Small oriented cycle double cover of graphs

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Abstract

A small oriented cycle double cover (SOCDC) of a bridgeless graph G on n vertices is a collection of at most $n-1$ directed cycles of the symmetric orientation, G_s, of G such that each arc of G_s lies in exactly one of the cycles. It is conjectured that every 2-connected graph except two complete graphs K_4 and K_6 has an SOCDC. In this paper, we study graphs with SOCDC and obtain some properties of the minimal counterexample to this conjecture.

Keywords: Cycle double cover, Small cycle double cover, Oriented cycle double cover, Small oriented cycle double cover.

1 SOCDC conjecture

We denote by G a finite undirected graph with vertex set V and edge set E with no loops or multiple edges. The symmetric orientation of G, denoted by G_s, is an oriented graph obtained from G by replacing each edge of G by a pair of opposite directed arcs. An even graph (odd graph) is a graph such that each vertex is incident to an even (odd) number of edges. A directed even graph is a graph such that for each vertex its out-degree equals to its in-degree. A cycle (a directed cycle) is a minimal non-empty even graph (directed even graph). We denote every directed cycle C and directed path P on n vertices with vertex set $\{v_1, \ldots, v_n\}$ and directed edge set $E(C) = \{v_iv_{i+1}, v_nv_1 : 1 \leq i \leq n-1\}$ and $E(P) = \{v_iv_{i+1} : 1 \leq i \leq n-1\}$ by $C = [v_1, \ldots, v_n]$, and $P = (v_1, \ldots, v_n)$, respectively.

A cycle double cover (CDC) $\mathcal{C}$ of a graph G is a collection of cycles in G such that every edge of G belongs to exactly two cycles of $\mathcal{C}$. Note that the cycles are not necessarily distinct. It can be easily seen that a necessary condition for a graph to have a CDC is that the graph has no cut edge which is called a bridgeless graph. Seymour [13] in 1979 conjectured that every bridgeless graph has a CDC. No counterexample to the CDC conjecture is known. It is proved that the minimal counterexample to the CDC conjecture is a bridgeless cubic graph with edge chromatic number equal to 4, which is called a snark.

A small cycle double cover (SCDC) of a graph on n vertices is a CDC with at most $n-1$ cycles. There exist simple graphs of order n for which any CDC requires at least $n-1$ cycles (e.g., $K_n, n \geq 3$). Furthermore, no simple bridgeless graph of order n is known to require more than $n-1$ cycles in a CDC. Note that clearly it is false if not restricted to simple graphs. Bondy [2] conjectured that every simple bridgeless graph has an SCDC. For more results on the CDC conjecture see [5, 15].

The CDC conjecture has many stronger forms. In this paper, we consider the oriented version of these conjectures.

An oriented cycle double cover (OCDC) is a CDC in which every cycle can be oriented in such a way that every edge of the graph is covered by two directed cycles in two different directions.

Conjecture 1.1 [6] (Oriented CDC conjecture) *Every bridgeless graph has an* OCDC.

No counterexample to this conjecture is known. It is clear that the validity of the OCDC conjecture implies the validity of the CDC conjecture. While there is a CDC of the Petersen graph that can not be oriented in such a way that forms an OCDC.

Definition 1.2 *A* small oriented cycle double cover (SOCDC) *of a graph on n vertices is an* OCDC *with at most $n-1$ directed cycles.*

The natural question is that which simple bridgeless graphs of order n have an OCDC with at most $n-1$ cycles (SOCDC)?

It can be proved that an OCDC for planar graphs can be obtained from their planar embedding and by the Euler's formula it can be seen that the sparse planar graphs have SOCDC. In fact, every bridgeless planar graph G with $|E(G)| < 2|V(G)| - 2$, has an SOCDC. Moreover, every simple triangle-free planar graph G with at least three vertices admits an SOCDC, since $|E(G)| \leq 2|V(G)| - 4$.

If $\mathcal{C}$ is a CDC of a cubic graph G of order n, then $|\mathcal{C}| \leq n/2 + 2$ [7]. Therefore, every OCDC of a cubic graph of order $n \geq 6$ is an SOCDC. Moreover, in cubic graph G, $\chi'(G) = 3$ implies the existence of an OCDC of G [15]. Thus, every cubic graph with edge chromatic number 3, $G \neq K_4$, has an SOCDC.

