INTERACTIONS OF ELEMENTARY WAVES FOR THE NONLINEAR CHROMATOGRAPHY EQUATIONS*

XIAOXIA WANG   LIHUI GUO †   GAN YIN

College of Mathematics and System Sciences, Xinjiang University, Urumqi, 830046, P.R.China

Abstract. In this article, we study the global solution of the elementary waves interaction problem for the nonlinear chromatography equations. We constructively obtain the solutions when the initial data are three piecewise constant states. The global structures and large time-asymptotic behaviors of the solutions are analyzed case by case. During the process of the interaction, it is easy to see that the solutions of the perturbed Riemann problem converge to nothing but the corresponding Riemann solutions as $\epsilon \to 0$, from which the stability of the Riemann solutions with respect to this local small perturbation of the Riemann initial data are obtained.

1. Introduction:

In this paper, we are concerned with the one-dimensional nonlinear chromatography equations

$$\begin{cases} u_t + \left(\left(1 + \dfrac{1}{1-u+v}\right) u\right)_x = 0, \\ v_t + \left(\left(1 + \dfrac{1}{1-u+v}\right) v\right)_x = 0, \end{cases} \tag{1.1}$$

where $u \geq 0$ and $v \geq 0$ are functions of the variables $(x,t) \in R \times R^+$, which express the concentrations of the species to be separated, and we consider system (1.1) under the situation $1-u+v > 0$. It is easy to see that the system (1.1) belongs to the Temple class, i.e., the shock curves coincide with the rarefaction curves in the phase plane, we can refer to [3, 5, 9, 16, 17] and the references cited therein.

Chromatography is not only a common analytical tool but also a powerful and efficient tool for preparative separations in the pharmaceutical, food, and agrochemical industries. Both single-column and multi-column operating modes of various degrees of complexity have been developed [7, 8, 12]. So it is necessary to study different chromatography equations. Mazzotti et al.[10, 11] have studied the more general nonlinear chromatography equations

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† Corresponding author.

Email address: xiaoxia_wang529@163.com (Xiaoxia Wang), lihguo@126.com (Lihui Guo), ganyinxj@gmail.com (Gan Yin) .

of the system (1.1), which can be read

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial}{\partial t}\left(u + \dfrac{au}{1-u+v}\right) = 0, \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial}{\partial t}\left(v + \dfrac{bv}{1-u+v}\right) = 0, \end{cases} \tag{1.2}$$

where u and v are the concentrations of the two absorbing species, with $u, v \geq 0$, $1-u+v > 0$ and $b > a > 0$ are constants. The difference between (1.1) and (1.2) is that the system (1.2) is hyperbolic in the region of the (u, v) plane where $(a(1+v)+b(1-u))^2-4ab(1-u+v) > 0$ and elliptic in the remaining part of it, while (1.1) is always hyperbolic in the whole composition space.

Recently, Shen [13] has studied the wave interactions and stability of the Riemann solutions for another chromatography equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{u}{1+u+v}\right) = 0, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{v}{1+u+v}\right) = 0. \end{cases} \tag{1.3}$$

This chromatography equations is widely used by chemists and engineers to study the separation of two chemical components in a fluid phase.

Ambrosio et al. [2] introduced the change of variables $w = u + v$ and $z = u - v$, then the system (1.3) can be written as

$$\begin{cases} \partial_t z + \partial_x(\frac{z}{1+w}) = 0, \\ \partial_t w + \partial_x(\frac{w}{1+w}) = 0. \end{cases} \tag{1.4}$$

They studied the system (1.4) as an example by using new well-posedness results for continuity and transport equations, so that exploited the transport equation techniques [1] heavily. Then, Sun [14] proved the existence and uniqueness of solutions involving the delta shock of (1.4) by employing the self-smilar viscosity vanishing method. Recently, Sun [15] has studied the interactions of delta shock waves for the system (1.4). In 1998, Bressan and Shen [4] adopted another change of variables $w = u + v$ and $\theta = v/u$, then the system (1.3) can be changed to

$$\begin{cases} w_t + \left(\dfrac{w}{1+w}\right)_x = 0, \\ \theta_t + \dfrac{1}{1+w}\theta_x = 0. \end{cases} \tag{1.5}$$

In that article their attentions were mainly drawn on the study of ODES with discontinuous vector fields.

The Riemann problem for system (1.1) was solved by Cheng and Yang completely in [6]. We find it is essential to study the interactions of elementary waves for (1.1) not only because of their significance in practical applications of the chromatography systems, such as comparison with the numerical and experimental results, separated the two chemical components in the chemical fields, etc., but also because of their basis for the general mathematical theory of the chromatography systems. In the present paper, we mainly study the interactions of the classical elementary waves with three piecewise constant initial data for system (1.1). In order to cover all the cases completely, the discussion should be divided into twelve cases. By analyzing the interactions of elementary waves case by case, we can prove that the solutions of the perturbed initial value problem converge to the corresponding Riemann solutions.

This paper is organized as follows. In Section 2, we present some preliminary knowledge for the system (1.1) and display the Riemann solutions of (1.1) with constant initial data. In Section 3, the interactions of all kinds of elementary waves are concerned, the global solutions are constructed and the stability of the Riemann solutions is analyzed case by case. Our conclusion is drawn in Section 4.

2. Preliminaries

In this section, we briefly review the Riemann solutions of (1.1) with initial data

$$(u(x,0),v(x,0))=(u_\pm,v_\pm),\ \pm x>0, \tag{2.1}$$

where $u_\pm>0$ and $v_\pm>0$, the detailed study of which can be found in [6].

It is seen that the nonlinear chromatography equations (1.1) have two eigenvalues

$$\lambda_1=1+\frac{1}{1-u+v},\ \lambda_2=1+\frac{1}{(1-u+v)^2}, \tag{2.2}$$

with corresponding right eigenvectors

$$r_1=(1,1)^T,\ r_2=(u,v)^T. \tag{2.3}$$

By simple calculation, we get $\nabla\lambda_1\cdot r_1{=}0$ and $\nabla\lambda_2\cdot r_2=2(u-v)/(1-u+v)^3$. So system (1.1) is nonstrictly hyperbolic. λ_1 is always linearly degenerate, λ_2 is genuinely nonlinear if $u\neq v$ and linearly degenerate if $u=v$. In this paper we will consider the case of $u\neq v$.

