website of Los Angeles Almanac: http://www.laalmanac.com/weather/we13.htm). The data are as follows:

```
08.38, 07.93, 13.68, 20.44, 22.00, 16.58, 27.47, 07.74, 12.32,
07.17, 21.26, 14.92, 14.35, 07.21, 12.30, 33.44, 19.67, 26.98,
08.96, 10.71, 31.28, 10.43, 12.82, 17.86, 07.66, 12.48, 08.08,
07.35, 11.99, 21.00, 27.36, 08.11, 24.35, 12.44, 12.40, 31.01,
09.09, 11.57, 17.94, 04.42, 16.42, 09.25, 37.96, 13.19, 03.21,
13.53, 09.08, 16.36, 20.20, 08.69.
```

Here, we checked the validity of the Gumbel based on the parameters $\widehat{\mu} = 11.604, \widehat{\sigma} = 6.2657$, using the Kolmogorov Smirnov (K-S) test. It is observed that the K-S distance is $K - S = 0.09227$ with a corresponding $p - value = 0.76863$. So, the Gumbel model provides a good fit to the above data.

If only the lower record values of the seasonal rainfall have been observed, these are

```
8.38, 7.93, 7.74, 7.17, 4.42, 3.21.
```

By Theorem 3.1, the 95% CI for σ is (0.9776454, 7.948644), with confidence length 6.970999. By Theorem 3.2, the 95% JCR for μ and σ is determined by the following inequalities:

$$0.8659263 < \sigma < 9.56348,$$
$$3.21 + \sigma \ln\left(\frac{3.7659}{2}\right) < \mu < 3.21 + \sigma \ln\left(\frac{25.4812}{2}\right)$$

with area 86.71719. Figure 2 shows the above joint confidence region. In this example, a 95% lower confidence bound for the expected value $E(X)$ is:

$$\inf_{\sigma}[3.21 + \sigma(\ln(4.420471/2) + 0.57722)] = \inf_{\sigma}[3.21 + 1.370319\sigma]$$

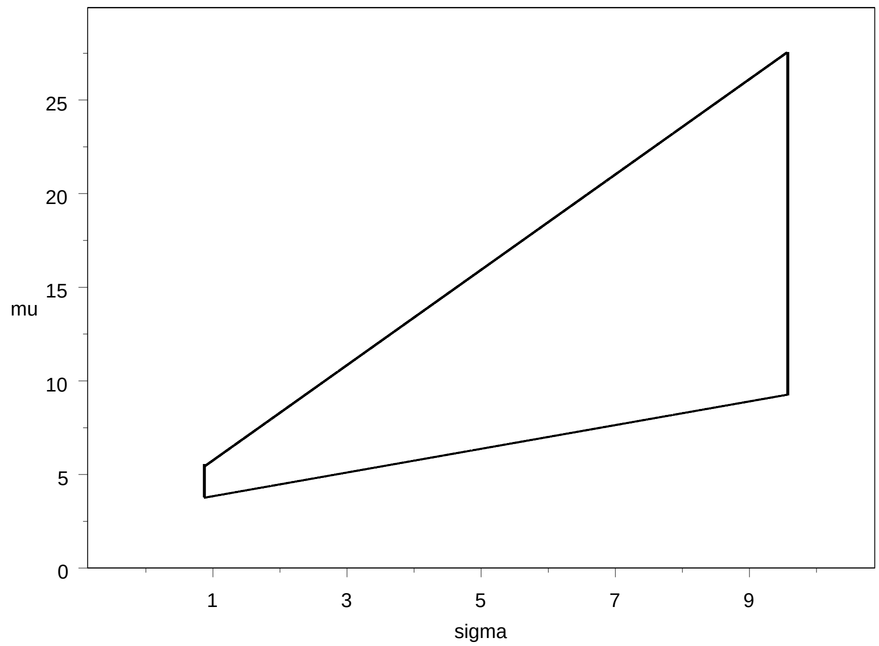

Figure 2: Joint confidence region for parameters in Example 2.

where the infimum is taken over $0.8659263 < \sigma < 9.56348$. Thus, we obtain the 95% lower confidence bound for $E(X)$ as $3.21 + (1.370319)(0.8659263) = 4.396657$.