An oriented perfect path double cover (OPPDC) of a graph G is a collection of directed paths in the symmetric orientation G_s such that each arc of G_s lies in exactly one of the paths and each vertex of G appears just once as a beginning and just once as end of a directed path. Maxová and Nešetřil in [10] showed that two complete graphs K_3 and K_5 have no OPPDC and in [9], they conjectured every connected graph except K_3 and K_5 has an OPPDC.

The join of two simple graphs G and H, $G \vee H$, is the graph obtained from the disjoint union of G and H by adding the edges $\{uv : u \in V(G),\ v \in V(H)\}$.

In [10], it is shown that if G is a connected graph, then graph G has an OPPDC if and only if $G \vee K_1$ has an SOCDC. Also, a list of some families of graphs that admit an

OPPDC are presented in [1, 10]. Therefore, the join of those graphs and K_1 admit an SOCDC.

In [11] infinite classes of graphs with an SCDC are obtained using the Cartesian product $G \square H$, for some classes of G and H. Applying the same method, one can obtain the similar results in the oriented version, adding the assumption that G or H has an OPPDC, if it is necessary.

Since K_3 and K_5 have no OPPDC, K_4 and K_6 have no SOCDC. It is known that every $K_{2n-1}, n \geq 4$, has an OPPDC [1], thus every $K_{2n},\ n \geq 4$, has an SOCDC. Moreover, every K_{2n+1} has an SOCDC, since K_{2n+1} has a Hamiltonian cycle decomposition [14]. It can be observed that if every block of a graph G has an SOCDC, then G has also an SOCDC.

This fact motivates us to present the following conjecture.

Conjecture 1.3 (SOCDC conjecture) *Every simple* 2*-connected graph except* K_4 *and* K_6 *admits an* SOCDC*.*

In the following proposition, we construct some graphs with no SOCDC. In fact, we show that the difference $|\mathcal{C}| - (n-1)$ could be large enough for every OCDC, $\mathcal{C}$ of some bridgeless graph of order n.

Let $V(K_4) = \{v_1, v_2, v_3, v_4\}$. The collection $\mathcal{C} = \{[v_1, v_2, v_4], [v_2, v_1, v_3], [v_3, v_4, v_2], [v_4, v_3, v_1]\}$ is an OCDC of K_4. Since K_4 has six edges, if $\mathcal{C}$ is an arbitrary OCDC of K_4, then $|\mathcal{C}| \leq (2 \times 6)/3 = 4$. Thus, every OCDC of K_4 is of size 4.

Let $V(K_6) = \{v_1, \dots, v_6\}$. The collection $\mathcal{C} = \{[v_1, v_2, v_3, v_4, v_5, v_6], [v_2, v_6, v_3, v_5, v_4], [v_1, v_5, v_2, v_4, v_3], [v_1, v_4, v_6, v_2, v_5], [v_1, v_6, v_5, v_3, v_2], [v_1, v_3, v_6, v_4]\}$ is an OCDC of K_6 of size 6.

Proposition 1.4 *For every integer $r \geq 1$, there exists a bridgeless graph G of order n such that every* OCDC *of G has $(n-1) + r$ directed cycles.*

Proof. Let P be a path of length r with $V(P) = \{v_1, \dots, v_{r+1}\}$ and $E(P) = \{v_i v_{i+1} : 1 \leq i \leq r\}$. Assume that G is a graph obtained from P by replacing each edge $v_i v_{i+1}$ of P with a clique K_4, say K_4^i, where $V(K_4^i) = \{v_i, v_i', v_{i+1}, v_{i+1}'\}$, $1 \leq i \leq r$. Every OCDC of G is decomposable to r OCDC of K_4. Moreover, every OCDC of K_4 has four cycles. Therefore, every OCDC of G has $4r$ cycles. Note that $|V(G)| = 3r+1$, thus every OCDC of G has $(|V(G)| - 1) + r$ cycles. ■

The above conjecture has a close relation to the following conjecture.

Conjecture 1.5 [3] (Hajós' conjecture) *If G is a simple, even graph of order n, then G can be decomposed into $\lfloor (n-1)/2 \rfloor$ cycles.*

If the Hajós' conjecture holds, then every even graph has an SOCDC obtained by taking two copies of the cycles used in its decomposition, in two opposite directions.

