On Chen ideal submanifolds satisfying some conditions of pseudo-symmetry type

Ryszard Deszcz [1], Miroslava Petrović-Torgašev [2], Leopold Verstraelen [3] and Georges Zafindratafa

Dedicated to the memory of Professor Franki Dillen

Abstract In this paper, we study Chen ideal submanifolds M^n of dimension n in Euclidean spaces $\mathbb{E}^{n+m}$ ($n \geq 4$, $m \geq 1$) satisfying curvature conditions of pseudo-symmetry type of the form: the difference tensor $R \cdot C - C \cdot R$ is expressed by some Tachibana tensors. Precisely, we consider one of the following three conditions: $R \cdot C - C \cdot R$ is expressed as a linear combination of $Q(g, R)$ and $Q(S, R)$, $R \cdot C - C \cdot R$ is expressed as a linear combination of $Q(g, C)$ and $Q(S, C)$ and $R \cdot C - C \cdot R$ is expressed as a linear combination of $Q(g, g \wedge S)$ and $Q(S, g \wedge S)$. We then characterize Chen ideal submanifolds M^n of dimension n in Euclidean spaces $\mathbb{E}^{n+m}$ ($n \geq 4$, $m \geq 1$) which satisfy one of the following six conditions of pseudo-symmetry type: $R \cdot C - C \cdot R$ and $Q(g, R)$ are linearly dependent, $R \cdot C - C \cdot R$ and $Q(S, R)$ are linearly dependent, $R \cdot C - C \cdot R$ and $Q(g, C)$ are linearly dependent, $R \cdot C - C \cdot R$ and $Q(S, C)$ are linearly dependent, $R \cdot C - C \cdot R$ and $Q(g, g \wedge S)$ are linearly dependent and $R \cdot C - C \cdot R$ and $Q(S, g \wedge S)$ are linearly dependent. We also prove that the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are linearly at every point of M^n at which its Weyl tensor C is non-zero.

Mathematics Subject Classification (2010). Primary 53B20, 53B25, 53B30, 53B50; Secondary 53C25, 53C40.

Key words and phrases: submanifold, condition of pseudo-symmetry type, generalized Einstein metric condition, Chen ideal submanifold, Roter space, Tachibana tensor.

1 Some generalized Einstein metric conditions

As it was presented in [61]: Elie Cartan in his book [5] defined the axiom of r-planes as follows: a Riemannian manifold M of dimension $n > 3$ satisfies the axiom of r-planes, where r is a fixed integer $2 < r < n$, if for each point p of M and any r-dimensional subspace S of the tangent space $T_p(M)$ there exists an r-dimensional totally geodesic submanifold V containing p such that $T_p(V) = S$. He proved that if M satisfies the axiom of r-planes for some r, then M has constant sectional curvature ([5]). In [61] it was proposed the following axiom called *axiom of r-spheres*: for each point p of M and any r-dimensional subspace S of $T_p(M)$, there exists an r-dimensional umbilical submanifold V with parallel mean curvature vector field such that $p \in V$ and $T_p(V) = S$. In [61](Theorem) it was proved that a Riemannian manifold M of dimension $n > 3$ satisfies the axiom of r-spheres for some r, $2 < r < n$, then M has constant sectional curvature. Further, axioms of this kind (i.e. related to properties of submanifolds) were introduced and investigated by several authors, e.g. see [62] and [62]. [7](Chapter 3, section 20) contains a survey related to this subject. For recent results we refer to [59] and [68] and references therein.

Other kind of investigation on submanifolds in Riemannian manifolds was proposed by Bang-Yen Chen in the early 1990's, introducing a family of Riemannian invariants $\delta(n_1, \ldots, n_k)$, known also as the δ-invariants, δ-curvatures or Chen invariants. At the same time he established for arbitrary Riemannian submanifolds general optimal inequalities involving those new intrinsic invariants (cf. [16]). As it was stated in [12]: the δ-curvatures are very different in nature from the standard scalar and Ricci curvatures; simply due to the fact that both scalar and Ricci curvatures are the "total sum" of sectional curvatures on a Riemannian manifold. In contrast, the δ-curvature invariants are obtained from the scalar curvature by throwing away a certain amount of sectional curvatures. In this way we can obtain other invariants also called δ-invariants ([9], p. 253): Kählerian δ-invariants (see, e.g. [13]), affine δ-invariants ([14]) contact δ-invariants ([16]), submersion δ-invariant ([8], [11]), etc. We mention that in [15], by an application of some δ-invariants, a characterizations of Einstein spaces and conformally flat spaces were found, generalizing two well-known results of I.M. Singer - J.A. Thorpe and of R.S. Kulkarni. δ-invariants were investigated by several authors. We refer to [6], [8] and [9] as fundamental works on δ-invariants. We also refer to recent survey articles [10] and [11] related to that subject.

[1]The first named author was supported by the Université de Valenciennes et du Hainaut-Cambrésis, France.

[2]The second named author was supported by a grant 174012 of the Ministry of Sciences of Republic of Serbia.

[3]The second and the third named author thank the Center for Scientific Research of the Serbian Academy of Sciences and Arts and the University of Kragujevac for their partial support of their research done for this paper.

Our paper is related to the above mentioned Chen's theory. Namely, we investigate curvature properties of pseudo-symmetry type of submanifolds M^n in Euclidean ambient spaces $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$, which realise an optimal equality between their squared mean curvature, i.e. the extrinsic scalar valued curvature, and their δ curvature, more precisely, $\delta(2)$ curvature of Chen, which is one of main intrinsic scalar valued curvature invariants. Submanifolds having that property are called *Chen ideal submanifolds*.

Let (M, g), $\dim M = n \geq 3$, be a semi-Riemannian manifold and let ∇ be its Levi-Civita connection. The manifold (M, g) is said to be an *Einstein manifold* ([3]) if at every point of M its Ricci tensor S is proportional to the metric tensor g, i.e.

$$S = \frac{\tau}{n}\, g \tag{1}$$

on M, where τ is the scalar curvature of (M, g). In particular, if S vanishes on M then (M, g) it is called a *Ricci flat manifold*. According to [3](p. 432), the condition (1) is called the *Einstein metric condition*. Evidently, if a manifold (M, g), $n \geq 3$, is a non-Einstein manifold then the set $\mathcal{U}_S$ of all points at which S is not proportional to g is an open and non-empty subset of M. Further, (M, g) is said to be a *quasi-Einstein manifold* if at every point $p \in \mathcal{U}_S$ we have $\operatorname{rank}(S - \alpha\, g) = 1$, for some $\alpha \in \mathbb{R}$, i.e. $S = \alpha\, g + \varepsilon\, w \otimes w$, for some $\alpha \in \mathbb{R}$, where $\varepsilon = \pm 1$ and w is a non-zero covector at p. It is known that quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and investigation on quasi-umbilical hypersurfaces of conformally flat spaces (e.g. see [18], [26], and references therein).
An extension of the class of Einstein semi-Riemannian manifolds form manifolds with parallel Ricci tensor S, i.e. $\nabla S = 0$. Such manifolds are called *Ricci-symmetric*. A very important subclass of the class of Ricci-symmetric manifolds form *locally symmetric manifolds*, i.e. manifolds with parallel Riemann-Christoffel curvature tensor R, i.e. $\nabla R = 0$. This implies the following integrability condition

$$\mathcal{R}(X, Y) \cdot R = 0, \tag{2}$$

where $\mathcal{R}(X, Y)\cdot$ denotes the derivation obtained from the curvature endomorphism $\mathcal{R}(X, Y)$ and X, Y are vector fields on M. From (2) we get immediately

$$\mathcal{R}(X, Y) \cdot C = 0, \tag{3}$$

where C is the Weyl conformal curvature tensor of (M, g). We refer to Section 2 for precise definitions of the symbols used. Manifolds satisfying (2), resp. (3), are called *semi-symmetric manifolds* ([66]), resp. *Weyl-semi-symmetric manifolds* ([23]). We denote by $\mathcal{U}_C$ the set of all points of a semi-Riemannian manifold (M, g), $n \geq 4$, at which its Weyl conformal curvature tensor C is non-zero. In [52] it was proved that (2) and (3) are equivalent at every point of $\mathcal{U}_C$ of a manifold (M, g), $n \geq 5$. That result is not true when $n = 4$ ([20], [70]). We also mention that hypersurfaces satisfying (3) or

$$\mathcal{C}(X, Y) \cdot R = 0, \tag{4}$$

were investigated in [4]. $\mathcal{C}(X, Y)\cdot$ denotes the derivation obtained from the Weyl conformal curvature endomorphism $\mathcal{C}(X, Y)$.
An extension of the class of semi-symmetric, resp. Weyl-semi-symmetric, manifolds form pseudo-symmetric, resp. Weyl-pseudo-symmetric, manifolds.
A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be *pseudo-symmetric* ([23], [26], [34], [35]) if the tensors $\mathcal{R}(X, Y) \cdot R$ and $(X \wedge_g Y) \cdot R$ are linearly dependent at every point of M. This is equivalent to

$$\mathcal{R}(X, Y) \cdot R = L_R\, (X \wedge_g Y) \cdot R \tag{5}$$

on $\mathcal{U}_R = \{x \in M \,|\, R - (\kappa/((n-1)n))\, G \neq 0 \text{ at } x\}$, where L_R is some function on this set and the $(0,4)$-tensor G is defined by $G(X, Y, W, Z) = g((X \wedge_g Y)Z, W)$. It is easy to see that the function L_R is uniquely determined on $\mathcal{U}_R$. We note that $\mathcal{U}_S \cup \mathcal{U}_C = \mathcal{U}_R$.
In [46] it was shown that hypersurfaces in spaces of constant curvature, with exactly two distinct principal curvatures at every point, are pseudo-symmetric. Thus in particular, Cartan's and Schouten's investigations of quasi-umbilical hypersurfaces in spaces of constant curvature are closely related to pseudo-symmetric manifolds (see [35]). It is clear that every semi-symmetric manifold is pseudo-symmetric. However, the converse statement is not true. For instance, the Schwarzschild spacetime, the Kottler spacetime and the Reissner-Nordström spacetime satisfy (5) with non-zero function L_R [45] (see also [53]). The Schwarzschild spacetime was discovered in 1916 by K. Schwarzschild, during his study on solutions of Einstein's equations. It seems that the Schwarzschild spacetime is the "oldest" example of a non semi-symmetric, pseudo-symmetric warped product (see [35]). A similar remark is related to Friedmann-Lemaître-Robertson-Walker spacetimes (cf. [35]). We refer to [23], [35], [54] and [55] for a more detailed presentation on the class of pseudo-symmetric manifolds. A geometric interpretation of the notion of the pseudo-symmetry is given in [54], see also [55].
A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be *Weyl-pseudo-symmetric* ([23], [26], [35]) if the tensors $\mathcal{R}(X, Y) \cdot C$ and $(X \wedge_g Y) \cdot C$ are linearly dependent at every point of M. This is equivalent to