For a given left state (u_-,v_-), it is easy to check that the self-similar waves $(u,v)(\xi)$ ($\xi=x/t$) are the rarefaction wave curves that can be connected on the right as:

$$R(u_-,v_-):\quad \begin{cases}\dfrac{x}{t}=\lambda_2=1+\dfrac{1}{(1+u+v)^2},\\ \dfrac{u}{v}=\dfrac{u_-}{v_-},\quad -u+v<-u_-+v_-,\end{cases} \tag{2.4}$$

and the shock wave that can be connected on the right is

$$S(u_-,v_-):\quad \begin{cases}\dfrac{x}{t}=\sigma=1+\dfrac{1}{(1-u+v)(1-u_-+v_-)},\\ \dfrac{u}{v}=\dfrac{u_-}{v_-},\quad 0<-u_-+v_-<-u+v\quad or\quad -u_-+v_-<-u+v<0.\end{cases} \tag{2.5}$$

Since λ_1 is linearly degenerate, the sets of states which can be connected to a given left state (u_-,v_-) by a contact discontinuity on the right if and only if

$$J(u_-,v_-):\quad \begin{cases}\dfrac{x}{t}=1+\dfrac{1}{1-u+v}=1+\dfrac{1}{1-u_-+v_-},\\ -u+v=-u_-+v_-.\end{cases} \tag{2.6}$$

From (2.4)-(2.6), the solutions of (1.1) and (2.1) can be constructed by employing the method of phase plane analysis. The Riemann solutions contain a single classical wave when $-u_++v_+=-u_-+v_-$ or $u_+/v_+=u_-/v_-$. For the other cases, we can construct the solutions

except the delta-shock wave solution as follows:
(1) $S+J$, when $0<-u_-+v_-<-u_++v_+$; (2) $R+J$, when $0\le -u_++v_+<-u_-+v_-$;
(3) $R+R$, when $-u_++v_+<0<-u_-+v_-$; (4) $J+R$, when $-u_++v_+<-u_-+v_-\le 0$;
(5) $J+S$, when $-u_-+v_-<-u_++v_+<0$.

3. Interactions of elementary waves for the nonlinear chromatography equations

In this section, we consider the initial value problem (1.1) with three pieces constant initial data as follows:

$$(u,v)(x,t)=\begin{cases}(u_-,v_-), & -\infty<x<-\epsilon,\\ (u_m,v_m), & -\epsilon<x<\epsilon,\\ (u_+,v_+), & \epsilon<x<+\infty,\end{cases} \tag{3.1}$$

where $\epsilon>0$ is arbitrarily small. The data (3.1) is a small perturbation of the corresponding Riemann initial data (2.1). The interactions of elementary waves are analyzed and the global solutions are constructed here. Then we face the question of determining whether the solutions $(u_\epsilon,v_\epsilon)(x,t)$ of perturbation Riemann problem converge to the corresponding Riemann solutions as $\epsilon\to 0$.

In order to cover all the cases completely, we divide our discussion into twelve cases according to the different combinations of the Riemann solutions starting from $(-\varepsilon,0)$ and $(\varepsilon,0)$ as follows:
(1) $S+J$ and $R+R$; (2) $R+J$ and $R+R$; (3) $S+J$ and $S+J$; (4) $R+J$ and $S+J$; (5) $R+J$ and $R+J$; (6) $S+J$ and $R+J$; (7) $R+R$ and $J+R$; (8) $R+R$ and $J+S$; (9) $J+S$ and $J+R$; (10) $J+S$ and $J+S$; (11) $J+R$ and $J+S$; (12) $J+R$ and $J+R$.

Case 1: $S+J$ and $R+R$

In this case, when t is small enough and $-u_++v_+<0<-u_-+v_-<-u_m+v_m$, the solution of the initial value problem (1.1)-(3.1) can be expressed briefly as follows (see Fig. 3.1):

$$(u_-,v_-)+S_1+(u_1,v_1)+J_1+(u_m,v_m)+R_1+R_2+(u_+,v_+),$$

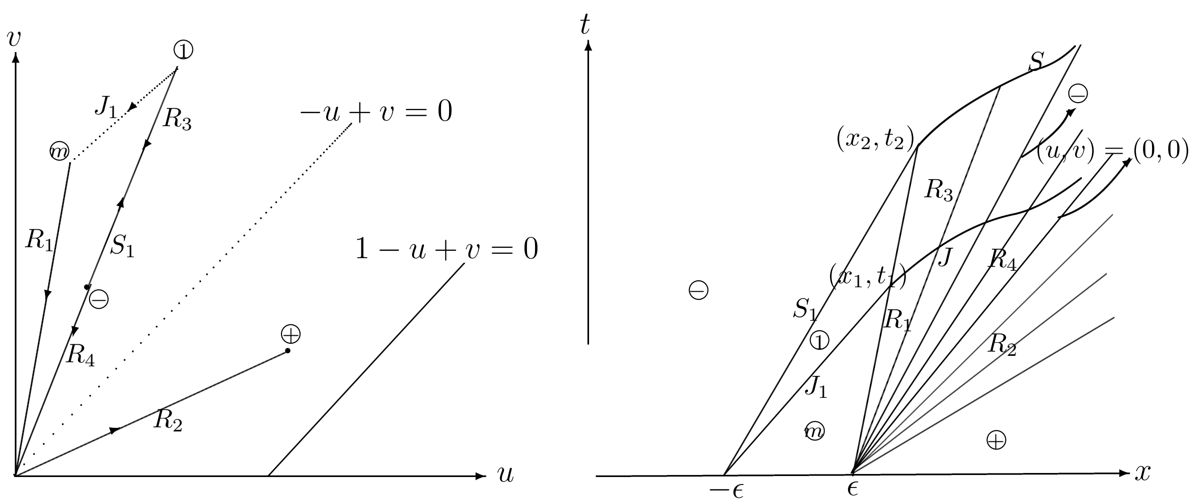

Fig. 3.1 $-u_++v_+<0<-u_-+v_-<-u_m+v_m$

where "+" means "followed by". The propagation speed of J_1 and that of the wave back in the rarefaction wave R_1 are $\tau_1 = 1 + 1/(1 - u_m + v_m)$ and $\xi = 1 + 1/(1 - u_m + v_m)^2$ respectively. It is easy to see $\tau_1 > \xi$ which means J_1 will overtake R_1 at a finite time t_1. The intersection point (x_1, t_1) is determined by

$$\begin{cases} x_1 + \epsilon = \left(1 + \dfrac{1}{1 - u_m + v_m}\right) t_1, \\ x_1 - \epsilon = \left(1 + \dfrac{1}{(1 - u_m + v_m)^2}\right) t_1, \end{cases} \tag{3.2}$$

which leads to

$$(x_1, t_1) = \left(\frac{2\epsilon(2 - u_m + v_m)(1 - u_m + v_m)}{-u_m + v_m} - \epsilon, \frac{2\epsilon(1 - u_m + v_m)^2}{-u_m + v_m} \right). \tag{3.3}$$

After interaction of J_1 and R_1, a new rarefaction wave R_3 and a new contact discontinuity J will appear. Meanwhile, the direction of R_1 is unchanged and J_1 will cross the rarefaction wave R_1 with a varying speed of propagation during the penetration, that is, the contact discontinuity $J : x = x(t)$ is no longer a straight line when $t > t_1$. This process is determined by

$$\begin{cases} \dfrac{dx}{dt} = 1 + \dfrac{1}{1 - u + v}, \\ x - \epsilon = \left(1 + \dfrac{1}{(1 - u + v)^2}\right) t, \\ \dfrac{u}{v} = \dfrac{u_m}{v_m}, \quad 0 \le -u + v < -u_m + v_m, \\ x(t_1) = x_1. \end{cases} \tag{3.4}$$

