Distributions of codimension 2 in Kenmotsu geometry

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Abstract. Given a 2-codimensional distribution normal to the structural vector field ξ on a Kenmotsu manifold the necessary and sufficient conditions for the normality of this distribution are studied. A main result is the existence of a total umbilical foliation and of bundle-like metrics. Under certain circumstances a new foliation arises and its properties are investigated.

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1 Introduction

The geometry of Riemannian foliations has been intensively studied in the latest years and many interesting results have been obtained; therefore, some monographs are dedicated to this subject: [14] and [15]. In their book [2], Bejancu and Farran gives a new approach by studying the foliations defined on a Riemannian manifold by using two adapted linear connections. The main notion of this theory is that of *bundle-like metric* introduced by Reinhart in [13] and intensively studied by several authors; see for example [15] and the related references cited therein. It was subsequently proved in [2, p. 32] that there exists a bundle-like metric on a Riemannian manifold (M, g) endowed with two complementary orthogonal non-integrable distributions.

The purpose of this paper is to study several properties of a distribution of codimension two in a Kenmotsu manifold. The given distribution is supposed to be normal to the structural vector field ξ since the case when the structural distribution in a generalized quasi-Sasakian manifold is tangent to the structure vector field was studied in [4] and [5] where was proved the existence of the bundle-like structure and other interesting properties. Also, the case of foliation induced by the structural distribution was treated in [10].

The structure of the paper is as follows. In the second section, several general results regarding quasi-Sasakian manifolds are stated for later use. We introduce the notion of a generalized quasi-Sasakian manifold (on short a G.Q.S manifold) defined as a manifold

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endowed with an almost contact metric structure enjoying property (2.5). Some important results of the G.Q.S manifolds are proved for later use (Proposition 2.2). In the third section it is proved the existence of an almost contact metric structure which satisfy the Eum condition (2.6) on any 2-codimension distribution normal to the structural vector field of a Kenmotsu manifold. In the last section we study the existence of normal metric structure. This existence is proved by verifying the necessary and sufficient conditions for the vanishing of a tensor field of $(1,2)$-type (Theorem 4.2, Theorem 4.4). Next, it is shown in Theorem 4.9 that the existence of a normal metric structure implies the existence of a foliation of dimension two. Also, a bundle-like metric and a totally umbilical property are studied in Theorems 4.8 and 4.9 respectively.

2 Preliminaries

Let (M,g) be an m-dimensional Riemannian manifold and TM its tangent bundle; all our objects are smooth differentiable. If $\mathcal{F}(M)$ is the algebra of the smooth functions on M then $\Gamma(E)$ denotes the $\mathcal{F}(M)$-module of the sections of a vector bundle E over M.

Now suppose that there exists a pair of complementary orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on M i.e. TM has the decomposition $TM = \mathcal{D} \oplus \mathcal{D}^\perp$ with respect to the metric g. We denote by Q and Q' the projection morphisms of TM on $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively.

Based on [2, p. 97] we consider two connections denoted by D and $D^\perp$ on the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$, called *intrinsic linear connections* and defined by:

$$D_X QY = Q\tilde{\nabla}_{QX} QY + Q[Q'X, QY], D^\perp_X Q'Y = Q'\tilde{\nabla}_{Q'X} Q'Y + Q'[QX, Q'Y], \qquad (2.1)$$

with $X, Y \in \Gamma(TM)$ where $\tilde{\nabla}$ is the Levi-Civita connection of (M,g). Suppose that the distribution $\mathcal{D}$ is integrable; then it defines a foliation on M which we denote by $\mathcal{F}_D$. The distribution $\mathcal{D}$ is called also and the *structural distribution* of $\mathcal{F}_D$. The Riemannian metric g is called *bundle-like* for the foliation $\mathcal{F}_D$ ([15]) if each geodesic in (M,g) that is tangent to the normal distribution to $\mathcal{F}_D$ at one point remains tangent for its entire length. Bejancu-Farran ([2, p. 110]) gave a characterization for a bundle-like metric on $(M,\mathcal{F}_D)$: the Riemannian metric g is bundle-like for the foliation $\mathcal{F}_D$ if the Riemannian metric induced by g on $\mathcal{D}^\perp$, denoted by the same symbol g, is parallel with respect to the intrinsic connection $D^\perp$:

$$(D^\perp_X g)(Q'Z, Q'Y) = 0, \quad \forall X, Y, Z \in \Gamma(TM).$$

Also, from [2, p. 112] we recall the following result:

Theorem 2.1. *If $(M, g, \mathcal{F}_D)$ is a foliated Riemannian manifold, then the following assertions are equivalent:*
a) g is bundle-like metric for $\mathcal{F}_D$,
b) QX is a $\mathcal{D}^\perp$-Killing vector field, that is for all $X, Y, Z \in \Gamma(TM)$:

$$g(\tilde{\nabla}_{Q'Y} QX, Q'Z) + g(\tilde{\nabla}_{Q'Z} QX, Q'Y) = 0.$$

Now, we denote by ∇ respectively $\nabla^\perp$ the connection induced by $\tilde{\nabla}$ on $\mathcal{D}$ (resp. $\mathcal{D}^\perp$) and by h, h' the $\mathcal{F}(M)$-bilinear mappings $h : \Gamma(TM) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}^\perp)$, $h' : \Gamma(TM) \times \Gamma(\mathcal{D}^\perp) \to \Gamma(\mathcal{D})$ given by:

$$\nabla_X QY = Q\tilde{\nabla}_X QY, \quad \nabla^\perp_X Q'Y = Q'\tilde{\nabla}_X Q'Y,$$

$$h(X,QY) = Q'\tilde{\nabla}_X QY, \quad h'(X,Q'Y) = Q\tilde{\nabla}_X Q'Y,$$

for any $X, Y \in \Gamma(TM)$.

In connection with the decomposition (2.1) we have for all $X, Y \in \Gamma(TM)$ ([2, p. 27]):

$$\tilde{\nabla}_X QY = \nabla_X QY + h(X,QY), \quad \tilde{\nabla}_X Q'Y = \nabla^{\perp}_X Q'Y + h'(X,Q'Y), \tag{2.2}$$

relations which are called *the Gauss formulae* for the Riemannian distributions $(\mathcal{D}, g)$ and $(\mathcal{D}^{\perp}, g)$, respectively. For any $Q'X \in \Gamma(D^{\perp})$ and $QX \in \Gamma(D)$ we define two $\mathcal{F}(M)$-linear operators ([2, p. 27]):

$$A_{Q'X} : \Gamma(D) \to \Gamma(D), \quad A_{QX} : \Gamma(D^{\perp}) \to \Gamma(D^{\perp}),$$

by:

$$A_{Q'X}QY = -h'(QY, Q'X), \quad A'_{QX}Q'Y = -h(Q'Y, QX), \quad \forall X, Y \in \Gamma(TM).$$

According to the theory of submanifolds, $A_{Q'X}$, and A'_{QX} are called *the shape operators* of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ with respect to the normal sections $Q'X$ and QX, respectively.

It is easy to see that ([2, p. 28]) for $X, Y, Z \in \Gamma(TM)$:

$$g(h(QX,QY), Q'Z) = g(A_{Q'Z}QX, QY), [QX,QY] \in \Gamma(\mathcal{D}) \Leftrightarrow h(QX,QY) = h(QY,QX). \tag{2.3}$$