5.3 Example 3. (Floods data)

Consider the data given by Dumonceaux and Antle (1973), represents the maximum flood levels (in millions of cubic feet per second) of the Susquehenna River at Harrisburg, Pennsylvenia over 20 four-year periods (1890-1969) as:

```
0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379,
0.324, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392,
0.484, 0.265
```

Maswadah (2003) showed that the inverse weibull distribution with cdf and pdf

$$G(y, \beta, \lambda) = e^{-\lambda y^{-\beta}}, \quad y > 0, \quad \lambda > 0, \quad \beta > 0,$$

$$g(y, \beta, \lambda) = \lambda \beta y^{-\beta-1} e^{-\lambda y^{-\beta}}$$

provides a very good fit to the given data set. Using the method described in Section 4, we can obtain CI and JCR for the parameters β and λ.

Now the lower records of the maximum flood level are as follows:

```
0.654, 0.613, 0.315, 0.297, 0.269, 0.265.
```

By Theorem 4.1, the 95% CI for β is (0.7200, 5.8548), with confidence length 5.1338 and the 95% JCR for β and λ is determined by the following inequalities:

$$0.5984 < \beta < 6.6091,$$

$$\frac{3.7659}{2} e^{\beta \ln 0.265} < \lambda < \frac{25.4812}{2} e^{\beta \ln 0.265},$$

with area 3.6918. Figure 3 shows the above joint confidence region.

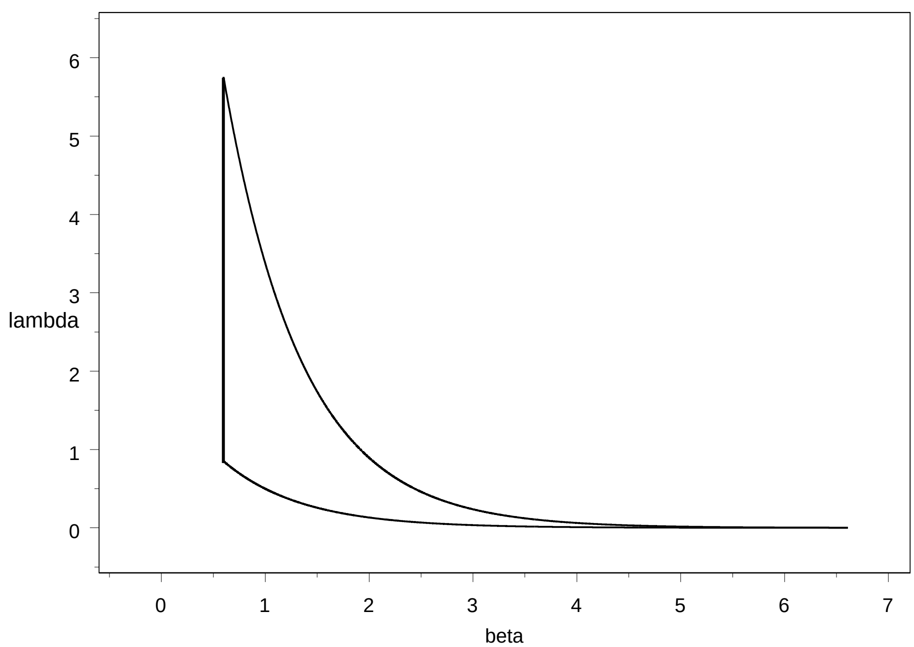

Figure 3: Joint confidence region for parameters in Example 3.

6 Simulation Study

In this section, a Monte Carlo simulation is conducted to study the performance of the proposed confidence interval and region. In this simulation, we randomly generate lower record sample $L_1, L_2, \ldots, L_m$ from the Gumbel distribution with the value of parameters $(\mu = 0, \sigma = 1)$, and then computed 95% confidence intervals and regions using the results presented in Section 3. We then replicated the process 10,000 times. We presented in Table 2, the simulated average confidence length for parameter σ, confidence area for the parameters (μ, σ), and the 95% coverage probabilities of the proposed confidence intervals and regions.

