

HARMONIC K-QUASICONFORMAL MAPPINGS FROM UNIT DISK ONTO HALF PLANES

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ABSTRACT. We consider the analytic properties for sense preserving univalent harmonic mappings on the unit disk D onto half planes $\Omega = \{w : \operatorname{Re} w > a, -\infty < a < a_0 < +\infty\}$ normalized at the origin by $f(0) = a_0, f_z(0) > 0, f_{\bar{z}}(0) \in R$. By deriving an analytic representing formula for these univalent harmonic mappings, we obtain one necessary and sufficient condition for these sense preserving univalent harmonic mappings to be harmonic K-quasiconformal mappings, we also obtain sharp coefficient estimates for these harmonic K-quasiconformal mappings. As an application, distortion theorems on length and area for their images are also obtained. Our results improve and generalize the one made by M. Öztürk.

1. INTRODUCTION

For a given complex-valued function $f(z)$ defined on a plane domain Ω is called harmonic if $f(z)$ is a twice continuously differentiable and satisfies $\Delta f = 4f_{z\bar{z}} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. The Jacobian of $f(z)$ is given by $J_f = |f_z|^2 - |f_{\bar{z}}|^2$. By Lewy's result [1], $f(z)$ is locally univalent and sense preserving on Ω if and only if its Jacobian $J_f > 0$. We known that any harmonic mapping $f(z)$ defined on simply connected domain Ω can be written as $f(z) = h(z) + g(z)$, where $h(z)$ and $g(z)$ are holomorphic on Ω . Thus, $f(z) = h(z) + \overline{g(z)}$ is a locally univalent and orientation-preserving harmonic function on Ω if and only if $|g'| < |h'|, z \in \Omega$. If univalent harmonic mapping $f(z) = h(z) + \overline{g(z)}$ satisfies $|\frac{g'(z)}{h'(z)}| \leq k < 1, z \in \Omega$, then $f(z)$ is called harmonic K-quasiconformal mapping on Ω , where $K = \frac{1+k}{1-k}$.

In [2-5], the analytic representing formula, coefficient estimates and the boundary characteristics were studied for univalent harmonic mappings defined on the unit disk $D = \{z : |z| < 1\}$. Recently, there were many researches on the conditions for univalent harmonic mappings to be harmonic K-quasiconformal mappings. In [6], Parlović studied the analytic representing expression for the univalent harmonic mappings from D onto itself and the conditions for the univalent harmonic mappings $f = h + \bar{g}$ to be harmonic K-quasiconformal mappings, that is, f satisfying $k = \sup_{z \in D} |\frac{g'(z)}{h'(z)}| < 1$. In [8], further researches were made to improve the results in [6] and some sufficient conditions for univalent harmonic mappings to be harmonic K-quasiconformal mappings were obtained. Later, Kalaj and Parlović

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[7] considered the conditions for univalent harmonic mappings from upper half plane $U = \{z | \text{Im}z > 0\}$ onto itself to be harmonic K -quasiconformal mappings and their boundary properties. In [9, 10, 11, 12], the geometric properties for the image domains of harmonic mappings were studied, several criterions were obtained to judge if a univalent harmonic mapping is a harmonic K -quasiconformal mapping or not. From this point of view, it is a very active topic to study harmonic K -quasiconformal mappings.

We know that the composition $f \circ \varphi$ of a harmonic mapping f with an analytic function φ is harmonic, however, if φ is analytic, $\varphi \circ f$ is not in general harmonic. Compared with conformal mappings, univalent harmonic mappings are not determined (up to normalization) by their image domain. Thus, it is natural to study the characteristics of class $S_H(\Delta, \Pi)$ of harmonic sense preserving univalent mappings of one domain Δ to another domain Π .