Differentiating $(3.4)_1$ and $(3.4)_2$ with respect to t leads to

$$\frac{d^2x}{dt^2} = -\frac{1}{(1 - u + v)^2} \left(-\frac{du}{dt} + \frac{dv}{dt} \right), \tag{3.5}$$

$$\frac{dx}{dt} = 1 + \frac{1}{(1 - u + v)^2} - \frac{2t}{(1 - u + v)^3} \left(-\frac{du}{dt} + \frac{dv}{dt} \right). \tag{3.6}$$

Combine $(3.4)_1$ with (3.6), we have

$$-\frac{du}{dt} + \frac{dv}{dt} = -\frac{(-u + v)(1 - u + v)}{2t} < 0. \tag{3.7}$$

By (3.5) and (3.7), it is easy to see that $d^2x/dt^2 > 0$, which means that the propagation speed of the contact discontinuity J_1 will increase during the process of passing through R_1. We also get that the speed of J is $\tau = 2$ as $(u, v) \to (0, 0)$ and that of the wave front of R_1 is $\xi = 2$, it is illustrated that the contact discontinuity $J : x = x(t)$ can not cross the whole of R_1 completely, which means that $x = x(t)$ does not intersect with the characteristic $x - \epsilon = 2t$.

When $t > t_1$, we notice that the shock wave S_1 and the rarefaction wave R_3 will interact, due to the propagation speed of S_1 is greater than the wave back in the rarefaction wave R_3.

The intersection point (x_2, t_2) can be determined by

$$\begin{cases} x_2+\epsilon=\left(1+\dfrac{1}{(1-u_-+v_-)(1-u_m+v_m)}\right)t_2, \\ x_2-\epsilon=\left(1+\dfrac{1}{(1-u_m+v_m)^2}\right)t_2, \end{cases} \tag{3.8}$$

which means that

$$(x_2,t_2)=\left(\frac{2\epsilon(1-u_-+v_-)\left((1-u_m+v_m)^2+1\right)}{v_m-u_m+u_--v_-}+\epsilon,\ \frac{2\epsilon(1-u_-+v_-)(1-u_m+v_m)^2}{-u_m+v_m+u_--v_-}\right). \tag{3.9}$$

When $t>t_2$, the sets of states can be connected to a given left state (u_-, v_-) by a shock wave S with the method of phase plane analysis and it is no longer a straight line. The varying speed of S can be determined by

$$\begin{cases} \dfrac{dx}{dt}=1+\dfrac{1}{(1-u_-+v_-)(1-u+v)}, \\ x-\epsilon=\left(1+\dfrac{1}{(1-u+v)^2}\right)t, \\ \dfrac{u}{v}=\dfrac{u_1}{v_1}, \quad 0<-u+v<-u_1+v_1, \\ x(t_2)=x_2. \end{cases} \tag{3.10}$$

By $(3.10)_1$ and $(3.10)_2$, we obtain

$$\frac{d^2x}{dt^2}=-\frac{1}{(1-u_-+v_-)(1-u+v)^2}\left(-\frac{du}{dt}+\frac{dv}{dt}\right), \tag{3.11}$$

$$-\frac{du}{dt}+\frac{dv}{dt}=-\frac{\left((-u+v)-(-u_-+v_-)\right)(1-u+v)}{2t(1-u_-+v_-)}. \tag{3.12}$$

Due to $-u+v\geq -u_-+v_->0$, from (3.11) and (3.12), we have

$$-\frac{du}{dt}+\frac{dv}{dt}<0, \quad \frac{d^2x}{dt^2}>0, \tag{3.13}$$

which means that the shock wave S accelerates and passes through R_3. Differentiating $(3.10)_3$ of variable t, we have

$$\frac{du}{dt}=\frac{u_1}{v_1}\frac{dv}{dt}. \tag{3.14}$$

Substituting (3.14) into (3.12), we get

$$\frac{1}{t}\,dt=-\frac{2(1-u_-+v_-)(v_1-u_1)}{((-u+v)-(-u_-+v_-))(1-u+v)v_1}\,dv. \tag{3.15}$$

Integrating (3.15) from t_2 to t, we obtain

$$\ln\frac{t}{t_2}=\int_{v_1}^{v}-\frac{2(1-u_-+v_-)(v_1-u_1)}{((-u+v)-(-u_-+v_-))(1-u+v)v_1}\,dv. \tag{3.16}$$

It is obvious that $t\to\infty$ as $-u+v\to -u_-+v_-$. Due to $0<-u+v<-u_1+v_1$, it is impossible for the shock wave $S: x=x(t)$ to cross the whole of R_3 completely. Moreover, it can be shown that $x=x(t)$ does not intersect with characteristic line $x-\epsilon=(1+1/(1-u_-+v_-)^2)t$.

When $t \to \infty$, the final solution can be shown as (see Fig. 3.1)

$$(u_-, v_-) + R_4 + R_2 + (u_+, v_+).$$

It is easy to see that (x_1, t_1) and (x_2, t_2) tend to $(0, 0)$ as $\epsilon \to 0$ from (3.3) and (3.9). Thus, the limit of the solution of (1.1) and (3.1) is still a backward rarefaction wave plus a forward rarefaction wave, which is exactly the corresponding Riemann solution of (1.1) and (2.1) in this case.

Remark 1. The situation is similar to the case $R + R$ and $J + S$. The occurrence of this case depends on the situation $-u_m + v_m < -u_+ + v_+ < 0 < -u_- + v_-$.

Case 2: $R + R$ and $J + R$

In this case, when t is small enough and $-u_+ + v_+ < -u_m + v_m < 0 < -u_- + v_-$, the solution of the initial value problem (1.1)-(3.1) can be expressed briefly as follows (see Fig. 3.2):

$$(u_-, v_-) + R_1 + R_2 + (u_m, v_m) + J_1 + (u_1, v_1) + R_3 + (u_+, v_+).$$

The propagation speed of J_1 and that of the wave front in the rarefaction wave R_2 are $\tau_1 = 1 + 1/(1 - u_m + v_m)$ and $\xi = 1 + 1/(1 - u_m + v_m)^2$ respectively. By $-1 < -u_m + v_m < 0$, it is easy to see that $\xi > \tau_1$ which means R_2 will interact with J_1 at a finite time t_1.