Next, let $m = 2n+1$ and suppose that the manifold M is endowed with an almost contact metric structure (f, ξ, η, g) ([3] or [7]):

$$\begin{aligned} &a)\ \ f^2 = -I + \eta \otimes \xi, \quad b)\ \ \eta(\xi) = 1, \quad c)\ \ \eta \circ f = 0, \\ &d)\ \ f(\xi) = 0, \quad e)\ \ \eta = g(\cdot, \xi), \quad f)\ \ g(f\cdot, \cdot) + g(\cdot, f\cdot) = 0, \end{aligned} \tag{2.4}$$

where I is the identity on TM, f is a tensor field of $(1,1)$-type and η is the 1-form dual to the vector field ξ. The Nijenhuis tensor field of the *structural tensor field* f is:

$$N_f(X,Y) = [fX, fY] + f^2[X,Y] - f[fX,Y] - f[X,fY].$$

In the following we consider a class of almost contact metric manifolds for which the structural tensor field f is assumed to satisfy for all $X, Y \in \Gamma(TM)$:

$$(\tilde{\nabla}_X f)Y = g(\tilde{\nabla}_{fX}\xi, Y)\xi - \eta(Y)\tilde{\nabla}_{fX}\xi. \tag{2.5}$$

In the paper [9] S. S. Eum studied the integrability of invariant hypersurfaces immersed in an almost contact Riemannian manifold satisfying the condition:

$$g((\tilde{\nabla}_X f)Y, Z) = (\tilde{\nabla}_X \eta)(\eta(Y)fZ - \eta(Z)fY)$$

which is equivalent with:

$$(\tilde{\nabla}_X f)Y = g(f\tilde{\nabla}_X \xi, Y)\xi - \eta(Y)f\tilde{\nabla}_X \xi. \tag{2.6}$$

It is interesting to see that if (2.5) holds then (2.6) is also true but not conversely.

For convenience, we define a tensor field F of $(1,1)$-type by:

$$F(X) = -\tilde{\nabla}_X \xi. \tag{2.7}$$

Using (2.4)a, (2.5) and (2.7) one obtains the following result through direct calculation:

Proposition 2.2. *If M is an almost contact metric manifold with the property* (2.5) *then the following equalities hold:*
a) (M, f, ξ, η, g) *is normal and* $f \circ F = F \circ f$,
b) $F(\xi) = 0$ *and* $\eta \circ F = 0$,
c) $\tilde{\nabla}_\xi f = 0$.

Next we prove the following characterization result for quasi-Sasakian manifold:

Proposition 2.3. *The structural vector field ξ on an almost contact metric manifold M enjoying property* (2.5) *is a geodesic vector field, that is $\tilde{\nabla}_\xi \xi = 0$; then η is exact. Moreover, ξ is a Killing vector field if and only if M is a quasi-Sasakian manifold, that is $d\Phi = 0$, where Φ is the fundamental* 2*-form:* $\Phi(X, Y) = g(X, fY)$.

Examples 2.1. *It is easy to see that on an almost contact metric manifold M enjoying property* (2.5) *the structural vector field ξ is not necessarily a Killing vector field. Also, it is interesting to see that:*
1) if $F = -\alpha f$ then M is an α-Sasakian manifold ([11]),
2) if $F = \beta(-id + \eta \otimes \xi)$ then M becomes a β-Kenmotsu manifold ([6]),
3) if $F = 0$ then M is a cosymplectic manifold,
4) if $F = \alpha f + \beta f^2$ with $\alpha, \beta \in \mathcal{F}(M)$ then M is trans-Sasakian manifold ([8]).
This was the reason for which we called M satisfying (1.5) *a generalized quasi-Sasakian manifold, shortly G.Q.S manifold.*

3 Distributions of codimension 2 in Kenmotsu geometry

Suppose that the almost contact metric manifold (M, f, ξ, η, g) is a Kenmotsu one, [6]:

$$(\tilde{\nabla}_X f)Y = g(fX, Y)\xi - \eta(Y)fX, \quad \tilde{\nabla}_X \xi = X - \eta(X)\xi.$$