Table 2: The simulated average confidence length (CL), confidence area (CA) and 95% coverage probabilities (CP) for the parameters.

m	CL (σ)	CA(μ,σ)	CP(σ)	CP(μ,σ)
5	3.6551	35.3372	0.950	0.950
6	3.0851	21.7041	0.950	0.949
7	2.6974	14.9888	0.949	0.949
8	2.4411	11.1992	0.954	0.953
9	2.2645	9.0827	0.945	0.948
10	2.1241	7.4849	0.950	0.952
11	1.9912	6.2638	0.953	0.952
12	1.9044	5.5080	0.946	0.945
13	1.8283	4.8641	0.950	0.948
14	1.7568	4.3232	0.952	0.950
15	1.7027	3.9377	0.949	0.953
16	1.6574	3.6466	0.949	0.948
17	1.6153	3.3499	0.954	0.948
18	1.5747	3.1204	0.948	0.946
19	1.5404	2.9116	0.950	0.947
20	1.5085	2.7228	0.954	0.951
21	1.4816	2.5802	0.948	0.947
22	1.4608	2.4582	0.956	0.951
23	1.4337	2.3255	0.950	0.948
24	1.4170	2.2309	0.948	0.945
25	1.3830	2.0835	0.952	0.952
30	1.3031	1.7288	0.945	0.946

From Table 2, we observe when m increases, the average confidence length for σ, and the average confidence area for (μ, σ) are decreased.

The simulation results shows that the coverage probabilities of the exact confidence intervals for parameter σ and joint confidence regions for parameters (μ, σ) are close to the desired level of 0.95 for different sample sizes. Hence, our proposed methods for constructing exact confidence intervals and joint confidence regions can be used reliably.

Acknowledgments
The authors would like to thank the Editor and the referees for their useful comments which improved the paper.

References

[1] Ahmadi, J. and Arghami, N. R. (2003). Comparing the Fisher information in record values and iid observations. *Statistics*, 37(5), 435-441.

[2] Ahsanullah, M. (1990). Estimation of the parameters of the Gumbel distribution based on the m record values. *Computational Statistics Quarterly* , 6, 231-239.

[3] Ahsanullah, M. (1991). Inference and prediction of the Gumbel distribution based on record values. *Pakistan Journal of Statistics*, 7(3) B, 53-62.

[4] Ahsanullah, M. (1995), *Record Statistics*. Nova Science Publishers, Inc. Commack, New York.

[5] Ahsanullah, M. and Shakil, M. (2013). Characterizations of Rayleigh distribution based on order statistics and record values. *Bulletin of the Malaysian Mathematical Sciences Society, (2) 36 (3), 625-635.*

[6] Ali Mousa, M. A. M., Jaheen, Z. F. and Ahmad, A. A. (2002). Bayesian estimation, prediction and characterization for Gumbel distribution based on record data. *Statistics*, 36(1), 65-74.

[7] Arnold, B. C., Balakrishnan, N. and Nagaraja, H. N. (1998). *Records*. John Wiley & Sons, New York.

[8] Balakrishnan, N., Ahsanullah, M., and Chan, P. s. (1992). Relations for single and product moments of record values from Gumbel distribution. *Statistics and Probability Letters*, 15, 223-227.

[9] Chandler, K. N. (1952). The distribution and frequency of record values. *Journal of Royal Statist, Soc.* B14, 220-228.

[10] Cooray, K. (2010). Generalized Gumbel distribution. *Journal of Applied Statistics*. 37(1), 171-179.

[11] Dumonceaux, R. and Antle, C. E. (1973). Discrimination between the lognormal and Weibull distribution. *Technometrics*, 15, 923-926.

[12] Gumbel, E. J. (1958). *Statistics of Extremes*, Columbia University Press, New York.

[13] Johnson, N.L. Kotz, S. and Balakrishnan, N. (1994). *Continuous Univariate Distributions*, John Wiley & Sons, New York.

[14] Maswadah, M. (2003). Conditional confidence interval estimation for the inverse weibull distribution based on censored generalized order statistics. *Journal of Statistical Computation and Simulation*, 73(12), 887-898.

[15] Kotz S. and Nadarajah, S. (2000). *Extreme Value Distributions: Theory and Applications*, Imperial College Press, London.

[16] Nadarajah, S. (2006). The exponentiated Gumbel distribution with climate application. *Environmetrics*, 17, 1323.

[17] Nagaraja, H.N. (1988). Record values and related statistics-A review, *Communications in Statistics - Theory and Methods,* 17, 2223-2238.

[18] Nevzorov, V. B. (1988). Records, *Theory of Probability and its Applications,* 32, 201-228.

[19] Persson, K. and Rydén, J. (2010). Exponentiated Gumbel distribution for estimation of return levels of significant wave height. *Journal of Environmental Statistics*, 1(3), 1-12.