Let $S_H(D, \Omega)$ be the univalent harmonic mappings from the unit disk D to half plane domain $\Omega = \{w | \text{Re}w > a, -\infty < a < a_0 < +\infty\}$ with the normalization $f(0) = a_0, f_z(0) > 0, f_{\bar{z}}(0) = 0$. In [5], the representation of the class $S_H(D, \Omega)$ and their coefficient estimates were considered.

Let $F(D, \Omega) = \{f : f(z) = \text{Re}(\frac{cz+a_0}{1-z}) + i\text{Im}[2(a_0 - a) \int_{|\eta|=1} k(z, \eta) d\mu, \mu \in P]\}$, where $K(z, \eta) = \int_0^z \frac{1+\eta\zeta}{1-\eta\zeta} \frac{d\zeta}{(1-\zeta)^2}$ and let P be the set of probability measures μ on the Borel sets of $|\eta| = 1$. Following results were proved in [5]:

Theorem A. $S_H(D, \Omega) \subset F(D, \Omega)$.

Theorem B. If $f(z) = h(z) + \overline{g(z)} \in F(D, \Omega)$, then $f(z)$ is normalized harmonic by $f(0) = a_0, f_z(0) > 0, f_{\bar{z}}(0) = 0$, sense preserving and univalent from D into Ω .

Theorem C. Let $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} b_n z^n} \in S_H(D, \Omega)$, then

$$|a_n| \leq \frac{n+1}{2} a_1 \quad |b_n| \leq \frac{n-1}{2} a_1 \quad ||a_n| - |b_n|| \leq a_1,$$

Equality occurs in all cases for the functions

$$f(z) = \text{Re}(\frac{(a_0 - 2a)z + a_0}{1-z}) + ia_1 \text{Im}(\frac{z}{(1-z)^2}).$$

Theorem D. If $f(z) = h(z) + \overline{g(z)} \in S_H(D, \Omega)$ and $U_r = \{z : |z| \leq r < 1\}$, then the area of $f(U_r)$ satisfies the following

$$\text{Area}f(U_r) \leq 4\pi(a_0 - a)^2 \frac{r^2(1+r^2)}{(1-r^2)^3}.$$

The bound is sharp.

Given $K \geq 1$, let $F(z) = \frac{1}{2}(K+1)\frac{(a_0-2a)z+a_0}{1-z} + \frac{1}{2}(1-K)\frac{\overline{(a_0-2a)z+a_0}}{1-z}$, $z \in D$, it is easy to show that $F(z)$ is a harmonic K -quasiconformal mapping from the unit disk D onto the half plane domain $\Omega = \{w | \text{Re}w > a, -\infty < a < a_0 < +\infty\}$ with $F(0) = a_0, F_z(0) = (K+1)(a_0 - a) > 0, F_{\bar{z}}(0) = (1-K)(a_0 - a) \in \mathbb{R}$, thus, $F(z) \notin S_H(D, \Omega)$. Using the mapping $\varphi(z) = \frac{F(z) - F_{\bar{z}}(0)\overline{F(z)}}{1 - |F_{\bar{z}}(0)|^2}$, we can not obtain

that $\varphi(z) \in S_H(D, \Omega)$. Therefore, it is significant to consider the set of harmonic mappings which is bigger than the set $S_H(D, \Omega)$.

In this paper, we consider the sense preserving univalent harmonic mappings $f(z)$ from the unit disk D onto the half plane domain $\Omega = \{w \mid \operatorname{Re} w > a, -\infty < a < a_0 < +\infty\}$ with the normalization $f(0) = a_0, f_z(0) > 0, f_{\bar{z}}(0) = c, c \in R$. First, we derive an analytic representation for sense preserving univalent harmonic mappings $f(z)$ from the unit disk D onto half plane domain $\Omega = \{w \mid \operatorname{Re} w > a, -\infty < a < a_0 < +\infty\}$ with the normalization $f(0) = a_0, f_z(0) > 0, f_{\bar{z}}(0) = c, c \in R$, then we give one necessary and sufficient condition for such univalent harmonic mappings to be harmonic K -quasiconformal mappings and some sharp coefficient estimates. Owing to the unboundedness of image domain, as an application, distortion theorems on length and area for their images are also obtained. Our results improve and generalize the one made by M. Öztürk.