Let us given on M a distribution $\mathcal{D}_1$ of codimension 2, normal to the structural vector field ξ and consider $\mathcal{D}_2$ as the orthogonal complementary distribution:

$$TM = \mathcal{D}_1 \oplus \mathcal{D}_2. \tag{3.1}$$

From $\dim \mathcal{D}_2 = 2$ and $\xi \in \Gamma(\mathcal{D}_2)$ one deduces that $\Gamma(\mathcal{D}_2) = \{N, \xi\}$ with N a unit vector field orthogonal to ξ. By using (2.4)f we get $f(\mathcal{D}_2) = \{f(N) = U\} \subset \mathcal{D}_1$ and let $\mathcal{D}$ be the orthogonal complementary distribution of $f(\mathcal{D}_2)$ in $\mathcal{D}_1$. It results the orthogonal decomposition:

$$\mathcal{D}_1 = \mathcal{D} \oplus \mathcal{D}^\perp; \;\; \mathcal{D}^\perp = f(\mathcal{D}_2), \; \dim\mathcal{D}^\perp = 1. \tag{3.2}$$

From (2.4)f one deduces that U is also a unit vector field. The decomposition (3.1) have the detailed expression:

$$TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus f(\mathcal{D}^\perp) \oplus \{\xi\}$$

where $\{\xi\}$ is the 1-dimensional distribution generated by ξ and then $\mathcal{D}$ is an f-invariant distribution, $f(\mathcal{D}) = \mathcal{D}$, of dimension $2n - 2$. The restriction of f to $\mathcal{D}$ is an almost complex structure.

Let ∇ be the connection induced by $\tilde{\nabla}$ on $\mathcal{D}_1$. Relative to the decomposition (3.1) the formulae (2.2) have the expression ([1]):

$$\begin{cases} \tilde{\nabla}_X Y = \nabla_X Y + B(X,Y)N + C(X,Y)\xi, \\ \tilde{\nabla}_X N = -A_N X + b(X)\xi, \\ \tilde{\nabla}_X \xi = -A_\xi X - b(X)N. \end{cases} \tag{3.3}$$

where A_N, A_ξ are the shape operators with respect to the sections N, ξ and B and C are the bilinear forms:

$$\begin{cases} B(X,Y) = g(\tilde{\nabla}_X Y, N) = -g(\tilde{\nabla}_X N, Y) = g(A_N X, Y) \\ C(X,Y) = g(\tilde{\nabla}_X Y, \xi) = -g(\tilde{\nabla}_X \xi, Y) = -g(X,Y). \end{cases}$$

The 1-form b of (3.3) is vanishing: $b(X) = g(\tilde{\nabla}_X N, \xi) = -g(\tilde{\nabla}_X \xi, N) = -g(X - \eta(X)\xi, N) = 0$ since $X \in \Gamma(\mathcal{D}_1)$. Therefore, the formulae (3.3) becomes:

$$\begin{cases} i: \quad \tilde{\nabla}_X Y = \nabla_X Y + g(A_N X, Y)N - g(X,Y)\xi, \\ ii: \quad \tilde{\nabla}_X N = -A_N X =: -AX, \quad \tilde{\nabla}_X \xi = -A_\xi X = X. \end{cases} \tag{3.4}$$

which are very similar to that of hypersurfaces. Hence, $A_\xi = -id$.

Suppose that $n \geq 2$ and let us denote by P the projection morphism of TM on $\mathcal{D}$. Taking into account the decompositions (3.1) and (3.2) we may express $X \in \Gamma(\mathcal{D}_1)$ as:

$$X = PX + u(X)U, \tag{3.5}$$

where u is the 1-form defined by $u(X) = g(X, U)$. From (3.5) we see that:

$$fX = tX - u(X)N, \tag{3.6}$$

where t is the tensor field of $(1,1)$-type given by: $t = f \circ P$. By straightforward calculations, using (2.4) we obtain the following result:

Proposition 3.1. *(t, U, u, g) is an almost contact metric structure on the distribution $\mathcal{D}_1$:*

$$t^2 X = -X + u(X)U, \quad u \circ t = 0, \quad t(U) = 0, \quad g(tX, Y) + g(X, tY) = 0.$$

This almost contact metric structure is of a special type:

Theorem 3.2. *The almost contact metric structure (t, U, u, g) is of Eum's type i.e. the tensor field t satisfies:*

$$(\nabla_X t)Y = g(t\nabla_X U, Y)U - u(Y)t\nabla_X U$$

on the distribution $\mathcal{D}_1$.