$$\mathcal{R}(X, Y) \cdot C = L\, (X \wedge_g Y) \cdot C \tag{6}$$

on $\mathcal{U}_C$, where L is some function on this set. The function L is uniquely determined on $\mathcal{U}_C$. A geometric interpretation of the notion of the Weyl-pseudo-symmetry is given in [58]. Every pseudo-symmetric manifold is Weyl-pseudo-symmetric. The converse statement is not true. Precisely, (5) and (6) are equivalent at every point of $\mathcal{U}_C$ of a manifold (M,g), $n \geq 5$, and that result is not true when $n = 4$, see [35] and references therein.
A semi-Riemannian manifold (M,g), $n \geq 4$, is said to be *a manifold with pseudo-symmetric Weyl tensor* ([23], [26], [35], [47]) if the tensors $\mathcal{C}(X,Y) \cdot C$ and $(X \wedge_g Y) \cdot C$ are linearly dependent at every point of M. This is equivalent to

$$\mathcal{C}(X,Y) \cdot C \; = \; L_C \, (X \wedge_g Y) \cdot C \tag{7}$$

on $\mathcal{U}_C$, where L_C is some function on this set. The function L is uniquely determined on $\mathcal{U}_C$. It is clear that (7) is invariant under the conformal deformations of the metric tensor g. We say that (5), (6) and (7) are *pseudo-symmetry type curvature conditions* ([23], [26], [35], [55]). In Section 3 we present more information on pseudo-symmetric manifolds, as well as manifolds with pseudo-symmetric Weyl tensor.

In what follows, for a $(0,k)$-tensor T and a symmetric $(0,2)$-tensor A on a manifold (M,g) we will denote the tensors $\mathcal{R}(X,Y) \cdot T$, $\mathcal{C}(X,Y) \cdot T$ and $(X \wedge_A Y) \cdot T$ by $R \cdot T$, $C \cdot T$ and $Q(A,T)$, respectively. The tensor $Q(A,T)$ is called the *Tachibana tensor* (e.g. see [33]). In particular, we have the following $(0,6)$-tensors: $R \cdot R$, $R \cdot C$, $C \cdot R$, $C \cdot C$ and $R \cdot C - C \cdot R$, and the $(0,6)$-Tachibana tensors: $Q(g,R)$, $Q(S,R)$, $Q(g,C)$, $Q(S,C)$, $Q(g, g \wedge S)$ and $Q(S, g \wedge S)$. The tensor $R \cdot C - C \cdot R$ is called the *difference tensor*. Now we can present (5) and (7) in the form

$$R \cdot R \; = \; L_R \, Q(g,R), \tag{8}$$

$$C \cdot C \; = \; L_C \, Q(g,C), \tag{9}$$

respectively. We also note that $Q(g, g \wedge S)$ and $Q(S, g \wedge S)$ can be expressed by some other Tachibana tensors (e.g. see [33], p. 228)

$$\begin{aligned} Q(g, g \wedge S) &= -Q(S,G), \\ Q(S, g \wedge S) &= -\frac{1}{2} \, Q(g, S \wedge S). \end{aligned} \tag{10}$$

Let (M,g), $n \geq 4$, be a semi-Riemannian manifold. Trivially, if $R \cdot C = C \cdot R = 0$ then $R \cdot C - C \cdot R = 0$. Conversely, if $R \cdot C - C \cdot R$ is a zero tensor on $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$ then $C \cdot R = 0$ and $R \cdot R = 0$ (and in a consequence $R \cdot C = 0$) on $\mathcal{U}$ ([41], Corollary 4.1). It is also clear that the difference tensor $R \cdot C - C \cdot R$ vanishes identically on any Ricci-flat manifold. However, $R \cdot C - C \cdot R$ is a non-zero tensor on every non-Ricci flat Einstein manifold. This is a consequence of the fact that, on every Einstein manifold (M,g), $n \geq 4$, the following identity is satisfied ([41], Theorem 3.1)

$$R \cdot C - C \cdot R = \frac{\tau}{(n-1)n} \, Q(g,R). \tag{11}$$

We note that on any Einstein manifold (M,g), $n \geq 4$, we have

$$\begin{aligned} \frac{\tau}{n} \, Q(g,R) &= Q(S,R) \; = \; Q(S,C) \; = \; \frac{\tau}{n} \, Q(g,C), \\ Q(g, g \wedge S) &= Q(S, g \wedge S) \; = \; 0. \end{aligned} \tag{12}$$

Now from (11) and (12) it follows that on every Einstein manifold (M,g), $n \geq 4$, we have

$$R \cdot C - C \cdot R \; = \; \frac{\tau}{(n-1)n} \, Q(g,C) \; = \; \frac{1}{n-1} \, Q(S,R) \; = \; \frac{1}{n-1} \, Q(S,C).$$

We also can investigate curvature properties of non-Einstein and non-conformally flat semi-Riemannian manifolds of dimension ≥ 4 satisfying the following condition
$(*)$ the difference tensor $R \cdot C - C \cdot R$ is a linear combination of the Tachibana tensors: $Q(g,R)$, $Q(g,C)$, $Q(S,R)$, $Q(S,C)$, $Q(g, g \wedge S)$ and $Q(S, g \wedge S)$.
A survey of results on semi-Riemannian manifolds satisfying $(*)$ is given in [26]. Furthermore, results of [28] show that some particular cases of $(*)$ are realized on hypersurfaces in space forms. For instance

$$R \cdot C - C \cdot R \; = \; \frac{1}{n-2} \, Q(S,R) - \frac{2}{n-1} \, Q(g,R)$$

on the Cartan hypersurfaces in $S^{n+1}(1)$, $n = 6, 12, 24$ (see [28], Theorem 1.3, and references therein). For recent results on hypersurfaces in space forms satisfying particular cases of $(*)$ we refer to [42]. We also mention that hypersurfaces in space forms for which the tensor $R \cdot C$ or the tensor $C \cdot R$ is a linear combinatin of the tensors $Q(S,R)$, $Q(g,R)$, $Q(g, g \wedge S)$ and $Q(S, g \wedge S)$ were investigated in [33], [50] and [65].
As we presented above, some particular cases of $(*)$ are realized on Einstein manifolds. Therefore $(*)$ is called a *generalized Einstein metric condition*. Clearly, $(*)$ is also a condition of pseudo-symmetry type. A presentation of results on Riemannian manifolds satisfying certain generalized Einstein metric conditions is given in [3].
We present now some results on semi-Riemannian manifolds satisfying the following conditions:

(i) $R \cdot C - C \cdot R$ and $Q(g, R)$ are linearly dependent,

(ii) $R \cdot C - C \cdot R$ and $Q(S, R)$ are linearly dependent,

(iii) $R \cdot C - C \cdot R$ and $Q(g, C)$ are linearly dependent,

(iv) $R \cdot C - C \cdot R$ and $Q(S, C)$ are linearly dependent,

(v) $R \cdot C - C \cdot R$ and $Q(g, g \wedge S)$ are linearly dependent,

(vi) $R \cdot C - C \cdot R$ and $Q(S, g \wedge S)$ are linearly dependent.

Manifolds satisfying (i) and (ii) were investigated in [38] and [41], respectively. Examples of warped products manifolds satisfying (i), resp., (ii), are given in [27], resp., in [38] and [60]. Further, it seems that there is no result on manifolds satisfying (iii) or (vi) with non-zero tensors $R \cdot C - C \cdot R$, $Q(g, g \wedge S)$ and $Q(S, g \wedge S)$. Manifolds satisfying (iii) or (vi) will be investigated in [29]. Next, manifolds satisfying (v) were investigated in [1] and [57]. In particular, examples of warped products manifolds satisfying that condition are given in [1]. Section 6 of [26] contains some results on manifolds satisfying (iv). An example of a warped product manifold satisfying (iv) is given in [39](Example 5.1). For further results on manifolds satisfying (iv) we refer to [29]. We also mention that warped products manifolds satisfying (i), (ii), (iv) and (v) are quasi-Einstein or not. Recently some curvature properties of manifolds satisfying these conditions were obtained in [30].

Pseudo-symmetric Chen ideal submanifolds M^n of dimension n in Euclidean spaces $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$, were investigated in [32] and [43]. In particular, in Section 3 of [32] it was stated that non-Einstein and non-conformally flat pseudo-symmetric Chen ideal submanifolds M are Roter spaces. The difference tensor $R \cdot C - C \cdot R$ of a Roter space is a linear combination of the tensors $Q(g, g \wedge S)$ and $Q(S, g \wedge S)$ (e.g. see [28], Proposition 4.2). In Sections 3 and 4 we present more facts related to Chen ideal submanifolds and Roter spaces. In Section 3 we also recall that Chen ideal submanifolds M^n of codimension m in $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$, satisfy (7) ([32], [43]). Moreover, in that section we prove that on the set U_C of every Chen ideal submanifold M^n of dimension n in $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$, we have

$$R \cdot R - Q(S, R) \quad = \quad L\, Q(g, C),$$

where L is some function on this set. We mention that the last equation is satisfied on any hypersurface in space forms (e.g. see [28], eq. (22)). In particular, the tensor $R \cdot R - Q(S, R)$ vanishes on any hypersurface in a semi-Euclidean space.

With respect to the above presentation, in this paper we investigate Chen ideal submanifolds satisfying some particular cases of $(*)$. Precisely, we investigate Chen ideal submanifolds M^n in $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$, satisfying:

$$\begin{aligned} R \cdot C - C \cdot R &= L_1 Q(g, R) + L_2 Q(S, R), \\ R \cdot C - C \cdot R &= L_3 Q(g, C) + L_4 Q(S, C), \\ R \cdot C - C \cdot R &= L_5 Q(g, g \wedge S) + L_6 Q(S, g \wedge S), \end{aligned}$$

for some functions $L_1,\ L_2,\ \ldots,\ L_6 : M^n \to \mathbb{R}$. Then we characterize Chen ideal submanifolds M^n of dimension n in $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$, satisfying conditions (i)-(vi). Our main results are presented in Section 5. Finally, in Section 6 we give proof of those results.