The intersection (x_1, t_1) is determined by

$$\begin{cases} x_1 - \epsilon = \left(1 + \dfrac{1}{1 - u_m + v_m}\right) t_1, \\ x_1 + \epsilon = \left(1 + \dfrac{1}{(1 - u_m + v_m)^2}\right) t_1, \end{cases} \tag{3.17}$$

which yields

$$(x_1, t_1) = \left(\frac{2\epsilon(2 - u_m + v_m)(1 - u_m + v_m)}{-u_m + v_m} + \epsilon, \frac{2\epsilon(1 - u_m + v_m)^2}{-u_m + v_m} \right). \tag{3.18}$$

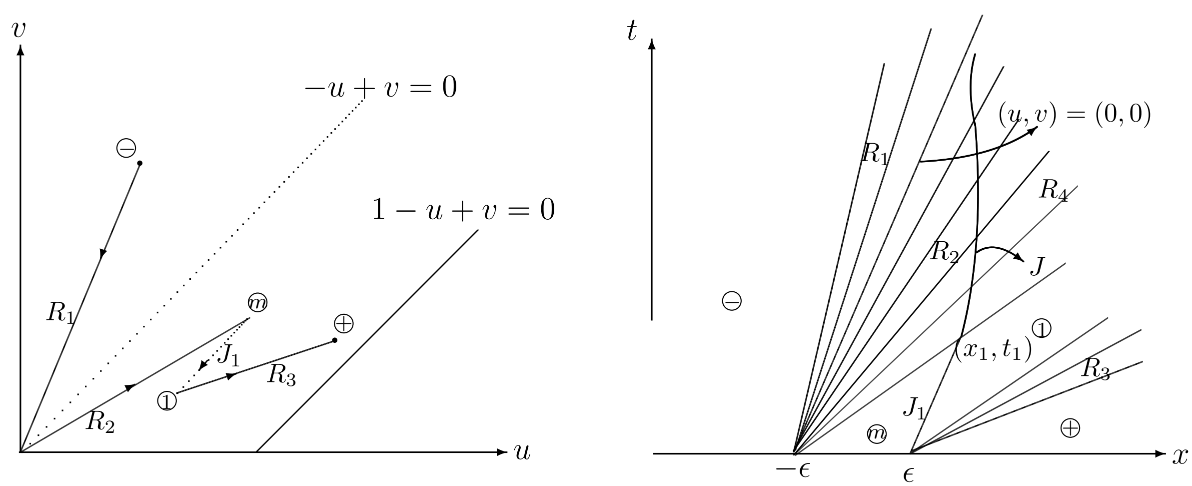

Fig. 3.2 $\quad -u_+ + v_+ < -u_m + v_m < 0 < -u_- + v_-$

Therefore, a new Riemann problem is formed at $t = t_1$, the interaction of R_2 and J_1 gives rise to a new contact discontinuity J and a new rarefaction wave R_4. Meanwhile, the propagation direction of R_2 is unchanged during the process of penetration.

The expression of the contact discontinuity $J : x = x(t)$ passing through R_2 is similar to (3.4). From (3.5)-(3.7) and $-1 < -u_m + v_m < -u + v \leq 0$ (see Fig. 3.2). It is obtained that $-du/dt + dv/dt = \left(-(-u+v)(1-u+v)\right)/2t \geq 0$, $d^2x/dt^2 \leq 0$. It illustrates that the contact discontinuity J decelerates and passes through R_2. The speed of J is $\tau = 2$ when $(u, v) \to 0$ and that of the wave back in the rarefaction wave R_2 is $\xi = 2$. It shows that contact discontinuity J cannot penetrate the whole rarefaction wave R_2 completely and ultimately has $x + \epsilon = 2t$ as its asymptote (see Fig. 3.2). The propagation speed of the wave front in the rarefaction wave R_4 is equivalent to that of the wave back in the rarefaction wave R_3. So R_4 will no longer interact with R_3 .

When $t \to \infty$, the solution can be expressed as (see Fig. 3.2)

$$(u_-, v_-) + R_1 + R_4 + (u_1, v_1) + R_3 + (u_+, v_+).$$

It is easy to obtain that (x_1, t_1) tend to (0, 0) as $\epsilon \to 0$ from (3.18). Thus, the solution of (1.1) and (3.1) is apparently converges to the solution of the Riemann initial value problem (1.1) and (2.1).

Remark 2. The situation is similar to the case $R + J$ and $R + R$. The occurrence of this case depends on the situation $-u_+ + v_+ < 0 < -u_m + v_m < -u_- + v_-$.

Case 3: $J + S$ and $J + R$

In this case, when t is small enough and $-u_- + v_- < -u_+ + v_+ < -u_m + v_m < 0$ or $-u_+ + v_+ < -u_- + v_- < -u_m + v_m < 0$, the solution of the initial value problem (1.1)-(3.1) can be expressed briefly as follows (see Fig. 3.3):

$$(u_-, v_-) + J_1 + (u_1, v_1) + S_1 + (u_m, v_m) + J_2 + (u_2, v_2) + R_1 + (u_+, v_+).$$

The propagation speeds of S_1 and J_2 are $\sigma_1 = 1 + 1/((1 - u_- + v_-)(1 - u_m + v_m))$ and $\tau_2 = 1 + 1/(1 - u_m + v_m)$ respectively. Due to $-1 < -u_- + v_- < 0$, we obtain $\sigma_1 > \tau_2$. It is illustrated that the shock wave S_1 will interact with J_2 at the point (x_1, t_1). The intersection (x_1, t_1) is given by

$$\begin{cases} x_1 + \epsilon = \left(1 + \dfrac{1}{(1 - u_- + v_-)(1 - u_m + v_m)}\right) t_1, \\ x_1 - \epsilon = \left(1 + \dfrac{1}{1 - u_m + v_m}\right) t_1, \end{cases} \tag{3.19}$$

which leads to

$$(x_1, t_1) = \left(\frac{2\epsilon(2 - u_m + v_m)(1 - u_- + v_-)}{u_- - v_-} + \epsilon, \frac{2\epsilon(1 - u_- + v_-)(1 - u_m + v_m)}{u_- - v_-} \right). \tag{3.20}$$