2. MAIN RESULTS

The expression of $f(z) \in F(D, \Omega)$ by [5] connecting with integral on the Borel sets of $|\eta| = 1$, it is difficult to consider the quasiconformality for them. For the sense preserving univalent harmonic mappings $f(z)$ from the unit disk D onto the half plane domain $\Omega = \{w \mid \operatorname{Re} w > a, -\infty < a < a_0 < +\infty\}$ with the normalization $f(0) = a_0, f_z(0) > 0, f_{\bar{z}}(0) = c, c \in R$, we will prove the following theorem.

Theorem 1. *Let $f(z)$ be a sense preserving univalent harmonic mapping from the unit disk D onto the half plane domain $\Omega = \{w \mid \operatorname{Re} w > a, -\infty < a < a_0 < +\infty\}$ with the normalization $f(0) = a_0, f_z(0) > 0, f_{\bar{z}}(0) = c, c \in R$. Then there is an analytic function $\varphi(z)$ such that*

$$f(z) = \frac{\varphi(z) + \zeta(z)}{2} + \frac{\overline{\varphi(z) - \zeta(z)}}{2}, \quad z \in D,$$

where $\varphi(z) = \frac{(a_0 - 2a)z + a_0}{1 - z}$ and $\zeta(z)$ is analytic with $\zeta(0) = 0, \zeta'(0) \in R$.

Furthermore, $f(z)$ is a harmonic K -quasiconformal mapping if and only if $\zeta(z)$ satisfies

$$\left| \frac{(1 - z)^2 \zeta'(z)}{2(a_0 - a)} - \frac{1 + k^2}{1 - k^2} \right| \leq \frac{2k}{1 - k^2}, \quad z \in D,$$

where $K = \frac{1+k}{1-k}$ for some $0 \leq k < 1$.

Proof. Suppose that $f(z)$ is a sense preserving univalent harmonic mapping from the unit disk D onto the half plane domain $\Omega = \{w \mid \operatorname{Re} w > a, -\infty < a < a_0 < +\infty\}$ with the normalization $f(0) = a_0, f_z(0) > 0, f_{\bar{z}}(0) = c, c \in R$, by [1], there exist two analytic functions $h(z), g(z)$ on D such that $f(z) = h(z) + \overline{g(z)}, g(0) = 0$. Since $J_f(0) = |f_z^2(0)| - |f_{\bar{z}}^2(0)| = f_z^2(0) - c^2 > 0$, we have $f_z^2(0) > c^2$. Because the domain Ω is convex in the direction of the imaginary axis, by [2], $\varphi(z) = h(z) + g(z)$ is a conformal univalent mapping of D with $\varphi(0) = h(0) = a_0, \varphi'(0) = h'(0) + g'(0) = f_z(0) + \overline{f_{\bar{z}}(0)} > 0$. Taking note of $\operatorname{Re} f(z) = \operatorname{Re}(h(z) + g(z))$, the boundary values of $h(z) + g(z)$ are all on the line $\partial\Omega$, thus $\varphi(z) = h(z) + g(z)$ also maps D onto $\Omega = \{w \mid \operatorname{Re} w > a, -\infty < a < a_0 < +\infty\}$ such that $\varphi(0) = a_0, \varphi'(0) = h'(0) + g'(0) > 0$, by Riemann mapping theorem, we obtain $\varphi(z) = \frac{(a_0 - 2a)z + a_0}{1 - z}$. On the other

hand, let $f(z) = u(z) + iv(z)$, to the harmonic function $iv(z)$, there exists an analytic function $\zeta(z)$, $\zeta(0) = 0$ such that $iv(z) = \frac{\zeta(z) - \overline{\zeta(z)}}{2}$. We obtain that

$$f(z) = \operatorname{Re}(\varphi(z)) + iv(z) = \frac{\varphi(z) + \zeta(z)}{2} + \frac{\overline{\varphi(z) - \zeta(z)}}{2}.$$

By the normalization $f(0) = a_0$, $f_z(0) > 0$, $f_{\bar{z}}(0) = c$, $c \in R$, we have $\zeta'(0) \in R$.