Proof From (3.4) and the first above equation we deduce that on $\Gamma(\mathcal{D}_1)$:

$$\nabla_X U = -tAX. \tag{3.7}$$

The covariant derivatives of t and u on $\mathcal{D}_1$ are given by:

$$(\nabla_X t)Y = g(AX, Y)U - u(Y)AX, \quad (\nabla_X u)Y = g(AX, tY)U. \tag{3.8}$$

Then the conclusion follows from a direct computation. □

4 Normality of a distribution of codimension 2

The purpose of this section is to study the normality of the almost contact metric structure (t, U, u, g) on the distribution $\mathcal{D}_1$.

Recall that for an almost contact metric structure (t, U, u, g) its *normality tensor* S is:

$$S(X,Y) = N_t(X,Y) + 2du(X,Y)U. \tag{4.1}$$

This almost contact metric structure (t, U, u, g) is *normal* if $S = 0$; then, we are interested in the expression of S:

Proposition 4.1. *The tensor field S is expressed by:*

$$S(X,Y) = u(X)(AtY - tAY) - u(Y)(AtX - tAX)+$$

$$+[g(AtX,Y) - g(tX,AY) + g(AX,tY) - g(AtY,X)]U + [g(AtX,tY) - g(AtY,tX)]N. \tag{4.2}$$

Proof. Since $\tilde{\nabla}$ is a torsion free connection, by using (2.5) and (3.4)i we get:

$$N_t(X,Y) = (\nabla_{tX}t)Y - (\nabla_{tY}t)X + t[(\nabla_Y t)X - (\nabla_X t)Y] + [g(AtX,tY) - g(AtY,tX)]N =$$

$$= u(X)(At - tA)(Y) - u(Y)(At - tA)(X) + [g(AtX,Y) - g(X,AtY)]U + [g(AtX,tY) - g(AtY,tX)]N \tag{4.3}$$

On the other hand, from (3.8) we deduce that:

$$2du(X,Y) = (\nabla_X u)Y - (\nabla_Y u)X = [g(AX,tY) - g(AY,tX)]U. \tag{4.4}$$

Finally, the relation (4.2) comes from (4.3) and (4.4). □

We now give a characterization for the normality of almost contact metric structure (t, U, u, g) defined on the distribution $\mathcal{D}_1$.

Theorem 4.2. *The almost contact metric structure (t, U, u, g) is normal if and only if $\mathcal{D}$ is integrable and for all $X \in \Gamma(\mathcal{D})$:*

$$AtX = tAX + u(AtX)U. \tag{4.5}$$

Proof. For $X, Y \in \Gamma(\mathcal{D})$ the relation (4.2) becomes:

$$S(X,Y) = [g(AtX,Y) - g(tX,AY) + g(AX,tY) - g(X,AtY)]U + [g(AtX,tY) - g(AtY,tX)]N \tag{4.6}$$

and:

$$S(X,U) = tAX - AtX + [u(AtX) - g(AU,tX)]U. \tag{4.7}$$

Now, suppose that $S = 0$ on $\mathcal{D}_1$. Using the fact that $S(X,U) = 0$ for all $X \in \Gamma(\mathcal{D})$ and (4.7) we obtain:

$$AtX = tAX + [u(AtX) - g(AU,tX)]U,$$

which implies $g(AU,tX) = 0$ and then $tAU = 0$. Therefore the above relation can be expressed as follows:

$$AtX = tAX + u(AtX)U, \quad \forall\ X \ \in\ \Gamma(\mathcal{D}),$$

and the relation (4.5) is proved. By using the fact that $tAU = 0$ we deduce that:

$$\nabla_U U = 0. \tag{4.8}$$

Next, from the relation (4.6) we obtain that for $X, Y \in \Gamma(\mathcal{D})$:

$$g(AtX, tY) = g(AtY, tX). \tag{4.9}$$

Since $\tilde{\nabla}$ is the Levi-Civita connection one obtains from (3.3)ii that $B(X, Y) = B(Y, X)$ ($\Leftrightarrow g(AX, Y) = g(X, AY)$) for $X, Y \in \Gamma(\mathcal{D})$ if and only if the distribution $\mathcal{D}$ is involutive.