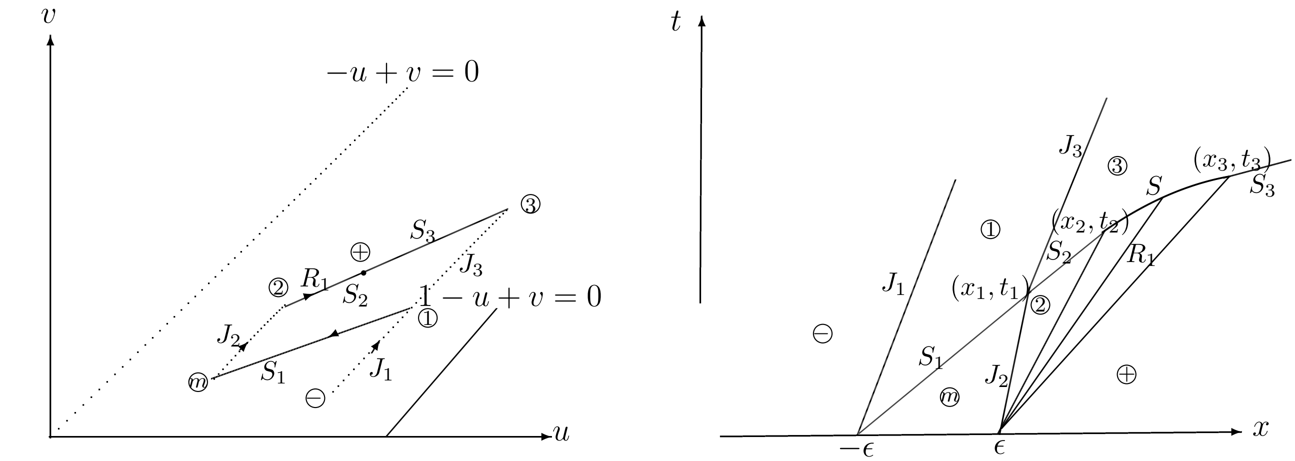

Fig. 3.3(a) $\quad -u_- + v_- < -u_+ + v_+ < -u_m + v_m < 0$

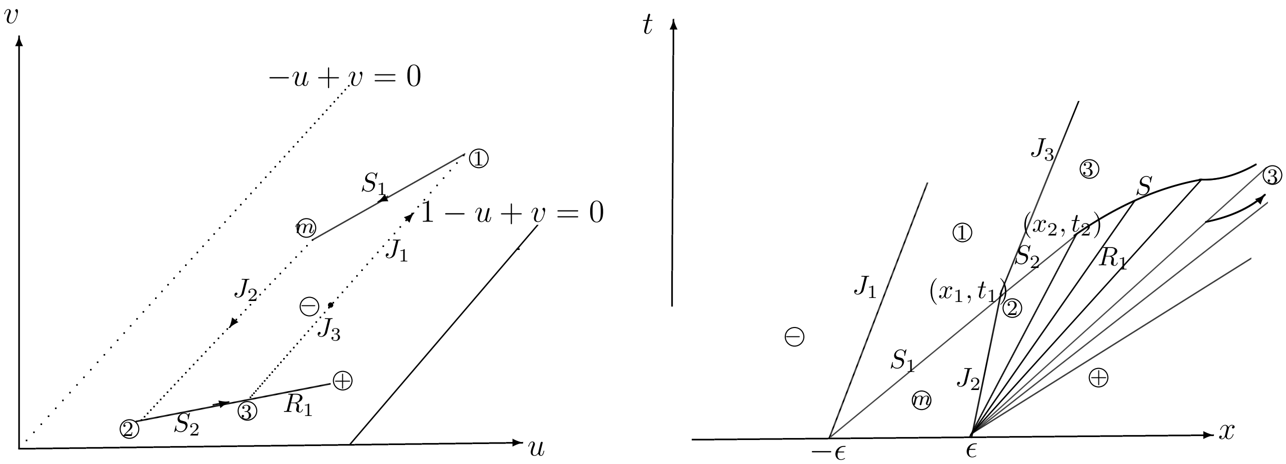

Fig. 3.3(b) $\quad -u_+ + v_+ < -u_- + v_- < -u_m + v_m < 0$

After the interaction of S_1 and J_2, a new contact discontinuity J_3 and a new shock S_2 will generate. The propagation speed of the shock wave S_2 is equivalent to that of the shock wave S_1.

We also get that the propagation speeds of J_1 and J_3 are $\tau_1 = 1 + 1/(1 - u_- + v_-)$ and $\tau_3 = 1 + 1/(1 - u_- + v_-)$ respectively, which means that J_3 is parallel to J_1. By $-1 < -u_- + v_- < -u_m + v_m < 0$, it is easy to see that the propagation speed of J_3 is greater than that of J_2. The propagation speed of the wave back in the rarefaction wave R_1 is $\xi = 1 + 1/(1 - u_m + v_m)^2$. It is obvious that S_2 will overtake R_1 at the point (x_2, t_2), which is determined by

$$\begin{cases} x_2 + \epsilon = \left(1 + \dfrac{1}{(1 - u_- + v_-)(1 - u_m + v_m)}\right) t_2, \\ x_2 - \epsilon = \left(1 + \dfrac{1}{(1 - u_m + v_m)^2}\right) t_2, \end{cases} \tag{3.21}$$

it means that

$$(x_2, t_2) = \left(\frac{2\epsilon(1-u_-+v_-)\left((1-u_m+v_m)^2+1\right)}{v_m-u_m+u_--v_-}+\epsilon, \frac{2\epsilon(1-u_-+v_-)(1-u_m+v_m)^2}{-u_m+v_m+u_--v_-} \right). \tag{3.22}$$

When $t > t_2$, the sets of states can be connected to a given left state (u_3, v_3) by a shock wave S with the method of phase plane analysis and it is no longer a straight line (see Fig. 3.3). The expression of the shock $S : x = x(t)$ during the penetration of R_1 is similar to (3.10). So we also obtain the results of (3.11)-(3.13), which means that the propagation speed of the shock wave increases during the process of S_2 passing through R_1. In order to analyze whether the shock wave S will penetrate the whole rarefaction wave R_1 completely or not, the corresponding discussion should be divided into the following two subcases.

If $-u_-+v_- < -u_++v_+ < -u_m+v_m < 0$, the shock wave S: $x = x(t)$ will cross the whole of R_1 completely at a finite time t_3 (see Fig. 3.3(a)), a new shock S_3 will appear, which is determined by

$$t_3 = t_2 exp\left(\int_{v_2}^{v_+} \frac{2(1-u_3+v_3)(v_+-u_+)}{-((-u+v)-(-u_3+v_3))(1-u+v)v_+}dv\right).$$

When $t > t_3$, the solution can be expressed as (see Fig 3.3(a))

$$(u_-, v_-) + J_1 + (u_1, v_1) + J_3 + (u_3, v_3) + S_3 + (u_+, v_+).$$

If $-u_+ + v_+ < -u_- + v_- < -u_m + v_m < 0$, it is impossible for the shock wave S to cross the whole rarefaction wave R_1 completely, it ultimately has $x - \epsilon = (1 + 1/(1 - u_3 + v_3)^2)\, t$ as its asymptote. And the solution can be expressed as

$$(u_-, v_-) + J_1 + (u_1, v_1) + J_3 + (u_3, v_3) + R + (u_+, v_+).$$

It is easy to see that (x_1, t_1) and (x_2, t_2) tend to (0, 0) as $\epsilon \to 0$ from (3.20) and (3.22). Thus, the limit of (1.1) and (3.1) is $J + S$ for $-u_- + v_- < -u_+ + v_+ < -u_m + v_m < 0$ or $J + R$ for $-u_+ + v_+ < -u_- + v_- < -u_m + v_m < 0$, which is exactly the corresponding Riemann solution of (1.1) and (2.1) in this case.

Remark 3. The situation is similar to the case $R + J$ and $S + J$. The occurrence of this case depends on the situation $0 < -u_m + v_m < -u_- + v_- < -u_+ + v_+$ or $0 < -u_m + v_m < -u_+ + v_+ < -u_- + v_-$.