2 Notations

Let (M, g) be a connected Riemannian $\mathcal{C}^\infty$-manifold of dimension $n \geq 3$ and let ∇ be its Levi-Civita connection, $\mathcal{X}(M)$ the Lie algebra of vector fields on M. For vector fields X, Y, Z on M, we define *the endomorphism* $\mathcal{R}(X, Y)$ *on* $\mathcal{X}(M)$ by:

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The Riemann-Christoffel curvature $(0, 4)$*-tensor* R is defined as follows: $R(X, Y, Z, W) = g(\mathcal{R}(X, Y)Z, W)$. *The Ricci* $(0, 2)$*-tensor* S and *the Ricci operator* $\mathcal{S}$ are related by: $S(X, Y) = g(\mathcal{S}X, Y)$. With respect to an orthonormal framefield $\{e_1, \cdots, e_n\}$, one has: $S(X, Y) = \sum_{i=1}^n R(X, e_i, e_i, Y)$. *The scalar curvature* τ is given by $\tau = \mathrm{tr}(S)$. With respect to an orthonormal framefield $\{e_1, \cdots, e_n\}$, one has: $\tau = \sum_{i=1}^n S(e_i, e_i)$. Let A be a symmetric $(0, 2)$-tensor. To every couple (X, Y) of vector fields on M, one can associate *an endomorphism* $X \wedge_A Y$ *on* $\mathcal{X}(M)$ by putting:

$$(X \wedge_A Y)(Z) = A(Y, Z)X - A(X, Z)Y.$$

In particular, when $A = g$,

$$(X \wedge_g Y)(Z) = g(Y, Z)X - g(X, Z)Y.$$

Let A, B be two symmetric $(0, 2)$-tensors on M. Their *Kulkarni-Nomizu product* $A \wedge B$ is defined on $(\mathcal{X}(M))^4$ by:

$$\begin{aligned} (A \wedge B)(X, Y, Z, W) = {} & A(X, W)B(Y, Z) + A(Y, Z)B(X, W) \\ & - A(X, Z)B(Y, W) + A(Y, W)B(X, Z). \end{aligned}$$

In particular, when $A = B = g$, we have *the Kulkarni-Nomizu squared* $g \wedge g$:

$$(g \wedge g)(X, Y, Z, W) = 2\left[g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\right].$$

We notice that

$$(g \wedge g)(X, Y, Z, W) = 2g\left((X \wedge_g Y)(Z), W\right).$$

This leads to the $(0, 4)$-tensor $G = (1/2)(g \wedge g)$; it is defined as follows:

$$G(X, Y, Z, W) = g(X, W)g(Y, Z) - g(X, Z)g(Y, W).$$

It is well-known that M is of constant curvature c if and only if $R = cG$.
For every vector fields X, Y on M, *the endomorphism* $\mathcal{C}(X, Y)$ *on* $\mathcal{X}(M)$ is given by:

$$\mathcal{C}(X, Y) = \mathcal{R}(X, Y) - \frac{1}{n-2}\left[X \wedge_g (\mathcal{S}Y) + (\mathcal{S}X) \wedge_g Y\right] + \frac{\tau}{(n-1)(n-2)} X \wedge_g Y.$$

The Weyl conformal curvature $(0, 4)$*-tensor* C associated to $\mathcal{C}$ is defined by:

$$C(X, Y, Z, W) = g\left(\mathcal{C}(X, Y)Z, W\right).$$

This gives the following relation:

$$C = R - \frac{1}{n-2}(g \wedge S) + \frac{\tau}{(n-1)(n-2)} G. \tag{13}$$

For every vector fields X, Y on M, we consider a skew-symmetric endomorphism $\mathcal{B}(X, Y)$ on $\mathcal{X}(M)$. We define the $(0, 4)$-tensor B associated to $\mathcal{B}$ by:

$$B(X, Y, Z, W) = g\left(\mathcal{B}(X, Y)Z, W\right).$$

This tensor B is called *a generalized curvature tensor* if the following two conditions are fulfilled:

$$\begin{aligned}
&B(X_1, X_2, X_3, X_4) = B(X_3, X_4, X_1, X_2);\\
&B(X_1, X_2, X_3, X_4) + B(X_2, X_3, X_1, X_4) + B(X_3, X_1, X_2, X_4) = 0.
\end{aligned}$$

Now let us extend the endomorphism $\mathcal{B}(X, Y)$ to a derivation $\mathcal{B}(X, Y)\cdot$ of the algebra of tensor fields on M, assuming that it commutes with contractions and $\mathcal{B}(X, Y) \cdot f = 0$ for any smooth real valued function f on M. Furthermore consider a $(0, k)$-tensor T, for $k \geq 1$. We define the $(0, k + 2)$-tensor $B \cdot T$ by putting:

$$\begin{aligned}
(B \cdot T)(X_1, \cdots, X_k; X, Y) = &-T\left(\mathcal{B}(X, Y)X_1, \cdots, X_k\right)\\
&- T\left(X_1, \mathcal{B}(X, Y)X_2, \cdots, X_k\right)\\
&- \cdots - T\left(X_1, X_2, \cdots, \mathcal{B}(X, Y)X_k\right).
\end{aligned}$$

Substituting $\mathcal{B} = \mathcal{R}$ or $\mathcal{B} = \mathcal{C}$, and $T = C$ or $T = R$ in the above formulas, we get the tensors: $R \cdot R$, $C \cdot C$, $R \cdot C$ and $C \cdot R$. The two latest lead to the *difference tensor* $R \cdot C - C \cdot R$. Further, let A be a symmetric $(0, 2)$-tensor. Denote by $\mathcal{A}$ the endomorphism associated to A by: $g(\mathcal{A}X, Y) = A(X, Y)$. Now we consider a $(0, k)$-tensor T, for $k \geq 2$. *The Tachibana tensor of* A *and* T (or for short, *the Tachibana tensor*) $Q(A, T)$ is defined on $(\mathcal{X}(M))^k \times (\mathcal{X}(M))^2$ by:

$$\begin{aligned}
Q(A, T)(X_1, \cdots, X_k; X, Y) &= \left((X \wedge_A Y) \cdot T\right)(X_1, \cdots, X_k)\\
&= -T\left((X \wedge_A Y) X_1, \cdots, X_k\right)\\
&\quad - T\left(X_1, (X \wedge_A Y) X_2, \cdots, X_k\right)\\
&\quad - \cdots - T\left(X_1, X_2, \cdots, (X \wedge_A Y) X_k\right).
\end{aligned}$$

Substituting $A = g$ or $A = S$, and $T = C$ or $T = R$ or $T = g \wedge S$ in the above formulas, we get one of the following $(0, 6)$-*Tachibana tensors* [33] which may not vanish identically: $Q(g, C)$, $Q(g, R)$, $Q(g, g \wedge S)$, $Q(S, C)$, $Q(S, R)$ and $Q(S, g \wedge S)$. We also have the following identity (e.g. see [26]):

$$(n-2)(R \cdot C - C \cdot R) = Q\left(S - \frac{\tau}{n-1} g, R\right) - g \wedge (R \cdot S) + P$$

where the $(0, 6)$-tensor P is defined by:

$$\begin{aligned}
P(X_1, X_2, X_3, X_4; X, Y) = &\, g(X, X_1)R\left(\mathcal{S}(Y), X_2, X_3, X_4\right) - g(Y, X_1)R\left(\mathcal{S}(X), X_2, X_3, X_4\right)\\
&+ g(X, X_2)R\left(X_1, \mathcal{S}(Y), X_3, X_4\right) - g(Y, X_2)R\left(X_1, \mathcal{S}(X), X_3, X_4\right)\\
&+ g(X, X_3)R\left(X_1, X_2, \mathcal{S}(Y), X_4\right) - g(Y, X_3)R\left(X_1, X_2, \mathcal{S}(X), X_4\right)\\
&+ g(X, X_4)R\left(X_1, X_2, X_3, \mathcal{S}(Y)\right) - g(Y, X_4)R\left(X_1, X_2, X_3, \mathcal{S}(X)\right).
\end{aligned}$$

It is well-known that the *conharmonic curvature tensor* $conh(R)$ of a semi-Riemannian manifold (M, g), $n \geq 4$, is defined by

$$conh(R) \quad = \quad R - \frac{1}{n-2}\, g \wedge S.$$

Evidently, $conh(R)$ is a generalized curvature tensor. In addition (13) yields

$$conh(R) \quad = \quad C - \frac{\tau}{(n-1)(n-2)}\, G. \tag{14}$$

It is clear that

$$conh(R) = 0 \iff (C = 0 \text{ and } \tau = 0)\,.$$

We also have

Proposition 1. [25] *For any semi-Riemannian manifold (M, g), $n \geq 4$, the following identities hold good:*

$$\left\{ \begin{aligned} conh(R) \cdot S &= C \cdot S - \frac{\tau}{(n-2)(n-1)}\, Q(g, S);\\ R \cdot conh(R) &= R \cdot C;\\ conh(R) \cdot R &= C \cdot R - \frac{\tau}{(n-1)(n-2)}\, Q(g, R);\\ conh(R) \cdot conh(R) &= C \cdot C - \frac{\tau}{(n-1(n-2)}\, Q(g, C).\ \blacksquare \end{aligned} \right. \tag{15}$$

Using the the above presented relations, we get immediately

$$\left\{ \begin{aligned} &R \cdot conh(R) - conh(R) \cdot R = R \cdot C - C \cdot R + \frac{\tau}{(n-1)(n-2)}\, Q(g, R),\\ &C \cdot conh(R) - conh(R) \cdot C = C \cdot (conh(R) + \frac{\tau}{(n-1)(n-2)}\, G) - (C - \frac{\tau}{(n-1)(n-2)}\, G) \cdot C\\ &\qquad = \frac{\tau}{(n-1)(n-2)}\, Q(g, C). \end{aligned} \right. \tag{16}$$

We mention that quasi-Einstein manifolds satisfying some curvature conditions were investigated by several authors (e.g. see [18], [26], [27], [38], [42], [51], [67], [71], and references therein). In particular, [67] contains results on quasi-Einstein manifolds satisfying curvature conditions involving the conharmonic tensor $conh(R)$.