Conversely, if the relation (4.5) are true then it is easy to see that from (4.7) and (4.9) we obtain that $S = 0$ on $\mathcal{D}_1$. □

Remark 4.1. *The last result justifies the fact that the normality of the almost contact metric structure (t, U, u, g) implies the involutivity of the distribution $\mathcal{D}_1$ and consequently the existence of a foliation which we shall denote by $\mathcal{F}_1$. We shall also say that the foliation $\mathcal{F}_1$ is normal provided that Theorem 4.2 is true.*

We consider the distribution $\mathcal{D}'_1 = \mathcal{D} \oplus \{N\}$ and therefore we have the following decomposition $TM = \mathcal{D}'_1 \oplus \{U\} \oplus \{\xi\}$. Next, we denote by A' the shape operator with respect to the section U and by $\mathcal{F}'_1$ the foliation corresponding to the distribution $\mathcal{D}'_1$. Through direct calculation, we obtain the next result:

Proposition 4.3. *The next equivalence holds for any $X \in \Gamma(\mathcal{D})$:*

$$AtX = tAX + u(AtX)U \Leftrightarrow A'tX = tA'X + g(A'tX, N)N.$$

Therefore the foliation $\mathcal{F}_1$ is normal if and only if the foliation $\mathcal{F}'_1$ is normal, too.

Next we obtain a new characterization of the normality of foliation $\mathcal{F}_1$.

Theorem 4.4. *The foliation $\mathcal{F}_1$ on a Kenmotsu manifold M is normal if and only if the following conditions are fulfilled:*
a) $\mathcal{D}$ is involutive,
b) U is a $\mathcal{D}_1$-Killing vector field and a geodesic vector field: $\nabla_U U = 0$.

Proof. Using (3.7) and (3.8) we obtain through direct calculation that for $X, Y \in \Gamma(\mathcal{D}_1)$:

$$g(\nabla_X U, Y) + g(\nabla_Y U, X) = g(tX, AY) - g(tAX, Y). \tag{4.10}$$

Suppose that the foliation $\mathcal{F}_1$ is normal. From (4.5) and (4.10) it follows that if $X \in \Gamma(\mathcal{D})$ and $Y \in \Gamma(\mathcal{D}_1)$ then:

$$g(\nabla_X U, Y) + g(\nabla_Y U, X) = 0.$$

Taking into account (4.8) it follows that the assertion c) is proved. Conversely, suppose that U is a $\mathcal{D}_1$-Killing vector field, then from (4.10) we deduce that (4.5) is proved. Thus the proof of the Theorem is complete. □

The Theorem 4.4 justifies the following consequence:

Corollary 4.5. *Let M be a Kenmotsu manifold with normal foliation $\mathcal{F}_1$ (or equivalently normal foliation $\mathcal{F}'_1$, respectively). Then:*
a) any integral curve of the $\mathcal{D}_1$-Killing vector field U is a geodesic,
b) any integral curve of the $\mathcal{D}'_1$-Killing vector field N is a geodesic.

The next result can be proved:

Theorem 4.6. *If $\mathcal{F}_1$ is a normal foliation then for all $X, Y \in \Gamma(\mathcal{D}_1)$:*
a) $\nabla_{tX} U = t\nabla_X U$,
b) $(\nabla_{tX} t)Y = t(\nabla_X t)Y - g(\nabla_X U, Y)U$,
c) $(\nabla_X t)Y = g(\nabla_{tX} U, Y)U - u(Y)\nabla_{tX} U$.