Furthermore, suppose that $f(z)$ is a sense preserving univalent harmonic mapping of the unit disk D onto $\Omega = \{w | \operatorname{Re} w > a, -\infty < a < a_0 < +\infty\}$ such that $f(0) = a_0$, $f_z(0) > 0$, $f_{\bar{z}}(0) = c$, $c \in R$, let $f(z) = \frac{\varphi(z) + \zeta(z)}{2} + \frac{\overline{\varphi(z) - \zeta(z)}}{2}$, where $\varphi(z) = \frac{(a_0 - 2a)z + a_0}{1 - z}$ and $\zeta(z)$ is an analytic function of D satisfying $\zeta(0) = 0$, $\zeta'(0) \in R$. Since $f_z(z) = \frac{(a_0 - a)}{(1 - z)^2} + \frac{1}{2}\zeta'(z)$ and $f_{\bar{z}}(z) = \frac{(a_0 - a)}{(1 - z)^2} - \frac{1}{2}\zeta'(z)$. Therefore, $f(z)$ is a harmonic K -quasiconformal mapping if and only if

$$\left| \frac{f_{\bar{z}}(z)}{f_z(z)} \right| = \left| \frac{1 - \frac{(1-z)^2 \zeta'(z)}{2(a_0 - a)}}{1 + \frac{(1-z)^2 \zeta'(z)}{2(a_0 - a)}} \right| \leq k, \quad 0 \leq k = \frac{K - 1}{K + 1} < 1.$$

Which is equivalent to $\left| \frac{(1-z)^2 \zeta'(z)}{2(a_0 - a)} - \frac{1+k^2}{1-k^2} \right| \leq \frac{2k}{1-k^2}$, $z \in D$, for some $0 \leq k < 1$ and $K = \frac{1+k}{1-k}$. The proof of Theorem 1 is complete.

Following two examples show that using Theorem 1, it is very convenient to judge whether a univalent harmonic mapping $f(z)$ from D onto $\Omega = \{w | \operatorname{Re} w > a, -\infty < a < a_0 < +\infty\}$ with $f(0) = a_0$, $f_z(0) > 0$, $f_{\bar{z}}(0) = c$, $c \in R$ is a harmonic K -quasiconformal mapping and the condition $\left| \frac{(1-z)^2 \zeta'(z)}{2(a_0 - a)} - \frac{1+k^2}{1-k^2} \right| \leq \frac{2k}{1-k^2}$, $z \in D$ is sharp for quasiconformality.

Example 1. Let $f(z) = 2(a_0 - a) \left(\frac{2z - z^2}{2(1-z)^2} + \frac{\overline{-z^2}}{2(1-z)^2} \right) + a_0 = \frac{1}{2} \left(\frac{(a_0 - 2a)z + a_0}{1-z} + \frac{2(a_0 - a)z}{(1-z)^2} \right) + \frac{1}{2} \left(\frac{(a_0 - 2a)z + a_0}{1-z} - \frac{2(a_0 - a)z}{(1-z)^2} \right)$, $z \in D$, then $f(z)$ is a univalent harmonic mapping from D onto $\Omega = \{w | \operatorname{Re} w > a, -\infty < a < a_0 < +\infty\}$ such that $f(0) = a_0$, $f_z(0) = 2(a_0 - a) > 0$, $f_{\bar{z}}(0) = 0$.