As the Hajós' conjecture is true for even graphs with maximum degree four [4], planar graphs [12], projective graphs and K_6^--minor free graphs [3], these graphs have an SOCDC.

In the next section, we study the properties of the minimal counterexample to the SOCDC cojecture.

2 The minimal counterexample to the SOCDC conjecture

If the SOCDC conjecture is false, then it must has a minimal counterexample. In this section, we study the properties of the minimal counterexample to the SOCDC conjecture.

Observation 2.1 *If G is a graph with an* SOCDC *and G' is the graph obtained from G by subdividing one edge of G, then G' also admits an* SOCDC.

Corollary 2.2 *Let G be the minimal counterexample to the* SOCDC *conjecture, then the minimum degree of G is at least* 3.

Theorem 2.3 *The minimal counterexample to the* SOCDC *conjecture is* 3*-connected.*

Proof. Let G, the minimal counterexample to the SOCDC conjecture be a 2-connected graph of order n with vertex cut $\{v_1, v_2\}$ and $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{v_1, v_2\}$ and $|V(G_i)| = n_i$, $i = 1, 2$. Assume that $G_i \cup \{v_1v_2\}$ has an SOCDC, $\mathcal{C}_\text{i}$, i $= 1, 2$. Let $C_i^j, j = 1, 2$, be the two directed cycles in $\mathcal{C}_i$, $i = 1, 2$, which include the directed edge v_jv_{j+1}, where subscripts are reduced modulo 2. In each of the following cases, we show that G admits an SOCDC, which is a contradiction.

(I) If $v_1v_2 \in E(G)$, then we define

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{C_1^1 \Delta C_2^2\} \setminus \{C_1^1, C_2^2\}.$$

The collection $\mathcal{C}$ is an OCDC of G, where

$$\begin{aligned} |\mathcal{C}| = |\mathcal{C}_1| + |\mathcal{C}_2| - 1 &\leq (n_1 - 1) + (n_2 - 1) - 1 \\ &\leq (n_1 + n_2) - 3 \\ &\leq (n + 2) - 3 = n - 1. \end{aligned}$$

If $G_1 \cup \{v_1v_2\} = K_4$ with $V(K_4) = \{v_1, v_2, v_3, v_4\}$, and $G_2 \cup \{v_1v_2\}$ has an SOCDC, say $\mathcal{C}_2$, then let $C_1 = [v_1, v_2, v_4]$, $C_2 = [v_1, v_4, v_3, v_2]$, $C_3 = C_2^1 \cup (v_1, v_3, v_4, v_2) \setminus \{v_1v_2\}$, and $C_4 = C_2^2 \cup (v_2, v_3, v_1) \setminus \{v_2v_1\}$. Therefore,

$$\mathcal{C} = \mathcal{C}_2 \cup \{C_1, C_2, C_3, C_4\} \setminus \{C_1^1, C_2^2\}$$

is an SOCDC of G.

If $G_1 \cup \{v_1v_2\} = K_6$ with $V(K_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, and $G_2 \cup \{v_1v_2\}$ has an SOCDC, say $\mathcal{C}_2$, then let $C_1 = [v_1, v_2, v_4, v_6, v_3, v_5]$, $C_2 = [v_1, v_3, v_6, v_2]$, $C_3 = [v_1, v_4, v_2, v_5, v_6]$, $C_4 = [v_1, v_5, v_2, v_3, v_4]$, $C_5 = C_2^1 \cup (v_1, v_6, v_5, v_4, v_3, v_2) \setminus \{v_1v_2\}$, and $C_6 = C_2^2 \cup (v_2, v_6, v_4, v_5, v_3, v_1) \setminus \{v_2v_1\}$. Therefore,

$$\mathcal{C} = \mathcal{C}_2 \cup \{C_1, C_2, C_3, C_4, C_5, C_6\} \setminus \{C_1^1, C_2^2\}$$

is an SOCDC of G.

If $G_1 \cup \{v_1v_2\} = G_2 \cup \{v_1v_2\} = K_4$ or $G_1 \cup \{v_1v_2\} = K_4$ and $G_2 \cup \{v_1v_2\} = K_6$ or $G_1 \cup \{v_1v_2\} = G_2 \cup \{v_1v_2\} = K_6$, then by Theorem 1 in [1], $G \setminus v_1$ admits an OPPDC, thus G has an SOCDC.