Case 4: $S + J$ and $S + J$

When t is small enough and $0 < -u_- + v_- < -u_m + v_m < -u_+ + v_+$, the solution of the initial value problem (1.1)-(3.1) can be expressed briefly as follows (see Fig. 3.4):

$$(u_-, v_-) + S_1 + (u_1, v_1) + J_1 + (u_m, v_m) + S_2 + (u_2, v_2) + J_2 + (u_+, v_+).$$

The propagation speeds of J_1 and S_2 are

$$\tau_1 = 1 + 1/(1 - u_m + v_m), \quad \sigma_2 = 1 + 1/((1 - u_m + v_m)(1 - u_+ + v_+))$$

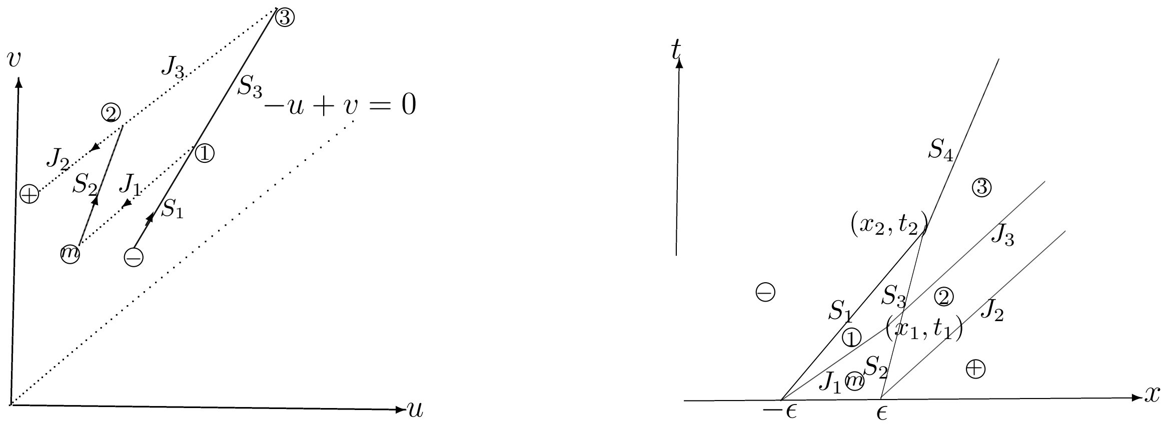

Fig. 3.4 $0 < -u_- + v_- < -u_m + v_m < -u_+ + v_+$

respectively, by $-u_+ + v_+ > 0$ we obtain $\tau_1 > \sigma_2$, which means J_1 will catch up with S_2 in finite time. The intersection (x_1, t_1) is determined by

$$\begin{cases} x_1 + \epsilon = \left(1 + \dfrac{1}{1 - u_m + v_m}\right) t_1, \\ x_1 - \epsilon = \left(1 + \dfrac{1}{(1 - u_m + v_m)(1 - u_+ + v_+)}\right) t_1, \end{cases} \tag{3.23}$$

which means that

$$(x_1, t_1) = \left(\frac{2\epsilon(1 - u_+ + v_+)(2 - u_m + v_m)}{-u_+ + v_+} - \epsilon, \frac{2\epsilon(1 - u_+ + v_+)(1 - u_m + v_m)}{-u_+ + v_+}\right). \tag{3.24}$$

After interaction, a new shock S_3 and a new contact discontinuity J_3 will appear. It is obvious that J_3 is parallel to J_2. The propagation speeds of S_1 and S_3 are $\sigma_1 = 1 + 1/((1 - u_m + v_m)(1 - u_- + v_-))$ and $\sigma_3 = 1 + 1/((1 - u_m + v_m)(1 - u_+ + v_+))$ respectively. Due to $-u_+ + v_+ > -u_- + v_- > 0$, we are easy to get $\sigma_1 > \sigma_3$ which means S_1 will overtake S_3 in finite time. At the point $t = t_2$ a new shock wave S_4 will appear. The intersection (x_2, t_2) is determined by

$$\begin{cases} x_2 + \epsilon = \left(1 + \dfrac{1}{(1 - u_m + v_m)(1 - u_- + v_-)}\right) t_2, \\ x_2 - \epsilon = \left(1 + \dfrac{1}{(1 - u_m + v_m)(1 - u_+ + v_+)}\right) t_2, \end{cases} \tag{3.25}$$

which yields

$$\begin{cases} x_2 = \dfrac{2\epsilon(1 - u_- + v_-)\left((1 - u_m + v_m)(1 - u_+ + v_+) + 1\right)}{-u_+ + v_+ + u_- - v_-} + \epsilon, \\ t_2 = \dfrac{2\epsilon(1 - u_- + v_-)(1 - u_m + v_m)(1 - u_+ + v_+)}{-u_+ + v_+ + u_- - v_-}. \end{cases} \tag{3.26}$$

When $t > t_2$, the propagating speed of the shock wave S_4 is

$$\sigma_4 = 1 + 1/((1 - u_- + v_-)(1 - u_+ + v_+))$$

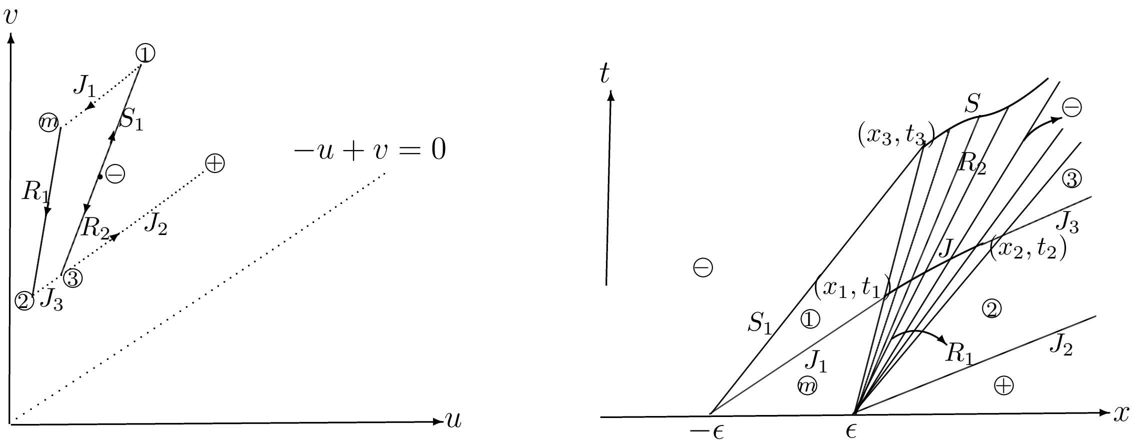

Fig 3.5(a) $\quad 0 < -u_+ + v_+ < -u_- + v_- < -u_m + v_m$

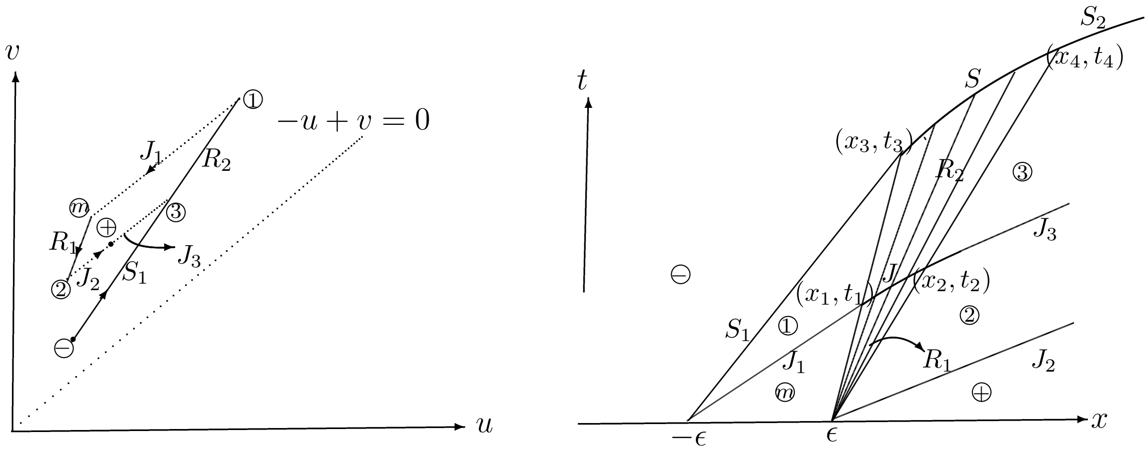

Fig 3.5(b) $\quad 0 < -u_- + v_- < -u_+ + v_+ < -u_m + v_m$

and satisfies $\sigma_4 < \tau_3$, which implies S_4 and J_3 will not interact forever. So, when $t > t_2$, the solution can be expressed as (see Fig. 3.4)