We will use the following

Proposition 2. ([21], Proposition 4.1; [37], Lemma 3.4) *Let (M, g), $n \geq 3$, be a semi-Riemannian manifold. Let a non-zero symmetric $(0, 2)$-tensor A and a generalized curvature tensor B, defined at $p \in M$, satisfy at this point $Q(A, B) = 0$. In addition, let Y be a vector at p such that the scalar $\rho = w(Y)$ is non-zero, where w is a covector defined by $w(X) = A(X, Y)$, $X \in T_p(M)$. Then we have:*
(i) $A - \rho\, w \otimes w \neq 0$ and $B = \lambda\, A \wedge A$, $\lambda \in \mathbb{R}$, or (ii) $A = \rho\, w \otimes w$ and

$$\begin{aligned} &w(X)\, B(Y, Z, X_1, X_2) + w(Y)\, B(Z, X, X_1, X_2)\\ + \quad &w(Z)\, B(X, Y, X_1, X_2) \;=\; 0, \quad X, Y, Z, X_1, X_2 \in T_pM. \end{aligned} \tag{17}$$

Moreover, in both cases the following condition holds at p:

$$B \cdot B \quad = \quad Q\,(\mathrm{Ric}(B), B) \tag{18}$$

where Ric(B) *is the Ricci tensor of B.* $\blacksquare$

As an immediate consequence of Proposition 2, using the definition of the tensors R and $Q(g, R)$, resp., C and $Q(g, C)$, we can easily check that $Q(g, R) = 0$ at a point of a manifold (M, g), $n \geq 4$, if and only if $R = (\tau/\,((n-1)n))\, G$ at this point, resp., $Q(g, C) = 0$ at a point of M if and only if $C = 0$ at this point. On another hand, it is also easy to check that $Q(g, g \wedge S)$ vanishes at a point of a manifold (M, g), $n \geq 4$, if and only if S is proportional to g, i.e. $S = (\tau/n)\, g$ holds at this point.
It is clear that $Q(S, g \wedge S)$ vanishes at all points at which $S = (\tau/n)\, g$. Proposition 1.1, Lemma 3.1 of [49] and (10) lead to the following: $Q(S, g \wedge S)$ vanishes at a point of $\mathcal{U}_S \subset M$ of a manifold (M, g), $n \geq 4$, if and only if rank $S = 1$ at this point.

With respect to the presented above material, we restrict our investigation to the set $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$ of a manifold (M, g), $n \geq 4$. Further, if $Q(S, C)$ is a zero tensor on $\mathcal{U} \subset M$ then $R \cdot R = (\tau/(n-1))\, Q(g, R)$ (and in consequence $R \cdot C = (\tau/(n-1))\, Q(g, C)$) hold on $\mathcal{U}$ ([36], Theorem 3.1). Moreover, if $n = 4$ then $R \cdot R = 0$ and $\tau = 0$ hold on $\mathcal{U}$ ([40], Theorem 3.1). If $Q(S, R)$ is a zero tensor on $\mathcal{U} \subset M$ then $R \cdot R = 0$ hold on $\mathcal{U}$ ([21], Theorem 4.1).

3 On Chen ideal submanifolds

3.1 Introduction

Let M be a submanifold of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 2$, $m \geq 1$. Let g be the Riemannian metric induced on M from the standard metric on $\mathbb{E}^{n+m}$, ∇ the corresponding Levi-Civita connection on M, and R, S, τ respectively the Riemann-Christoffel curvature tensor, the Ricci tensor and the scalar curvature of M. For the scalar curvature τ of (M, g) we use the calibration

$$\tau(p) := \sum_{i<j} K\left(p, e_i(p) \wedge e_j(p)\right)$$

where $K(p, \pi)$ denotes the *Riemannian sectional curvature* of (M, g) at the point p for a plane section π in the tangent space T_pM. For each point p in M, considering the number

$$(\inf K)(p) := \inf\left\{K(p, \pi) | \pi \text{ is a plane section in } T_pM\right\},$$

B.-Y. Chen (see [6], [9]) introduced the $\delta(2)$-curvature by

$$(\delta(2))(p) = \delta(p) := \tau(p) - (\inf K)(p).$$

This $\delta(2)$-, for short, *δ-curvature of Chen* thus is a well defined real function on M which clearly is a *Riemannian invariant* of (M, g). From [9] (see also [6], [8], [16]), we have the following basic result which, in particular, answered a question raised by S.S. Chern [17] long before, concerning *intrinsic obstructions* on Riemannian manifolds *in view of minimal immersibility* in Euclidean spaces.

Theorem 1. [6] *For any submanifold M of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 2$, $m \geq 1$,*

$$\delta \leq \frac{n^2(n-2)}{2(n-1)} H^2, \quad (*)$$

and in $(*)$ equality holds at a point $p \in M$ if and only if, with respect to some suitable adapted orthonormal frame $\{e_i, \xi_\alpha\}$ around p on M in $\mathbb{E}^{n+m}$, the shape operators are given by

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & z & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & z \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \beta > 1,$$

where $z = a + b$ and $(\inf K) = ab - \sum_{\beta>1}\left(c_\beta^2 + d_\beta^2\right) : M \to \mathbb{R}$. ■

Evidently, if $m = 1$ then $\inf K = ab$.
With respect to the above theorem, one has the following definition (see [6], [8], [32], [43], [69]).

Definition 1. *Let M be a submanifold of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 2$, $m \geq 1$. It is called a Chen ideal submanifold if, at each of its points, the Chen's basic inequality $(*)$ in the Theorem 1 is actually an equality.*

Let M be a Chen ideal submanifold of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$. We use the notations as in Theorem 1. The Riemann-Christoffel curvature tensor R satisfies:

$$(19) \quad \begin{cases} R\left(e_1, e_2, e_2, e_1\right) = \inf K = ab - \sum_{\beta=2}^{m}\left(c_\beta^2 + d_\beta^2\right); \\ R\left(e_1, e_i, e_i, e_1\right) = az \quad \text{for} \quad i \geq 3; \\ R\left(e_2, e_i, e_i, e_2\right) = bz \quad \text{for} \quad i \geq 3; \\ R\left(e_i, e_j, e_j, e_i\right) = z^2 \quad \text{for} \quad 3 \leq i < j \leq n. \end{cases}$$

The other values of $R\left(e_u, e_v, e_w, e_t\right)$ are null. The Ricci tensor S satisfies:

$$(20) \quad \begin{cases} S\left(e_1, e_1\right) = \inf K + (n-2)az; \\ S\left(e_2, e_2\right) = \inf K + (n-2)bz; \\ S\left(e_i, e_i\right) = (n-2)z^2 \quad \text{for} \quad 3 \leq i \leq n; \\ S\left(e_u, e_v\right) = 0 \quad \text{for} \quad 1 \leq u < v \leq n. \end{cases}$$

The scalar curvature τ is given by:

$$\tau = \sum_{i=1}^{n} \mathrm{S}(e_i, e_i) = 2 \inf K + (n-1)(n-2)z^2. \tag{21}$$

The Weyl conformal curvature tensor C is determined by the following relations:

$$\begin{cases} C\,(e_1, e_2, e_2, e_1) = \dfrac{(n-3)(\inf K)}{n-1}; \\ C\,(e_1, e_i, e_i, e_1) = -\dfrac{(n-3)(\inf K)}{(n-1)(n-2)} \quad \text{for} \quad i \geq 3; \\ C\,(e_2, e_i, e_i, e_2) = -\dfrac{(n-3)(\inf K)}{(n-1)(n-2)} \quad \text{for} \quad i \geq 3; \\ C\,(e_i, e_j, e_j, e_i) = \dfrac{2(\inf K)}{(n-1)(n-2)} \quad \text{for} \quad 3 \leq i < j \leq n. \end{cases} \tag{22}$$

From (22) it follows (cf. [32], [43], Theorem F) that every Chen ideal submanifold M of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$, has a pseudo-symmetric Weyl conformal curvature C, i.e. it satisfies the identity:

$$C \cdot C \;=\; L_C\, Q(g, C), \quad L_C \;=\; -\frac{(n-3)(\inf K)}{(n-1)(n-2)}. \tag{23}$$

Very recently semi-Riemannian manifolds satisfying (23) were investigated in [24].

It is known that at every point of a hypersurface N in a space form $\widetilde{N}^{n+1}(c)$, $n \geq 4$, the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are linearly dependent. Precisely, we have on N ([44])

$$R \cdot R - Q(S, R) = -(n-2)c\, Q(g, C).$$

Thus, in particular, $R \cdot R - Q(S, R) = 0$ on every Chen ideal hypersurface N in $\mathbb{E}^{n+1}$, $n \geq 4$.
Now let us compute the difference $R \cdot R - Q(S, R)$ on the Chen ideal submanifold M of codimension m in $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$. With respect to the notations in Theorem 1 and from the equalities (19), (20) and (22), we can prove the following

Theorem 2. *The identity*

$$R \cdot R - Q(S, R) = \frac{ab - \inf K}{\inf K}(n-2)z^2\, Q(g, C) \tag{24}$$

holds on the subset $\mathcal{U}_C$ (see section 1) of every Chen ideal submanifold M of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$. In addition, at each point $p \in M$ where C vanishes ($\inf K = 0$), the following equalities hold:

$$\begin{cases} (R \cdot R - Q(S, R))\,(e_1, e_2, Z, W; e_1, e_i) = (n-3)abz^2 \left\langle (e_2 \wedge_g e_i)\,(Z), W \right\rangle, \\ (R \cdot R - Q(S, R))\,(e_1, e_j, Z, W; e_1, e_i) = -abz^2 \left\langle (e_j \wedge_g e_i)\,(Z), W \right\rangle, \\ (R \cdot R - Q(S, R))\,(e_1, e_2, Z, W; e_2, e_i) = -(n-3)abz^2 \left\langle (e_1 \wedge_g e_i)\,(Z), W \right\rangle, \\ (R \cdot R - Q(S, R))\,(e_2, e_j, Z, W; e_2, e_i) = -abz^2 \left\langle (e_j \wedge_g e_i)\,(Z), W \right\rangle \end{cases} \tag{25}$$

the other values of $(R \cdot R - Q(S, R))\,(e_u, e_v, Z, W; e_w, e_t)$ being null. ■

As it was proved in [31], every warped product manifold $\overline{M} \times_F \widetilde{N}$ of a 2-dimensional base manifold $(\overline{M}, \overline{g})$ and an $(n-2)$-dimensional fibre, which is a space of constant curvature $(\widetilde{N}, \widetilde{g})$, $n \geq 4$, with the warping function F, satisfies

$$C \cdot C = -\frac{(n-3)\rho}{(n-2)(n-1)}\, Q(g, C), \qquad \rho = \frac{\overline{\tau}}{2} + \frac{\widetilde{\tau}}{(n-3)(n-2)F} + \frac{\Delta F}{2F} - \frac{\Delta_1 F}{2F^2}, \tag{26}$$

where $\Delta F = g^{ab}\nabla_b F_a$, $\Delta_1 F = g^{ab} F_a F_b$, and $\overline{\tau}$, $\widetilde{\tau}$ are the scalar curvatures of the base and the fibre, respectively, see to [31] for details. According to [19], every non-trivial and non-minimal Chen ideal submanifold M of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$ is isometric to an open subset of a warped product $\overline{M} \times_F \mathbb{S}^{n-2}$ of a 2-dimensional base manifold $(\overline{M}, \overline{g})$ and an $(n-2)$-dimensional unit sphere $\mathbb{S}^{n-2}$, where the warping function F is a solution of some second order quasilinear elliptical partial differential equation in the plane. Thus we see that (26) holds on M. Furthermore, from (23) and (26) it follows that $\inf K$ is expressed on M by

$$\inf K \;=\; \frac{\overline{\tau}}{2} + \frac{1}{F} + \frac{\Delta F}{2F} - \frac{\Delta_1 F}{2F^2}.$$

Since the scalar curvature τ of M is given by (21) and satisfies (e.g. see [31]):

$$\tau \quad = \quad \overline{\tau} + \frac{(n-3)(n-2)}{F} - \frac{(n-2)\Delta F}{F} - \frac{(n-2)(n-5)\Delta_1 F}{4F^2},$$

we get:

$$(n-2)z^2 \quad = \quad \frac{n-4}{F} - \frac{\Delta F}{F} - \frac{(n-6)\Delta_1 F}{4F^2}.$$

3.2 On pseudo-symmetric Chen ideal submanifolds

Semi-symmetric spaces has been investigated first by E. Cartan in 1946 ([5]). In 1982 ([66]), Z.I. Szabó established the classification of semi-symmetric spaces. In 1997 ([48]), F. Dillen and two of the present authors classified all Chen ideal submanifolds which are semi-symmetric.