(II) If $v_1v_2 \notin E(G)$, then we define

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{C_1^1 \Delta C_2^2, C_1^2 \Delta C_2^1\} \setminus \{C_1^1, C_1^2, C_2^1, C_2^2\}.$$

The collection $\mathcal{C}$ is an OCDC of G, where $|\mathcal{C}| \leq n-2$.
Furthermore, if $G_1 \cup \{v_1v_2\} = K_4$ or K_6, $v_1v_2 \notin E(G)$, and $G_2 \cup \{v_1v_2\}$ has an SOCDC, by the similar argument in above using the given SOCDC for K_4 and K_6 of size 4 and 6, an SOCDC for G is obtained.

If $G_1 \cup \{v_1v_2\} = G_2 \cup \{v_1v_2\} = K_4$ with $V(G_1) = \{v_1, v_2, v_3, v_4\}$ and $V(G_2) = \{v_1, v_2, v_5, v_6\}$, then

$\mathcal{C} = \{[v_1, v_4, v_3, v_2, v_5, v_6], [v_1, v_5, v_2, v_3], [v_1, v_3, v_4, v_2, v_6, v_5], [v_1, v_6, v_2, v_4]\}$

is an SOCDC of G.

If $G_1 \cup \{v_1v_2\} = K_4$ with $V(G_1) = \{v_1, v_2, v_3, v_4\}$ and $G_2 \cup \{v_1v_2\} = K_6$ $V(G_2) = \{v_1, v_2, v_5, v_6, v_7, v_8\}$, then

$\mathcal{C} = \{[v_1, v_6, v_5, v_7, v_8, v_2, v_3], [v_1, v_3, v_4, v_2, v_8], [v_1, v_7, v_6, v_8, v_5, v_2, v_4], [v_1, v_5, v_8, v_7, v_2, v_6], [v_1, v_8, v_6, v_2, v_7, v_5], [v_1, v_4, v_3, v_2, v_5, v_6, v_7]\}$

is an SOCDC of G.

If $G_1 \cup \{v_1v_2\} = G_2 \cup \{v_1v_2\} = K_6$ with $V(G_1) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $V(G_2) = \{v_1, v_2, v_7, v_8, v_9, v_{10}\}$, then

$\mathcal{C} = \{[v_1, v_6, v_4, v_5, v_3, v_2, v_7, v_9, v_8, v_{10}], [v_1, v_3, v_5, v_4, v_6, v_2, v_{10}, v_8, v_9, v_7], [v_1, v_4, v_3, v_6, v_5, v_2, v_9, v_{10}, v_7, v_8], [v_1, v_5, v_6, v_3, v_4, v_2, v_8, v_7, v_{10}, v_9], [v_1, v_8, v_2, v_4], [v_1, v_{10}, v_2, v_6], [v_1, v_9, v_2, v_5], [v_1, v_7, v_2, v_3]\}$

is an SOCDC of G. ■

Corollary 2.4 *The minimal counterexample to the* SOCDC *conjecture is* 3*-edge-connected.*

An edge cut F, is called trivial if one of the component in $G \setminus F$ be an isolated vertex.

Theorem 2.5 *The minimal counterexample to the* SOCDC *conjecture has no non-trivial edge cut of size* 3.

Proof. Let G be the minimal counterexample to the SOCDC conjecture. We know that G is 2-connected and 3-edge-connected. Assume that G has a non-trivial edge cut of size 3. We consider the following cases.

(I) $G = G_1 \cup G_2 \cup \{u_1v_1, u_2v_2, u_3v_3\}$, where $G_1 \cap G_2 = \emptyset$, the vertices u_i are distinct vertices of G_1, and the vertices v_i are distinct vertices of G_2, $i = 1, 2, 3$.