By Theorem 1, we have $\zeta(z) = \frac{2(a_0 - a)z}{(1-z)^2}$, $\frac{(1-z)^2 \zeta'(z)}{2(a_0 - a)} = \frac{1+z}{1-z}$, $z \in D$. For any constant k with $0 \leq k < 1$, the condition $\left| \frac{(1-z)^2 \zeta'(z)}{2(a_0 - a)} - \frac{1+k^2}{1-k^2} \right| = \left| \frac{1+z}{1-z} - \frac{1+k^2}{1-k^2} \right| \leq \frac{2k}{1-k^2}$, does not satisfy for all $z \in D$, so $f(z)$ is not a harmonic K -quasiconformal mapping from D onto $\Omega = \{w | \operatorname{Re} w > a, -\infty < a < a_0 < +\infty\}$.

Example 2. Given $K \geq 1$, let $F(z) = \frac{1}{2}(K+1) \frac{(a_0 - 2a)z + a_0}{1-z} + \frac{1}{2}(1-K) \frac{\overline{(a_0 - 2a)z + a_0}}{1-z}$, we know that $F(z)$ is a harmonic K -quasiconformal mapping from D onto $\Omega = \{w | \operatorname{Re} w > a, -\infty < a < a_0 < +\infty\}$ such that $F(0) = a_0$, $F_z(0) = (K+1)(a_0 - a) > 0$, $F_{\bar{z}}(0) = (1-K)(a_0 - a) \in R$, by Theorem 1, we have $\zeta(z) = K \frac{(a_0 - 2a)z + a_0}{1-z}$, let $k = \frac{K-1}{K+1}$, $\zeta(z)$ satisfies $\left| \frac{(1-z)^2 \zeta'(z)}{2(a_0 - a)} - \frac{1+k^2}{1-k^2} \right| = \frac{2k}{1-k^2}$, $z \in D$. Which shows that the condition $\left| \frac{(1-z)^2 \zeta'(z)}{2(a_0 - a)} - \frac{1+k^2}{1-k^2} \right| \leq \frac{2k}{1-k^2}$, $z \in D$ is sharp for quasiconformality.

Next, we will use Theorem 1 to solve some coefficient extremal problems for harmonic K -quasiconformal mappings. We will prove the following

Theorem 2. Let $f(z) = h(z) + \overline{g(z)} = a_0 + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ be a sense preserving harmonic K -quasiconformal mapping from D onto $\Omega = \{w \mid \text{Re} w > a, -\infty < a < a_0 < +\infty\}$ such that $f(0) = a_0, f_z(0) > 0, f_{\bar{z}}(0) = c, c \in \mathbb{R}$. Then $|b_1| \leq |a_1| \leq (1+K)(a_0 - a)$ and $\|a_n - b_n\| \leq 2(a_0 - a), k = 2, 3, \dots$. The estimates are sharp. The extremal mapping

$$F(z) = \frac{1}{2}(K+1) \frac{(a_0 - 2a)z + a_0}{1-z} + \frac{1}{2}(1-K) \frac{\overline{(a_0 - 2a)z + a_0}}{1-z}, z \in D,$$

shows their equalities.

Proof. Let $f(z) = h(z) + \overline{g(z)} = a_0 + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ be a sense preserving harmonic K -quasiconformal mapping from D onto $\Omega = \{w \mid \text{Re} w > a, -\infty < a < a_0 < +\infty\}$ such that $f(0) = a_0, f_z(0) > 0, f_{\bar{z}}(0) = c, c \in \mathbb{R}$. By theorem 1, there exists an analytic function $\zeta(z), \zeta(0) = 0$ of D such that

$$f(z) = \frac{\frac{(a_0 - 2a)z + a_0}{1-z} + \zeta(z)}{2} + \frac{\overline{\frac{(a_0 - 2a)z + a_0}{1-z} - \zeta(z)}}{2},$$

let $\zeta(z) = \sum_{n=1}^{\infty} c_n z^n$, we have

$$f(z) = a_0 + \sum_{n=1}^{\infty} (a_0 - a + \frac{c_n}{2}) z^n + \overline{\sum_{n=1}^{\infty} (a_0 - a - \frac{c_n}{2}) z^n}.$$