Theorem 3. [48] *A Chen ideal submanifold M of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 3$, $m \geq 1$, is semi-symmetric if and only if M is minimal (in which case M is $(n-2)$-ruled)) or M is a round hypercone in some totally geodesic subspace $\mathbb{E}^{n+1}$ of $\mathbb{E}^{n+m}$.* ■

In [32] and [43], Chen ideal pseudo-symmetric submanifolds were classified.

Theorem 4. ([32], [43]) *A Chen ideal submanifold M of codimension m in $\mathbb{E}^{n+m}$ $(n \geq 3,\ n \geq 1)$ is pseudo-symmetric if and only if:*

(i) either M is semi-symmetric (see Theorem 3),

(ii) or at every point p of M where $R \cdot R \neq 0$, the 2D normal section $\Sigma^2_{\tilde{\pi}} \subset \mathbb{E}^{2+m}$ of M^n at p in the direction of the tangent plane $\tilde{\pi} \subset T_pM^n$ for which the sectional curvature function $K(p,\pi)$ at p attains its minimal value $(\inf K)(p)$ *is pseudo-umbilical at p, or equivalently, if p is a spherical point of the projection $\overline{\Sigma}^2_{\tilde{\pi}} \subset \mathbb{E}^3$ of this 2D normal section $\overline{\Sigma}^2_{\tilde{\pi}}$ on the space $\mathbb{E}^3$ spanned by $\tilde{\pi}$ and the mean curvature vector $\overrightarrow{H}$ (p) of M^n in $\mathbb{E}^{n+m}$ at p (and in this case $L_R = \big(n^2/\big(2(n-1)^2\big)\big)$, where H is the mean curvature of M^n in $\mathbb{E}^{n+m}$).* ■

4 On Chen ideal submanifolds and Roter manifolds

A Riemannian manifold (M,g) of dimension n, $n \geq 4$, is said to be a *Roter manifold* or a *Roter space* (e.g. see [26], [32] and references therein) if

$$R \quad = \quad \frac{\phi}{2}\, S \wedge S + \mu\, g \wedge S + \eta\, G \tag{27}$$

holds on $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C \subset M$, where ϕ, μ and η are some functions on this set.
According to [32], from a geometric point of view, the pseudo-symmetric Riemannian manifolds can be seen as the most natural symmetric spaces after the real space forms, i.e. the spaces of constant Riemannian sectional curvature. From an algebraic point of view, the Roter manifolds can be seen as the Riemannian manifolds whose Riemann-Christoffel curvature tensor R has the most simple expression after the real space forms, the latter ones being characterisable as the Riemannian spaces (M^n, g) for which the $(0,4)$-tensor R is proportional to the Nomizu-Kulkarni square of their $(0,2)$-metric tensor g. As it was stated in [32], every Chen ideal submanifold M of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$, is a Roter manifold if and only if it is pseudo-symmetric.

As we already mentioned, in this paper we investigate Chen ideal submanifolds (in Euclidean spaces) satisfying some curvature conditions of pseudo-symmetry type. We prove that those submanifolds are pseudo-symmetric and, as a consequence, Roter manifolds too ([32]).
Using (27), (10) and Theorem 6.7 of [26] we can easy check that the difference tensor $R \cdot C - C \cdot R$ of every Roter manifold (M,g), $n \geq 4$, can be expressed on $\mathcal{U} = \mathcal{U}_S \cap \mathcal{U}_C$ as a linear combination of the tensors $Q(g, g \wedge S)$ and $Q(S, g \wedge S)$, precisely on this set we have

$$R \cdot C - C \cdot R \quad = \quad Q\Big(\big(\frac{1}{n-2} - \mu - \frac{\phi\tau}{n-1}\big)\, S + \big(\frac{\mu\tau}{n-1} + \eta\big)\, g, g \wedge S\Big). \tag{28}$$

We note that if (M,g), $n \geq 4$, is a Roter manifold then at every point of $\mathcal{U} \subset M$ we must have rank $(S - \alpha\, g) > 1$, for any real number $\alpha \in \mathbb{R}$.

5 Main results

Now we give our main results about Chen ideal submanifolds in Euclidean spaces whose difference tensor $R \cdot C - C \cdot R$ can be expressed in terms of some of the Tachibana tensors $Q(g,R)$, $Q(S,R)$, $Q(g,C)$, $Q(S,C)$, $Q(g, g \wedge S)$, $Q(S, g \wedge S)$.

Theorem 5. *Let M be a non-conformally flat Chen ideal submanifold of codimension m in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$. Then there exist two real valued functions L_1, L_2 on M such that*

$$R \cdot C - C \cdot R = L_1 Q(g,R) + L_2 Q(S,R),$$

if and only if there exists an orthonormal tangent framefield $\{e_1, \cdots, e_n\}$ and an orthonormal normal framefield $\{\xi_1, \cdots, \xi_m\}$ such that the shape operators

$$A_\alpha := A_{\xi_\alpha}, \quad 1 \leq \alpha \leq m,$$

are given by:

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & \epsilon a & 0 & \cdots & 0 \\ 0 & 0 & (1+\epsilon)a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (1+\epsilon)a \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \beta \geq 2,$$

where $\epsilon = \pm 1$, a, c_β , d_β (for $2 \leq \beta \leq m$) are real functions on M such that

$$\sum_{\beta=2}^{m} \left(c_\beta^2 + d_\beta^2\right) = \epsilon a^2 - \inf K, \ 2 \inf K - (1+\epsilon)a^2 \neq 0, \ \inf K \neq (n-2)(1+\epsilon)a^2$$

and

$$L_1 = \frac{(n-3)\inf K - (n-2)(1+\epsilon)a^2}{(n-1)(n-2)} \frac{2\inf K}{2\inf K - (1+\epsilon)a^2},$$
$$(1+\epsilon)a^2 \left[L_2 - \frac{1}{(n-2)} \frac{\inf K}{2\inf K - (1+\epsilon)a^2}\right] = 0.$$

In this case, M is a Roter space. In addition one has one of the following two situations.

(i) Either $\epsilon = -1$ and M is a semi-symmetric and minimal submanifold (see Theorem 3) such that

$$R \cdot C - C \cdot R = \frac{(n-3)\inf K}{(n-1)(n-2)} Q(g,R).$$

(ii) Or $\epsilon = +1$ and M is a properly pseudo-symmetric and non minimal submanifold (see Theorem 4) such that

$$R \cdot C - C \cdot R = \frac{(n-3)\inf K - 2(n-2)a^2}{(n-1)(n-2)} \frac{\inf K}{\inf K - a^2} Q(g,R) + \frac{1}{2(n-2)} \frac{\inf K}{\inf K - a^2} Q(S,R). \ \blacksquare$$

Corollary 1. *Let M be a non-conformally flat Chen ideal submanifold of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$. Then there exists a real valued function L on M such that*

$$R \cdot C - C \cdot R = L Q(g,R)$$

if and only if M is minimal and there exists an orthonormal tangent framefield $\{e_1, \cdots, e_n\}$ and an orthonormal normal framefield $\{\xi_1, \cdots, \xi_m\}$ such that the shape operators

$$A_\alpha := A_{\xi_\alpha}, \quad 1 \leq \alpha \leq m,$$

are given by:

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & -a & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \beta \geq 2,$$

where a, c_β , d_β (for $2 \leq \beta \leq m$) are real functions on M such that

$$\sum_{\beta=2}^{m} \left(c_\beta^2 + d_\beta^2\right) = -a^2 - \inf K$$

and

$$L = \frac{(n-3)(\inf K)}{(n-1)(n-2)}.$$

In this case, M is semi-symmetric (see Theorem 3). ■

Corollary 2. *Let M be a Chen ideal submanifold of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$. Then the difference tensor $R \cdot C - C \cdot R$ and the Tachibana tensor $Q(S, R)$ are linearly dependent if and only if M is conformally flat ($\inf K = 0$).* ■

Theorem 6. *Let M be a non-conformally flat Chen ideal submanifold of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$. Then there exists two real valued functions L_3, L_4 on M such that*

$$R \cdot C - C \cdot R = L_3 Q(g, C) + L_4 Q(S, C),$$

if and only if there exists an orthonormal tangent framefield $\{e_1, \cdots, e_n\}$ and an orthonormal normal framefield $\{\xi_1, \cdots, \xi_m\}$ such that the shape operators

$$A_\alpha := A_{\xi_\alpha}, \quad 1 \leq \alpha \leq m,$$

are given by:

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & \epsilon a & 0 & \cdots & 0 \\ 0 & 0 & (1+\epsilon)a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (1+\epsilon)a \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \beta \geq 2,$$

where $\epsilon = \pm 1$, a, c_β , d_β (for $2 \leq \beta \leq m$) are real functions on M such that

$$\sum_{\beta=2}^{m} \left(c_\beta^2 + d_\beta^2\right) = \epsilon a^2 - \inf K$$

and

$$L_3 = -\frac{2\inf K + 2(n-1)(n-2)(1+\epsilon)a^2}{n-1}, \quad L_4 = -1.$$

In this case, M is a Roter space. In addition one has one of the following two situations.