Proof. The first assertion is a direct consequence of (4.5). By using the formulae of Theorem 3.2. and again (4.5) we get also the next statements. □

The above theorem and Proposition 2.2 justifies the next result:

Corollary 4.7. *If the foliation $\mathcal{F}_1$ is normal then the almost contact metric structure (t, U, u, g) defines a quasi-Sasakian structure on each leaf of $\mathcal{F}_1$.*

Next, denote by $\mathcal{A} = \mathcal{D}^\perp \oplus f(\mathcal{D}^\perp)$. Based on the result from [2, p. 140], we say that the distribution $\mathcal{D}_1$ is *totally umbilical* if there exists a vector field, denoted by H and called *the mean curvature vector field*, such that:

$$h(X, Y) = g(X, Y)H, \quad \forall X, Y \in \Gamma(\mathcal{D}_1),$$

where h is the second fundamental form.

We shall now prove our next important result:

Theorem 4.8. *If the foliation $\mathcal{F}_1$ is normal then the distribution $\mathcal{A}$ defines a foliation $\mathcal{F}$ of dimension 2 on M. Moreover, the distribution $\mathcal{A}$ is totally umbilical and its mean curvature vector field H is $-\xi$.*

Proof. It is enough to prove that the distribution $\mathcal{A}$ is involutive. As $\mathcal{F}_1$ is normal, we have $\nabla_U U = 0$ and $\nabla_N N = 0$. Using the fact that $\tilde{\nabla}$ is a torsion free connection and Theorem 4.4 we obtain through direct calculation that:

$$g(\tilde{\nabla}_U N, tX) = -g(AU, tX) = g(tAU, X) = 0,$$

$$g(\tilde{\nabla}_N U, tX) = -g(A'N, tX) = g(tA'N, X) = 0, \quad \forall X \in \Gamma(\mathcal{D}).$$

Therefore $[N, U] \in \Gamma(\mathcal{A})$. Next, the mean curvature vector field of the foliation $\mathcal{F}$ is denoted by H and let h is the second fundamental form of distribution $\mathcal{A}$. In order to evaluate H, we suppose that the foliation $\mathcal{F}_1$ is normal. Then $h(X, Y) \in \Gamma(\mathcal{D} \oplus \{\xi\}), \ \forall X, Y \in \Gamma(\mathcal{A})$ since $\mathcal{D} \oplus \{\xi\}$ is the orthogonal complement of $\mathcal{A}$. Now, from Theorem 4.4 and Proposition 2.2 it is easy to see that $g(h(U, U), X) = 0, g(h(N, N), X) = 0 \ \ \forall X \in \Gamma(\mathcal{D})$. By direct calculation we deduce that for all $X, Y \in \Gamma(\mathcal{A})$:

$$g(h(X, Y), \xi) = g(\tilde{\nabla}_X Y, \xi) = -g(\tilde{\nabla}_X \xi, Y) = -g(X, Y).$$

Therefore we have $h(X, Y) = -g(X, Y)\xi$ which ends the proof of Theorem. □

Let us remark that this result is a variant of the Theorem 5.2. of [12, p. 616] from the distributions point of view. Next we denote by g_1 the induced metric tensor by g of a leaf of the integrable contact distribution.

Theorem 4.9. *Let $(M, g, \mathcal{F})$ be a 2-foliated Kenmotsu manifold with the normal foliation $\mathcal{F}_1$. Then the metric tensor g_1 is bundle-like for $\mathcal{F}$.*

Proof. It is enough to prove that U and N are $\mathcal{D}$-Killing vector fields, that is for all $X, Y \in \Gamma(\mathcal{D})$:

$$g_1(\nabla_X U, Y) + g_1(\nabla_Y U, X) = 0 = g_1(\nabla_X N, Y) + g_1(\nabla_Y N, X).$$

Since $\mathcal{F}_1$ is normal one deduce that if $X, Y \in \Gamma(\mathcal{D})$ then:

$$g_1(\nabla_X U, Y) + g_1(\nabla_Y U, X) = -g_1(tAX, Y) - g_1(tAY, X) = g_1(AtX - tAX, Y) = 0.$$

In the same way it can be proved that N is a $\mathcal{D}$-Killing vector field because the foliation $\mathcal{F}_1'$ is normal too. The proof is complete. □

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