Since $f(z)$ is a sense preserving harmonic K -quasiconformal mapping, by Theorem 1, then $|\frac{(1-z)^2 \zeta'(z)}{2(a_0 - a)} - \frac{1+k^2}{1-k^2}| \leq \frac{2k}{1-k^2}, z \in D$, thus, $|\frac{c_1}{2}| \leq K(a_0 - a)$. By the sense preserving property of $f(z)$, we have $|b_1| \leq |a_1| \leq (1+K)(a_0 - a)$. For any $n = 2, 3, \dots$, since $\|a_n - b_n\| = \|(a_0 - a + \frac{c_n}{2}) - (a_0 - a - \frac{c_n}{2})\|$, we obtain that $\|a_n - b_n\| \leq 2(a_0 - a)$.

According to example 2, It is very easy to check that the extremal mapping $F(z) = \frac{1}{2}(K+1) \frac{(a_0 - 2a)z + a_0}{1-z} + \frac{1}{2}(1-K) \frac{\overline{(a_0 - 2a)z + a_0}}{1-z}, z \in D$ satisfies the conditions in Theorem 2, by calculation, its coefficients show that $|b_1| < |a_1| = (1+K)(a_0 - a)$ and $\|a_n - b_n\| = 2(a_0 - a), k = 2, 3, \dots$. The proof of Theorem 2 is complete.

In the following, we will study the distortion problems for the harmonic K -quasiconformal mappings considered in this paper. We will prove the following

Theorem 3. Let $f(z) = h(z) + \overline{g(z)} = a_0 + \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ be a sense preserving harmonic K -quasiconformal mapping from D onto $\Omega = \{w \mid \text{Re} w > a, -\infty < a < a_0 < +\infty\}$ such that $f(0) = a_0, f_z(0) > 0, f_{\bar{z}}(0) = c, c \in \mathbb{R}$. Then, for any $z \in D$, we have

$$|f(z) - a_0| \leq \frac{2(a_0 - a)K|z|}{1 - |z|}.$$

For any measurable set E in D , then the area of $f(E)$ satisfies the following inequality

$$\text{Area} f(E) \leq 4K(a_0 - a)^2 \iint_E \frac{1}{|1-z|^4} dx dy.$$

In particular, if we choose that $E = U_r = \{z : |z| \leq r < 1\}$, then the area of $f(U_r)$ satisfies the following inequality

$$\text{Area}f(U_r) \leq 4K\pi(a_0 - a)^2 \frac{r^2}{(1 - r^2)^2}.$$

Equality occurs for the extremal mapping

$$F(z) = \frac{1}{2}(K + 1) \frac{(a_0 - 2a)z + a_0}{1 - z} + \frac{1}{2}(1 - K) \frac{\overline{(a_0 - 2a)z + a_0}}{1 - z}, z \in D.$$

Proof. Suppose $f(z) = h(z) + \overline{g(z)}$ is a sense preserving harmonic K -quasiconformal mapping from D onto $\Omega = \{w \mid \text{Re}w > a, -\infty < a < a_0 < +\infty\}$ such that $f(0) = a_0, f_z(0) > 0, f_{\bar{z}}(0) = c, c \in R$. By Theorem 1, there exists an analytic function $\zeta(z)$ of D with $\zeta(0) = 0, \zeta'(0) \in R$ such that

$$f(z) = \frac{\varphi(z) + \zeta(z)}{2} + \frac{\overline{\varphi(z) - \zeta(z)}}{2},$$

and $|\frac{(1-z)^2\zeta'(z)}{2(a_0-a)} - \frac{1+k^2}{1-k^2}| \leq \frac{2k}{1-k^2}, z \in D$, where $K = \frac{1+k}{1-k}, 0 \leq k < 1$. Thus, we obtain that $|\frac{(1-z)^2\zeta'(z)}{2(a_0-a)} + 1| \leq \frac{2}{1-k}, z \in D$. For any $z \in D$, then