$$(u_-, v_-) + S_4 + (u_3, v_3) + J_3 + (u_2, v_2) + J_2 + (u_+, v_+).$$

Letting $\epsilon \to 0$, the limit of the solution of (1.1) and (3.1) is exactly identical with the Riemann solution of (1.1) and (2.1) in this case.

Remark 4. The situation is similar to the case $J + S$ and $J + S$. The occurrence of this case depends on the situation $-u_- + v_- < -u_m + v_m < -u_+ + v_+ < 0$.

Case 5: $S + J$ and $R + J$

In this case, when t is small enough, $0 < -u_+ + v_+ < -u_- + v_- < -u_m + v_m$ or $0 < -u_- + v_- < -u_+ + v_+ < -u_m + v_m$, the solution of the initial value problem (1.1)-(3.1) can be presented briefly as follows (see Fig. 3.5):

$$(u_-, v_-) + S_1 + (u_1, v_1) + J_1 + (u_m, v_m) + R_1 + (u_2, v_2) + J_2 + (u_+, v_+).$$

The propagation speed of J_1 and that of the wave back in the rarefaction wave R_1 are $\tau_1 = 1 + 1/(1 - u_m + v_m)$ and $\xi = 1 + 1/(1 - u_m + v_m)^2$ respectively. It is easy to see that

$\tau_1 > \xi$, which means J_1 will catch up with R_1 at a finite time t_1. The intersection point (x_1, t_1) can be obtained similar as (3.3), we omit it.

After interaction of J_1 and R_1, a new rarefaction wave R_2 and a new contact discontinuity J_3 will appear. We also have J_3 is parallel to J_2 and the direction of R_1 is unchanged during the process of penetration.

Besides, the contact discontinuity J_1 crosses the rarefaction wave R_1 with a varying speed of propagation, the analysis is same as (3.4)-(3.7). Due to $-u_2 + v_2 \le -u + v \le -u_m + v_m$, we have $-du/dt + dv/dt = (-(-u+v)(1-u+v))/2t < 0$ and $d^2x/dt^2 > 0$, which means that the propagation speed of the contact discontinuity J will increase during the process of passing through R_1.

From $(3.4)_1$ and $(3.4)_2$, we obtain

$$\frac{dx}{dt} = \sqrt{\frac{x-\epsilon}{t} - 1} + 1. \tag{3.27}$$

By combining (3.3) with (3.27), the curve of contact discontinuity J is determined by

$$x = (\sqrt{t} - \sqrt{2\epsilon(-u_m + v_m)})^2 + t + \epsilon. \tag{3.28}$$

It is illustrated that the contact discontinuity J will penetrate the whole of the rarefaction wave R_1 completely and the ending point can be calculated by

$$\begin{cases} x_2 = \left(\sqrt{t_2} - \sqrt{2\epsilon(-u_m + v_m)}\right)^2 + t_2 + \epsilon, \\ x_2 - \epsilon = \left(1 + \dfrac{1}{(1 - u_+ + v_+)^2}\right) t_2. \end{cases} \tag{3.29}$$

After the time t_2, we notice that the shock wave S_1 and the rarefaction wave R_2 will interact, due to the propagation speed of S_1 is greater than the wave back in the rarefaction wave R_2. The intersection point (x_3, t_3) is same as the (3.8) and (3.9).

When $t > t_3$, the sets of states can be connected to a given left state (u_-, v_-) by a shock wave S with the method of phase plane analysis and it is no longer a straight line. The process of S penetrating R_2 is similar to (3.10), we also have the results of (3.11)-(3.13), which means that the shock wave S will accelerate during the process of passing through R_2. In order to analyze whether the shock wave S can penetrate the whole rarefaction wave R_2 or not, our discussion should be divided into the following two subcases.

If $0 < -u_+ + v_+ < -u_- + v_- < -u_m + v_m$, it is impossible for the shock wave $S: x = x(t)$ to cross the whole rarefaction wave R_2 completely, the analysis is same as case 1. From (3.14)-(3.15), combining with the initial value (x_3, t_3), it is easy to have

$$\ln \frac{t}{t_3} = \int_{v_1}^{v} -\frac{2(1 - u_- + v_-)(v_3 - u_3)}{((-u+v) - (-u_- + v_-))(1 - u + v)v_3}\, dv. \tag{3.30}$$

It is obvious that $t \to \infty$ as $-u + v \to -u_- + v_-$. Due to $-u_3 + v_3 < -u + v < -u_1 + v_1$, it is impossible for the shock wave $x = x(t)$ to cross the whole of R_2 completely . Moreover, it can be shown that the shock wave $x = x(t)$ does not intersect with characteristic line $x - \epsilon = (1 + 1/(1 - u_- + v_-)^2)\, t$ (see Fig. 3.5(a)).