(i) Either $\epsilon = -1$ and M is a semi-symmetric and minimal submanifold (see Theorem 3) such that

$$R \cdot C - C \cdot R = \frac{2\inf K}{n-1} Q(g, C) - Q(S, C).$$

(ii) Or $\epsilon = +1$ and M is a properly pseudo-symmetric and non minimal submanifold (see Theorem 4) such that

$$R \cdot C - C \cdot R = -\frac{2\inf K + 4(n-1)(n-2)a^2}{n-1} Q(g, C) - Q(S, C).$$ ■

Corollary 3. *Let M be a Chen ideal submanifold of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$. Then the difference tensor $R \cdot C - C \cdot R$ and the Tachibana tensor $Q(g, C)$ are linearly dependent if and only if M is conformally flat.* ■

Corollary 4. *Let M be a non-conformally flat Chen ideal submanifold of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$. Then there exists a real valued function L on M such that*

$$R \cdot C - C \cdot R = L Q(S, C),$$

if and only if M is not minimal, and there exists an orthonormal tangent framefield $\{e_1, \cdots, e_n\}$ and an orthonormal normal framefield $\{\xi_1, \cdots, \xi_m\}$ such that the shape operators

$$A_\alpha := A_{\xi_\alpha}, \quad 1 \leq \alpha \leq m,$$

are given by:

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & 2a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2a \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \beta \geq 2,$$

where a, c_β , d_β (for $2 \le \beta \le m$) are real functions on M such that $\sum_{\beta=2}^{m}\left(c_\beta^2 + d_\beta^2\right) = \left(2n^2 - 6n + 5\right)a^2 > 0$ *and*

$$L = -1.$$

In this case, M^n is properly pseudo-symmetric (see Theorem 4). ■

Theorem 7. *Let M be a non-conformally flat Chen ideal submanifold of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \ge 4$, $m \ge 1$. Then there exists two real valued functions L_5, L_6 on M such that*

$$R \cdot C - C \cdot R = L_5 Q(g, g \wedge S) + L_6 Q(S, g \wedge S),$$

if and only if there exists an orthonormal tangent framefield $\{e_1, \cdots, e_n\}$ and an orthonormal normal framefield $\{\xi_1, \cdots, \xi_m\}$ such that the shape operators

$$A_\alpha := A_{\xi_\alpha}, \quad 1 \le \alpha \le m,$$

are given by:

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & \epsilon a & 0 & \cdots & 0 \\ 0 & 0 & (1+\epsilon)a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (1+\epsilon)a \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \beta \ge 2,$$

where $\epsilon = \pm 1$, a, c_β , d_β (for $2 \le \beta \le m$) are real functions on M such that

$$\sum_{\beta=2}^{m}\left(c_\beta^2 + d_\beta^2\right) = \epsilon a^2 - \inf K, \ \inf K \ne (n-2)(1+\epsilon)a^2$$

and moreover

$$L_5 = -\frac{2(n-2)(1+\epsilon)a^2}{n-1}\frac{\inf K\left[\inf K + (n-2)(1+\epsilon)a^2\right]}{[\inf K - (n-2)(1+\epsilon)a^2]^2},$$
$$L_6 = -\frac{1}{(n-1)(n-2)}\frac{\inf K\left[(n-3)\inf K + (n-1)(n-2)(1+\epsilon)a^2\right]}{[\inf K - (n-2)(1+\epsilon)a^2]^2}.$$

In this case, M is a Roter space. In addition one has one of the following two situations.

(i) Either $\epsilon = -1$ and M is a semi-symmetric and minimal submanifold (see Theorem 3) such that

$$R \cdot C - C \cdot R = -\frac{n-3}{(n-1)(n-2)}Q(S, g \wedge S).$$

(ii) Or $\epsilon = +1$ and M is a properly pseudo-symmetric and non minimal submanifold (see Theorem 4) such that

$$R \cdot C - C \cdot R = -\frac{4(n-2)a^2}{n-1}\frac{\inf K\left[\inf K + 2(n-2)a^2\right]}{[\inf K - 2(n-2)a^2]^2}Q(g, g \wedge S)$$
$$- \frac{1}{(n-1)(n-2)}\frac{\inf K\left[(n-3)\inf K + 2(n-1)(n-2)a^2\right]}{[\inf K - 2(n-2)a^2]^2}Q(S, g \wedge S). \ ■$$

Corollary 5. *Let M be a non-conformally Chen ideal submanifold of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \ge 4$, $m \ge 1$. Then there exists a real valued function L on M such that*

$$R \cdot C - C \cdot R = LQ(g, g \wedge S),$$

if and only if M is not minimal, and there exists an orthonormal tangent framefield $\{e_1, \cdots, e_n\}$ and an orthonormal normal framefield $\{\xi_1, \cdots, \xi_m\}$ such that the shape operators

$$A_\alpha := A_{\xi_\alpha}, \quad 1 \le \alpha \le m,$$

are given by:

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & 2a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2a \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \beta \ge 2,$$

where a, c_β , d_β (for $2 \le \beta \le m$) are real functions on M such that

$$\sum_{\beta=2}^{m} \left(c_\beta^2 + d_\beta^2\right) = \frac{2n^2 - 5n + 1}{n-1} a^2 > 0$$

and

$$L = -\frac{2a^2}{n-2}.$$

In this case, M^n is a properly pseudo-symmetric manifold (see Theorem 4). ∎

Corollary 6. *Let M be a non-conformally flat Chen ideal submanifold of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \ge 4$, $m \ge 1$. Then there exists a real valued function L on M such that*

$$R \cdot C - C \cdot R = LQ(S, g \wedge S),$$

if and only if one has one of the two cases which follow.

(i) Either M is minimal, and there exists an orthonormal tangent framefield $\{e_1, \cdots, e_n\}$ and an orthonormal normal framefield $\{\xi_1, \cdots, \xi_m\}$ such that the shape operators

$$A_\alpha := A_{\xi_\alpha}, \quad 1 \le \alpha \le m,$$

are given by:

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & -a & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \beta \ge 2,$$

where a, c_β , d_β (for $2 \le \beta \le m$) are real functions on M such that

$$\sum_{\beta=2}^{m} \left(c_\beta^2 + d_\beta^2\right) = -a^2 - \inf K$$

and

$$L = -\frac{n-3}{(n-1)(n-2)}.$$

(ii) Or M is not minimal, and there exists an orthonormal tangent framefield $\{e_1, \cdots, e_n\}$ and an orthonormal normal framefield $\{\xi_1, \cdots, \xi_m\}$ such that the shape operators

$$A_\alpha := A_{\xi_\alpha}, \quad 1 \le \alpha \le m$$

are given by:

$$A_1 = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & 2a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2a \end{pmatrix}, \quad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \cdots & 0 \\ d_\beta & -c_\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ \beta \ge 2,$$

where a, c_β , d_β (for $2 \le \beta \le m$) are real functions on M such that $\sum_{\beta=2}^{m} \left(c_\beta^2 + d_\beta^2\right) = (2n-3)a^2$ and

$$L = \frac{1}{2\,(n-1)\,(n-2)}.$$

In the first case, M is a semi-symmetric manifold (see Theorem 3). In the second case, M is a properly pseudo-symmetric manifold (see Theorem 4). ∎

Corollary 7. *Let M be a non-conformally Chen ideal submanifold of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \ge 4$, $m \ge 1$. If M is minimal, then:*

$$R \cdot C - C \cdot R = -\frac{n-3}{(n-1)(n-2)} Q(S, g \wedge S).$$ ∎

Corollary 8. *A Chen ideal submanifold M of dimension $n \geq 4$ in the Euclidean space $\mathbb{E}^{n+m}$ satisfies the curvature condition*

$$R \cdot C - C \cdot R = 0$$

if and only if M is conformally flat. ■

With respect to (28), as an immmediate consequence of Theorem 7, for Chen ideal and Roter submanifolds, we express the difference tensor $R \cdot C - C \cdot R$ as a linear combination of the Tachibana tensors $Q(g, g \wedge S)$, $Q(S, g \wedge S)$, in terms of $\inf K$ (the infimum of all of its sectional curvatures) and the scalar curvature τ.

Corollary 9. *Let M be a non-conformally flat Chen ideal submanifold of dimension n in the Euclidean space $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$. If M is a Roter space, then:*

$$R \cdot C - C \cdot R = -\frac{2\left[\tau - 2\inf K\right]\left[\tau + 2(n-2)\inf K\right]}{(n-1)\left[\tau - 2n(\inf K)\right]^2} Q(g, g \wedge S) - \frac{2(n-1)(\inf K)\left[\tau + 2(n-4)\inf K\right]}{(n-2)\left[\tau - 2n(\inf K)\right]^2} Q(S, g \wedge S). \ ■$$

Finally, in addition to (16), as an immmediate consequence of (23), (15) and (14), we get the following theorem:

Theorem 8. *Every Chen ideal submanifold M of codimension m in $\mathbb{E}^{n+m}$, $n \geq 4$, $m \geq 1$, satisfies:*

$$conh(R) \cdot conh(R) = -\frac{\tau + (n-3)(\inf K)}{(n-1)(n-2)} Q(g, C) = -\frac{\tau + (n-3)(\inf K)}{(n-1)(n-2)} Q(g, conh(R)). \ ■$$

We refer to [56] and [64] for further results on Chen ideal submanifolds satisfying curvature conditions involving the conharmonic curvature tensor.

6 Proofs of main results

In this section, we consider a Chen ideal submanifold M of codimension m in the Euclidean space $\mathbb{E}^{n+m}$. We use the notations as in Theorem 1. To prove all our main results on M, we begin by computing the Tachibana tensor $Q(A, T)$ on M, for any symmetric $(0,2)$-tensor A and for any generalized curvature $(0,4)$-tensor T. We then determine the properties of the difference tensor $R \cdot C - C \cdot R$.