$$\begin{aligned} |f(z) - a_0| &= \left| \int_0^z f_z(z) dz + f_{\bar{z}}(z) d\bar{z} \right| \\ &\leq \int_0^{|z|} (|f_z(z)| + |f_{\bar{z}}(z)|) |dz| \\ &\leq \frac{(1+k)}{2} \int_0^{|z|} |\varphi'(z) + \zeta'(z)| |dz| \\ &= \frac{(1+k)}{2} \int_0^{|z|} \left| 1 + \frac{(1-z)^2\zeta'(z)}{2(a_0-a)} \right| \frac{2(a_0-a)|dz|}{|(1-z)^2|} \\ &\leq 2(a_0-a)K \int_0^{|z|} \frac{|dz|}{|1-z|^2} \\ &\leq \frac{2(a_0-a)K|z|}{1-|z|}. \end{aligned}$$

Next, for any measurable set E in D , let the area of $f(E)$ be $\text{Area} f(E)$, then

$$\begin{aligned} \text{Area}f(E) &= \iint_E (|h'(z)|^2 - |g'(z)|^2) dx dy \\ &= \frac{1}{4} \iint_E (|\varphi'(z) + \zeta'(z)|^2 - |\varphi'(z) - \zeta'(z)|^2) dx dy \\ &= 2(a_0 - a) \iint_E \text{Re} \frac{\overline{\zeta'(z)}}{(1-z)^2} dx dy \\ &= 2(a_0 - a) \iint_E \text{Re} \frac{\overline{\zeta'(z)(1-z)^2}}{|1-z|^4} dx dy, \end{aligned}$$

by Theorem 1, $|\frac{(1-z)^2\zeta'(z)}{2(a_0-a)} - \frac{1+k^2}{1-k^2}| \leq \frac{2k}{1-k^2}$, $z \in D$, then $\frac{1}{K} \leq \operatorname{Re} \frac{(1-z)^2\zeta'(z)}{2(a_0-a)} \leq K$, we obtain that

$$\operatorname{Area}f(E) \leq 4K(a_0 - a)^2 \iint_E \frac{1}{|1-z|^4} dx dy.$$

The harmonic K -quasiconformal mapping

$$F(z) = \frac{1}{2}(K+1) \frac{(a_0 - 2a)z + a_0}{1-z} + \frac{1}{2}(1-K) \frac{\overline{(a_0 - 2a)z + a_0}}{1-z}, z \in D,$$

satisfies the conditions in Theorem 3, for any measurable set E in D , we have

$$\begin{aligned} \operatorname{Area}F(E) &= \iint_E (|F_z(z)|^2 - |F_{\bar{z}}(z)|^2) dx dy \\ &= \iint_E \frac{(K+1)^2(a_0-a)^2}{|1-z|^4} dx dy - \iint_E \frac{(K-1)^2(a_0-a)^2}{|1-z|^4} dx dy \\ &= 4K(a_0-a)^2 \iint_E \frac{1}{|1-z|^4} dx dy. \end{aligned}$$

This shows that the above estimate is sharp.

In particular, if we choose that $E = U_r = \{z : |z| \leq r < 1\}$, to the harmonic K -quasiconformal mapping $F(z) = \frac{1}{2}(K+1) \frac{(a_0-2a)z+a_0}{1-z} + \frac{1}{2}(1-K) \frac{\overline{(a_0-2a)z+a_0}}{1-z}$, by calculation, then the area of $F(U_r)$ is

$$\operatorname{Area}F(U_r) = 4K\pi(a_0 - a)^2 \frac{r^2}{(1-r^2)^2}.$$

This also shows that the above estimate is sharp. The proof of Theorem 3 is complete.

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