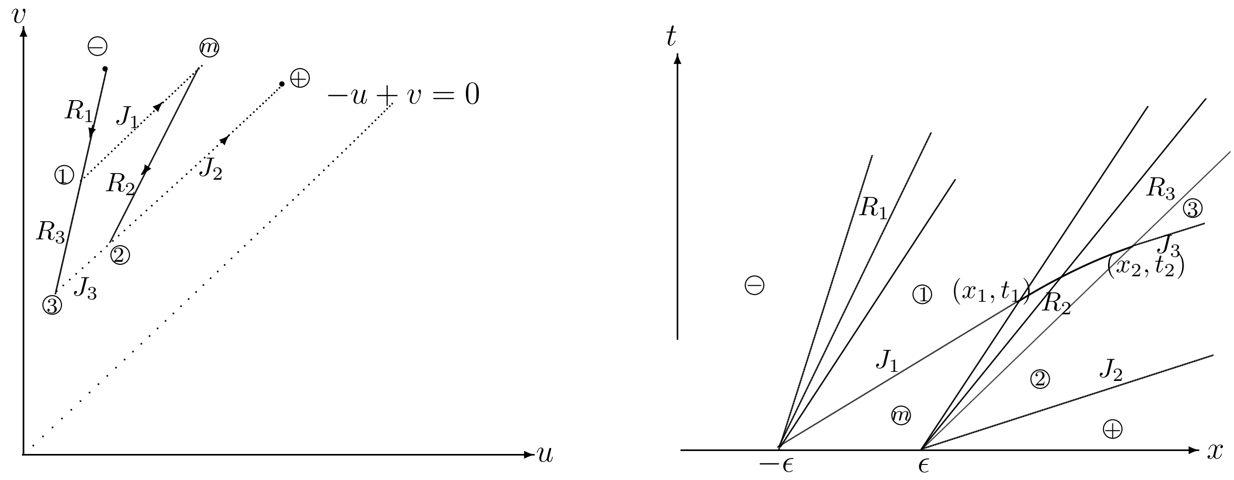

Fig. 3.6 $0 < -u_+ + v_+ < -u_m + v_m < -u_- + v_-$

When $t \to \infty$, the solution can be shown as

$$(u_-, v_-) + R + (u_3, v_3) + J_3 + (u_2, v_2) + J_2 + (u_+, v_+).$$

If $0 < -u_- + v_- < -u_+ + v_+ < -u_m + v_m$, the shock wave $S : x = x(t)$ will cross the whole of R_2 completely at a finite time t_4, due to $0 < -u_- + v_- < -u_3 + v_3 \leq -u + v < -u_1 + v_1$ (see Fig 3.5(b)). A new shock wave S_2 will occur at the time of $t = t_4$, and it is easy to obtain that $\sigma_2 < \tau_3$ which means that S_2 will not interact with J_3 forever.

From (3.30) we can calculate that

$$t_4 = t_3 exp\left(\int_{v_1}^{v_3} \frac{2(1 - u_- + v_-)(v_3 - u_3)}{-((-u + v) - (-u_- + v_-))(1 - u + v)v_3} dv\right).$$

When $t > t_4$, the solution can be presented as (see Fig 3.5(b))

$$(u_-, v_-) + S_2 + (u_3, v_3) + J_3 + (u_2, v_2) + J_2 + (u_+, v_+).$$

Letting $\epsilon \to 0$, we can see that the solution is obviously like as our assertion.

Remark 5. The situation is similar to the case $J + R$ and $J + S$. The occurrence of this case depends on the situation $-u_m + v_m < -u_- + v_- < -u_+ + v_+ < 0$ or $-u_m + v_m < -u_+ + v_+ < -u_- + v_- < 0$.

Case 6: $R + J$ and $R + J$

In this case, when t is small enough and $0 < -u_+ + v_+ < -u_m + v_m < -u_- + v_-$, the solution of the initial value problem (1.1) and (3.1) can be shown briefly as (see Fig. 3.6):

$$(u_-, v_-) + R_1 + (u_1, v_1) + J_1 + (u_m, v_m) + R_2 + (u_2, v_2) + J_2 + (u_+, v_+).$$

The interaction of J_1 and R_2 is similar as case 5 (see Fig. 3.5 J_1 and R_1). J_1 accelerates during the process of penetration and the propagation direction of R_2 is unchanged. A new rarefaction wave R_3 and a new contact discontinuity J_3 will appear, we also get J_3 is parallel to J_2. In addition, it is obvious that the wave front in R_1 and the wave back in R_3 have the same propagation speed $\xi = 1 + 1/(1 - u_m + v_m)^2$, which means that R_1 will not interact with R_3 forever.

As $\epsilon \to 0$, J_2 and J_3 will coincide with each other and the two rarefaction waves R_1 and R_3 will coalesce into one. So the limit situation is also a rarefaction wave plus a contact discontinuity which is corresponding to the Riemann solution of (1.1) and (2.1).

Remark 6. The situation is similar to the case $J + R$ and $J + R$. The occurrence of this case depends on the situation $-u_+ + v_+ < -u_m + v_m < -u_- + v_- < 0$.

4. Conclusions

So far, we have finished the discussion for all kinds of interactions of elementary waves. The global solutions for the perturbed initial value problem (1.1) and (3.1) have been constructed. We also notice that the propagation directions of the shock wave and the rarefaction wave are unchanged when they interact with the contact discontinuity. In addition, it is easy to see that the limits of the perturbed Riemann solutions are exactly the corresponding Riemann solutions of (1.1) and (2.1) by making the limit $\epsilon \to 0$ and the asymptotic behavior of the perturbed Riemann solutions is governed completely by the states $(u_\pm, v_\pm)$.

References

1. L. Ambrosio, *Transpot equation and Cauchy problem for BV vector fields*, Invent. Math., **158**(2004), pp: 227-260.
2. L. Ambrosio, G. Crippa, A. Figalli, L. A. Spinolo, *Some new well-posedness results for continuity and transport equations and applications to the chromatography system*, SIAM J. Math. Anal., **41**(2009), pp: 1890-1920.
3. A. Bressan, P. Goatin, *Stability of L^∞ solutions of Temple class systems*, Differential Integral Equations, **13**(2000), pp: 1503-1528.
4. A. Bressan, W. Shen, *Uniqueness of discontinuous ODE and conservation laws*, Nonlinear Anal., **34**(1998), pp: 637-652.
5. P. Barti, A. Bressan, *The semigroup generated by a Temple class system with large data*, Differential Integral Equations, **10**(1997), pp: 401-418.
6. H. Cheng, H. Yang, *Delta shock waves in chromatogrphy equations*, J. Math. Anal. Appl., **380**(2011), pp: 475-485.
7. G. Guiochon, *Preparative liquid chromatography*, J. Chromatogr. A., **965**(2002), pp: 129-161.
8. M. Juza, M. Mazzotti, M. Morbidelli, *Simulated moving-bed chromatography and its application to chirotechnology*, Trends in Biotechnology, **18**(2000), pp: 108-118.
9. T. P. Liu, T. Yang, *L^1 stability of conservation laws with coinciding Hugoniot and characteristic curves*, Indiana Univ. Math. J., **48**(1999), pp: 237-247.
10. M. Mazzotti, *Nonclassical composition fronts in nonlinear chromatography: delta-shock* , Ind. Eng. Chem. Res., **48**(2009), pp: 7733-7752.
11. M. Mazzotti, A. Tarafder, J. Cornel, F. Gritti, G. Guiochon, *Experimental evidence of a delta-shock in nonlinear chromatography*, J. Chromatogr. A., **1217**(2010), pp: 2002-2012.
12. A. Rajendran, G. Paredes, M. Mazzotti, *Simulated moving bed chromatography for the separation of enantiomers*, J. Chromatogr. A., **1216**(2009), pp: 709-738.
13. C. Shen, *Wave interactions and stability of the Riemann solutions for the chromatography equations*, J. Math. Anal. Appl., **365**(2010), pp: 609-618.
14. M. Sun, *Delta shock waves for the chromatography equations as self-similar viscosity limits*, Q. Appl. Math., **69**(2011), pp: 425-443.
15. M. Sun, *Interactions of delta shock waves for the chromatography equations*, Applied Mathematics Letters, **26**(2013), pp: 631-637.

16. B. Temple, *Systems of conservations laws with invariant submanifolds*, Trans. Amer. Math. Soc., **280**(1983), pp: 781-795.
17. B. Temple, *Global solution of the Cauchy problem for a class of 2×2 nonstrictly hyperbolic conservation laws*, Adv. in Appl. Math., **3**(1982), pp: 335-375.