Proposition 3. *Let M be a Chen ideal submanifold of codimension m in the Euclidean space $\mathbb{E}^{n+m}$. Consider a symmetric $(0,2)$-tensor A and a generalized curvature $(0,4)$-tensor T defined on M. With respect to the notations of Definition 1, we put*

$$T_{\alpha\beta\gamma\delta} = T\left(e_\alpha, e_\beta, e_\gamma, e_\delta\right), \quad A_{\alpha\beta} = A\left(e_\alpha, e_\beta\right)$$

for any different indices α, β, γ, $\delta \in \{1, \cdots, n\}$. Then, for any tangent vector field X, Y on M and for any different indices u, v, s, t, $\beta \in \{1, \cdots, n\}$, one has the following three identities:

$$\begin{aligned} Q(A,T)\left(e_u, e_s, X, Y; e_u, e_v\right) = {} & T_{ussu}\left[A_{uv}\Big\langle \left(e_s \wedge_g e_u\right)(X), Z\Big\rangle + \sum_{\alpha=1}^{n} A_{\alpha v}\Big\langle \left(e_s \wedge_g e_\alpha\right)(X), Y\Big\rangle\right] \\ & - T_{vssv}A_{uu}\Big\langle \left(e_s \wedge_g e_v\right)(X), Y\Big\rangle + T_{uvvu}A_{su}\Big\langle \left(e_u \wedge_g e_v\right)(X), Y\Big\rangle; \\ Q(A,T)\left(e_u, e_v, X, Y; e_u, e_v\right) = {} & 0; \\ Q(A,T)\left(e_s, e_t, X, Y; e_u, e_v\right) = {} & 0. \ ■ \end{aligned}$$

Proof of Proposition 3: Let X, Y be two tangent vector fields on M. It is usefull to recall here that, for any tangent vector fields U and V on M, one has:

$$\begin{aligned} Q(A,T)\left(U, V, X, Y; e_u, e_v\right) = {} & -T\left(\left(e_u \wedge_A e_v\right)(U), V, X, Y\right) - T\left(U, \left(e_u \wedge_A e_v\right)(V), X, Y\right) \\ & - T\left(U, V, \left(e_u \wedge_A e_v\right)(X), Y\right) - T\left(U, V, X, \left(e_u \wedge_A e_v\right)(Y)\right). \end{aligned}$$

Then for any three different indices u, v, s, $t \in \{1, \cdots, n\}$ and substituting (U, V, X, Y) successively by (e_u, e_s, X, Y), (e_u, e_v, X, Y) and (e_s, e_t, X, Y) in the above formula, we get straightforwardly all the formulas in Proposition 3. □

Proposition 4. *For all tangent vector field X, Y, Z, W and for all different indices u, $v \in \{1, \cdots, n\}$,*

$$(R \cdot C - C \cdot R)\left(X, Y, Z, W; e_u, e_v\right) = \left[R_{uvvu}Q(g, C) - C_{uvvu}Q(g, R)\right]\left(X, Y, Z, W; e_u, e_v\right). \ ■$$

Proof of Proposition 4: Let X, Y, Z, W be four tangent vector fields and u, v different indices such that $1 \leq u \leq n$ and $1 \leq v \leq n$. On the one hand,

$$\begin{aligned} (R \cdot C)\left(X, Y, Z, W; e_u, e_v\right) = {} & R_{uvvu}Q(g, C)\left(X, Y, Z, W; e_u, e_v\right); \\ (C \cdot R)\left(X, Y, Z, W; e_u, e_v\right) = {} & C_{uvvu}Q(g, R)\left(X, Y, Z, W; e_u, e_v\right). \end{aligned}$$

Then the difference gives directly the required equality. □

6.1 Proof of Theorem 5

To deduce Theorem 5, one can use directly the notations of Theorem 1 and the following Lemma 1.

Lemma 1. *Let M be a Chen ideal and non conformally flat submanifold of dimension $n \geq 4$ in the Euclidean space $\mathbb{E}^{n+m}$. There exists two real valued functions $L_1,\ L_2$ on M such that*

$$L_1 Q(g,R) + L_2 Q(S,R) = R \cdot C - C \cdot R$$

if and only if

$$\begin{cases} b = \epsilon a \\ (1+\epsilon)\, a^2 L_1 + 2\,(1+\epsilon)\, a^2\, (\inf K)\, L_2 = \dfrac{2\,(1+\epsilon)\, a^2 \inf K}{n-1} \\ (\inf K)\, L_1 + \Big[(2n-3)\,(1+\epsilon)\, a^2\, (\inf K) - 2(n-2)\,(1+\epsilon)\, a^4\Big] L_2 = \dfrac{(\inf K)\,\Big[(n-3)\,(\inf K) + (n-1)(n-2)\,(1+\epsilon)\, a^2\Big]}{(n-1)(n-2)} \end{cases}$$

where $\epsilon = \pm 1$. ■

Proof of Lemma 1: Using the relations given in Propositions 3 and 4, we obtain straightforwardly the required equivalences. □

6.2 Proof of Theorem 6

To deduce Theorem 6, one can use directly the notations of Theorem 1 and the following Lemma 2.

Lemma 2. *Let M be a Chen ideal and non conformally flat submanifold of dimension $n \geq 4$ in the Euclidean space $\mathbb{E}^{n+m}$. There exists two real valued functions $L_3,\ L_4$ on M such that*

$$L_3 Q(g,C) + L_4 Q(S,C) = R \cdot C - C \cdot R$$

if and only if

$$\begin{cases} b = \epsilon a \\ L_3 + \dfrac{\inf K + (n-2)(2n-3)\,(1+\epsilon)\, a^2}{n-1} L_4 = \dfrac{\inf K + (n-2)\,(1+\epsilon)\, a^2}{n-1} \\ L_3 + \dfrac{2 \inf K + 2(n-2)^2\,(1+\epsilon)\, a^2}{n-1} L_4 = \dfrac{2(n-2)\,(1+\epsilon)\, a^2}{n-1} \end{cases}$$

where $\epsilon = \pm 1$. ■

Proof of Lemma 2: Using the relations given in Propositions 3 and 4, we obtain straightforwardly the required equivalences. □

6.3 Proof of Theorem 7

To deduce Theorem 7, one can use directly the notations of Theorem 1 and the following Lemma 3.

Lemma 3. *Let M be a Chen ideal and non conformally flat submanifold of dimension $n \geq 4$ in the Euclidean space $\mathbb{E}^{n+m}$. There exists two real valued functions $L_5,\ L_6$ on M such that*

$$L_5 Q(g, g \wedge S) + L_6 Q(S, g \wedge S) = R \cdot C - C \cdot R$$

if and only if

$$\begin{cases} b = \epsilon a \\ L_5 - \left[\inf K + (n-2)\,(1+\epsilon)\, a^2\right] L_6 = (\inf K)\, \dfrac{n-3}{(n-1)(n-2)} \dfrac{\inf K + (n-2)\,(1+\epsilon)\, a^2}{\inf K - (n-2)\,(1+\epsilon)\, a^2} \\ L_5 - \left[2(n-2)\,(1+\epsilon)\, a^2\right] L_6 = -\dfrac{2a^2\,(1+\epsilon)}{n-1} \dfrac{\inf K}{\inf K - (n-2)\,(1+\epsilon)\, a^2} \end{cases}$$

where $\epsilon = \pm 1$. ■

Proof of Lemma 3: Using the relations given in Propositions 3 and 4, we obtain straightforwardly the required equivalences. □

References

[1] K. Arslan, R. Deszcz, R. Ezentaş, M. Hotloś and C. Murathan, *On generalized Robertson-Walker spacetimes satisfying some curvature condition*, Turk Math. J., in print.

[2] A.C. Asperti, G.A. Lobos, and F. Mercuri, *Pseudo-parallel submanifolds of a space form*, Adv. Geom., **2** (2002), 57–71.

[3] A.L. Besse, *Einstein Manifolds*, Ergeb. Math. Grenzgeb., 3. Folge, Bd. 10, Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[4] D.E. Blair, P. Verheyen and L. Verstraelen, *Hypersurfaces satisfaisant a $R \cdot C = 0$ ou $C \cdot R = 0$*, C. R. Acad. Bulgare Sci. **37** (1984), 459–462.

[5] E. Cartan, *LeÃ§ons sur la gÃ©omÃ©trie des espaces de Riemann*, 2nd ed., Paris: Gauthier-Villars, (1946).

[6] B.-Y. Chen, *Some pinching and classification theorems for minimal submanifolds*, Arch. Math. (Basel) **60** (1993), 568–578.

[7] B.-Y. Chen, *Riemannian submanifolds*, in: Handbook of Differential Geometry, edited by F. Dillen and L. Verstraelen, Vol. 1, North Holland, Amsterdam, 2000, pp. 187–418.

[8] B.-Y. Chen, *δ-invariants, inequalities of submanifolds and their applications*, Chap. 2, in: Topics in Diff. Geom., Rom. Acad. Sci., Bucharest (2008), pp. 29–155.

[9] B.-Y. Chen, *Pseudo-Riemannian geometry, δ-invariants and applications*, World Sci., Hackensack, New Jersey (2011).

[10] B.-Y. Chen, *A tour through δ-invariants: From Nash embedding theorem to ideal immersions, best ways of living and beyond*, Publ. Inst. Math. (Beograd) (N.S.), Proc. XVII Geom. Sem., Zlatibor 2012, to appear.

[11] B.-Y. Chen, *δ-Invariant, Inequalities of Submanifolds and Their Applications*, arXiv: 1307.1877v1 [math.DG] 7 Jul 2013.

[12] B.-Y. Chen, *Ideal hypersurfaces of Euclidean four-space*, arXiv: 1307.4772v1 [math.DG] 17 Jul 2013.

[13] B.-Y. Chen and F. Dillen, *Totally real bisectional curvature, Bochner-Kaehler and Einstein-Kaehler manifolds*, Diff. Geom. Appl. **10** (1999), 145–154.

[14] B.-Y. Chen, F. Dillen and L. Verstraelen, *δ-invariants and their applications to centroaffine geometry*, Diff. Geom. Appl. **22** (2005), 341–354.

[15] B.-Y. Chen, F. Dillen, L. Verstraelen, and L. Vrancken, *Characterizations of Riemannian space forms, Einstein spaces and conformally flat spaces*, Proc. Amer. Math. Soc. **128** (2000), 589–598.

[16] B.-Y. Chen and V. Martin-Molina, *Optimal inequalities, contact δ-invariants and their applications*, Bull. Malays. Math. Sci. Soc. (2) **36** (2013), no. 2, 263–276.

[17] S.S. Chern, *Minimal submanifolds in a Riemannian manifold*, Lect. Notes Technical Rep. 19, Dept. Math. Univ. Kansas (1968).

[18] J. Chojnacka-Dulas, R. Deszcz, M. Głogowska and M. Prvanović, *On warped products manifolds satisfying some curvature conditions*, J. Geom. Phys., **74** (2013), 328–341.

[19] M. Dajczer and L.A. Florit, *On Chen's basic equality*, Illinois J. Math. **42** (1998), 97–106.

[20] A. Derdziński, *Examples de métriques de Kaehler et d'Einstein autoduales sur le plan complexe*, Géométrie riemannienne en dimension 4 (Séminaire Arthur Besse 1978/79), Cedic/Fernand Nathan Paris 1981, 334–346.

[21] F. Defever and R. Deszcz, *On semi-Riemannian manifolds satisfying the condition $R \cdot R = Q(S, R)$*, in: Geometry and Topology of Submanifolds, **III**, World Sci., River Edge, NJ (1991), 108–130.

[22] J. Deprez, W. Roter and L. Verstraelen, *Conditions on the projective curvature tensor of conformally flat Riemannian manifolds*, Kyungpook Math. J. **29** (1989), 153–165.

[23] R. Deszcz, *On pseudo-symmetric spaces*, Bull. Belg. Math. Soc., Ser. A, **44** (1992), 1–34.

[24] R. Deszcz, M. Głogowska, H. Hashiguchi, M. Hotloś and M. Yawata, *On semi-Riemannian manifolds satisfying some conformally invariant curvature condition*, Colloq. Math. **131** (2013), 149–170.

[25] R. Deszcz, M. Głogowska and M. Hotloś, *Some identities on hypersurfaces in conformally flat spaces*, Proceedings of the International Conference XVI Geometrical Seminar, Vrnjačka banja, September, 20-25, 2010, Faculty of Science and Mathematics, University of Niš, Serbia, (2011), 34–39.

[26] R. Deszcz, M. Głogowska, M. Hotloś and K. Sawicz, *A survey on generalized Einstein metric conditions*, AMS/IP Studies in Advanced Mathematics **49** (2011), 27–45.

[27] R. Deszcz, M. Głogowska, M. Hotloś and Z. Șentürk, *On certain quasi-Einstein semi-symmetric hypersurfaces*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **41** (1998), 151–164.

[28] R. Deszcz, M. Głogowska, M. Hotloś, and L. Verstraelen, *On some generalized Einstein metric conditions on hypersurfaces in semi-Riemannian space forms*, Colloq. Math. **96** (2003), 149–166.

[29] R. Deszcz, M. Głogowska, M. Hotloś and G. Zafindratafa, *On some curvature conditions of pseudo-symmetry type*, to appear.

[30] R. Deszcz, M. Głogowska, J. Jełowicki, M. Petrović-Torgašev and G. Zafindratafa, *On Riemann and Weyl compatible tensors*, Publ. Inst. Math. (Beograd) (N.S.), in print.

[31] R. Deszcz, M. Głogowska, J. Jełowicki and G. Zafindratafa, *On some warped product manifolds with pseudo-symmetric Weyl tensor*, to appear.

[32] R. Deszcz, M. Głogowska, M. Petrović-Torgašev and L. Verstraelen, *On the Roter type of Chen ideal submanifolds*, Results. Math. **59** (2011), 401–413.

[33] R. Deszcz, M. Głogowska, M. Plaue, K. Sawicz and M. Scherfner, *On hypersurfaces in space forms satisfying particular curvature conditions of Tachibana type*, Kragujevac J. Math. **35** (2011), 223–247.

[34] R. Deszcz and W. Grycak, *On some class of warped product manifolds*, Bull. Inst. Math. Acad. Sinica, **15** (1987), 311–322.

[35] R. Deszcz, S. Haesen and L. Verstraelen, *On natural symmetries*, Chap. 6 in Topics in Differential Geometry, Ed. Rom. Acad. Sci., Bucharest (2008), 249–308.

[36] R. Deszcz and M. Hotloś, *On a certain extension of class of semi-symmetric manifolds*, Publ. Inst. Math. (Beograd) (N.S.) **63**(**77**) (1998), 115–130.

[37] R. Deszcz and M. Hotloś, *On certain subclass of pseudo-symmetric manifolds*, Publ. Math. Debrecen **53** (1998), 29–48.

[38] R. Deszcz and M. Hotloś, *On some pseudo-symmetry type curvature condition*, Tsukuba J. Math. **27** (2003), 13–30.

[39] R. Deszcz and M. Hotloś, *On hypersurfaces with type number two in spaces of constant curvature*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **46** (2003), 19–34.

[40] R. Deszcz, M. Hotloś and Z. Ṣentürk, *On a certain application of Patterson's curvature identity*, Publ. Math. Debrecen **58** (2001), 93–107.

[41] R. Deszcz, M. Hotloś and Z. Ṣentürk, *On some family of generalized Einstein metric conditions*, Demonstr. Math. **34** (2001), 943–954.

[42] R. Deszcz, M. Hotloś and Z. Ṣentürk, *On curvature properties of certain quasi-Einstein hypersurfaces*, Int. J. Math. **23** (2012), 1250073 (17 pages). DOI: 10.1142/S0129167X12500735

[43] R. Deszcz, M. Petrović-Torgašev, L. Verstraelen and G Zafindratafa, *On the intrinsic symmetries of Chen ideal submanifolds*, Bull. Transilvania Univ. Braşov, **1 (50)**, series III, 99–109 (2008).

[44] R. Deszcz and L. Verstraelen, *Hypersurfaces of semi-Riemannian conformally flat manifolds*, Geometry and Topology of Submanifolds, **V**, World Sci., River Edge, NJ, 109–131, 1993.

[45] R. Deszcz, L. Verstraelen and L. Vrancken, *The symmetry of warped product space-times*, Gen. Rel. Gravitation **23** (1991), 671–681.

[46] R. Deszcz, L. Verstraelen and S. Yaprak, *Pseudo-symmetric hypersurfaces in 4-dimensional spaces of constant curvature*, Bull. Inst. Math. Acad. Sinica, **22** (1994), 167–179.

[47] R. Deszcz and S. Yaprak, *Curvature properties of certain pseudosymmetric manifolds*, Publ. Math. Debrecen, **45** (1994), 333–345.

[48] F. Dillen, M. Petrović-Torgašev and L. Verstraelen, *Einstein, conformally flat and semi-symmetric sumbmanifolds satisfying Chen's equality*, Israel J. Math. **100** (1997), 163–169.

[49] M. Głogowska, *Semi-Riemannian manifolds whose Weyl tensor is a Kulkarni-Nomizu square*, Publ. Inst. Math. (Beograd) (N.S.) **72** (**86**) (2002), 95–106.

[50] M. Głogowska, *On a curvature characterization of Ricci-pseudosymmetric hypersurfaces*, Acta Math. Scientia **24** B (2004), 361–375.

[51] M. Głogowska, *On quasi-Einstein Cartan type hypersurfaces*, J. Geom. Phys. **58** (2008), 599–614.

[52] W. Grycak, *Riemannian manifolds with a symmetry condition imposed on the* 2*-end derivative of the conformal curvature tensor*, Tensor (N.S.) **46** (1987), 287–290.

[53] S. Haesen and L. Verstraelen, *Classification of the pseudo-symmetric space-times*, J. Math. Phys. **45** (2004), 2343–2346.

[54] S. Haesen and L. Verstraelen, *Properties of a scalar curvature invariant depending on two planes*, Manuscripta Math. **122** (2007), 59–72.

[55] S. Haesen and L. Verstraelen, *Natural Intrinsic Geometrical Symmetries*, Symmetry, Integrability and Geometry: Methods and Applications SIGMA **5** (2009), 086, 15 pages.

[56] A. Hinić, *Some curvature conditions of the type* 2×4 *on the submanifolds satisfying Chen's equality*, Mat. Vesnik **59** (2007), 189–196.

[57] M. Hotloś, *On certain curvature condition of pseudo-symmetry type*, to appear.

[58] B. Jahanara, S. Haesen, M. Petrović-Torgašev and L. Verstraelen, *On the Weyl curvature of Deszcz*, Publ. Math. Debrecen **74** (2009), 417–431.

[59] M. Jamali, J.-B. Jun, M.H. Shahid, M. Ahmad. *A remark on the axiom of spheres with lightlike submanifolds and application of Jacobi equation*, Bull. Malays. Math. Sci. Soc. (2), accepted.

[60] D. Kowalczyk, *On some subclass of semi-symmetric manifolds*, Soochow J. Math. **27** (2001), 445–461.

[61] D.S. Leung and K. Nomizu, *The axiom of spheres in Riemannian Geometry*, J. Differential Geom., **5** (1971) 487–489.

[62] D. van Lindt and L. Verstraelen, *Some axioms of Einsteinian and conformally fiat hypersurfaces*, J. Differential Geom. **16** (1981), 205–212.

[63] D. van Lindt and L. Verstraelen, *A survey on axioms of submanifolds in Riemannian and Kaehlerian geometry*, Colloq. Math. **54** (1987), 193–213.

[64] M. Petrović-Torgašev and A. Hinić, *Some curvature conditions of the type* 4×2 *on the submanifolds satisfying Chen's equality*, Math. Vesnik **59** (2007), 143–150.

[65] K. Sawicz, *On some class of hypersurfaces with three distinct principal curvatures*, Banach Center Publications **69**, Inst. Math. Polish Acad. Sci. 145–156, 2005.

[66] Z.I. Szabó, *Structure theorems on Riemannian spaces satisfying* $R(X,Y) \cdot R = 0$. *I. The local version*, J. Differential Geom. **17** (1982), 531–582.

[67] A. Taleshian and A.A. Hosseinzadeh, *Investigation of some conditions on* $N(k)$*-quasi Einstein manifolds*, Bull. Malays. Math. Sci. Soc. (2) **34** (2011), no. 3, 455–464.

[68] H.M. Taștan, *The axiom of hemi-slant* 3*-spheres in almost Hermitian geometry*, Bull. Malays. Math. Sci. Soc. (2), accepted.

[69] L. Verstraelen, *Some comments on the* δ*-curvatures of Bang-Yen Chen*, Rend. Sem. Matem. Messina, 1998 International Congress in honour of R. Calapso, 331–337.

[70] G.K. Zafindratafa, *Sous-variÃ©tÃ©s soumises Ã des conditions de courbure*, Doctoral Thesis, K.U. Leuven, 1991.

[71] A. Yildiz, U.C. De and A.C. Çetinkaya, $N(k)$*-quasi Einstein manifolds satisfying certain curvature conditions*, Bull. Malays. Math. Sci. Soc. (2), accepted.

Ryszard Deszcz
Wrocław University of Environmental and Life Sciences
Department of Mathematics
Grunwaldzka 53, 50-357 Wrocław, Poland
E-mail: Ryszard.Deszcz@up.wroc.pl
Miroslava Petrović-Torgasěv
University of Kragujevac
Faculty of Science, Department of Mathematics,
Radoja Domanovića 12
34000 Kragujevac, Serbia
E-mail: mirapt@kg.ac.rs
Leopold Verstraelen
Katholieke Universiteit Leuven
Fakulteit Wetenschappen, Departement Wiskunde
Celestijnenlaan 200B
B-3001 Heverlee, Belgium
E-mail: Leopold.Verstraelen@wis.kuleuven.be
Georges Zafindratafa
Université de Valenciennes et du Hainaut-Cambrésis
Laboratoire LAMAV - ISTV2
F-59313 Valenciennes Cedex 9, France
E-mail: Georges.Zafindratafa@univ-valenciennes.fr