Denote by H_i the graph obtained by contracting the subgraph G_{i+1} to a single vertex w_i, $i = 1, 2$, where subscripts are reduced modulo 2. Since $\deg(w_i) = 3$, $H_i \neq K_6$, $i = 1, 2$. By the minimality of G, H_i has an SOCDC or $H_i = K_4$. Therefore, H_i has an OCDC, $\mathcal{C}_i$, $i = 1, 2$. Let C_i^j, $j = 1, 2, 3$, be the three directed cycles in $\mathcal{C}_i$ which include w_i, $i = 1, 2$, where without loss of generality, we assume that C_1^j includes directed path (u_{j-1}, w_1, u_{j+1}), and C_2^j includes directed path (v_{j+1}, w_2, v_{j-1}), where subscripts are reduced modulo 3, $j = 1, 2, 3$. Let $P_i^j = C_i^j \setminus w_i$, $i = 1, 2$, $j = 1, 2, 3$. Define $C^j = P_1^j \cup P_2^j \cup \{u_{j-1}v_{j-1}, v_{j+1}u_{j+1}\}$, $\mathcal{C}' = \{C^j : j = 1, 2, 3\}$, and $\mathcal{C}'' = \{C_i^j : i = 1, 2,\ j = 1, 2, 3\}$. Thus, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}' \setminus \mathcal{C}''$ is an OCDC of G, where $|\mathcal{C}| = |\mathcal{C}_1| + |\mathcal{C}_2| - 3$. Note that every OCDC of K_4 has 4 cycles, therefore, in both cases $|\mathcal{C}| \leq |V(G)| - 1$, which is a contradiction.

(II) $G = G_1 \cup G_2 \cup \{u_1v_1, u_1v_2, u_2v_3\}$, where $G_1 \cap G_2 = \emptyset$, the vertices u_1 and u_2 are distinct vertices of G_1, and the vertices v_i are distinct vertices of G_2, $i = 1, 2, 3$.

Denote by H_i the graph obtained by contracting the subgraph G_{i+1} to a single vertex w_i, $i = 1, 2$, and removing the multiple edge in H_1, where subscripts are reduced modulo 2. Since $\deg(w_i) = 2$ or 3, $H_1 \neq K_4$ and $H_i \neq K_6$, $i = 1, 2$. By the minimality of G, H_i has an SOCDC or $H_2 = K_4$. Therefore, H_i has an OCDC, $\mathcal{C}_i$, $i = 1, 2$. Let C_1^1 and C_1^2 be two directed cycles in $\mathcal{C}_1$ which include w_1, where without loss of generality, we assume that C_1^j includes directed path (u_j, w_1, u_{j+1}), where subscripts are reduced modulo 2, $j = 1, 2$, and C_2^k, $k = 1, 2, 3$, be the three directed cycles in $\mathcal{C}_2$ which include w_2, where without loss of generality, we assume that C_2^k includes directed path (v_k, w_2, v_{k-1}), where subscripts are reduced modulo 3, $k = 1, 2, 3$. Let $P_1^j = C_1^j \setminus w_1$, $j = 1, 2$, and $P_2^k = C_2^k \setminus w_2$, $k = 1, 2, 3$. Define $C^1 = P_1^1 \cup P_2^3 \cup \{u_1v_2, v_3u_2\}$, $C^2 = P_1^2 \cup P_2^1 \cup \{u_2v_3, v_1u_1\}$, and $C^3 = P_2^2 \cup \{u_1v_1, v_2u_1\}$. Let $\mathcal{C}' = \{C^1, C^2, C^3\}$, and $\mathcal{C}'' = \{C_1^1, C_1^2, C_2^1, C_2^2, C_2^3\}$. Thus, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}' \setminus \mathcal{C}''$ is an OCDC of G, where $|\mathcal{C}| = |\mathcal{C}_1| + |\mathcal{C}_2| - 2$. Note that every OCDC of K_4 has 4 cycles, therefore, in both cases $|\mathcal{C}| \leq |V(G)| - 1$, which is a contradiction.

(III) $G = G_1 \cup G_2 \cup \{u_1v_1, u_1v_2, u_2v_2\}$, where $G_1 \cap G_2 = \emptyset$, the vertices u_1 and u_2 are distinct vertices of G_1, and the vertices v_1 and v_2 are distinct vertices of G_2.

Denote by H_i the graph obtained by contracting the subgraph G_{i+1} to a single vertex w_i, $i = 1, 2$, and removing the multiple edges, where subscripts are reduced modulo 2. Since $\deg(w_i) = 2$, $H_i \neq K_4$ or K_6, $i = 1, 2$. By the minimality of G, H_i has an SOCDC, $\mathcal{C}_i$, $i = 1, 2$.

Let C_i^j, $j = 1,2$, be the two directed cycles in $\mathcal{C}_i$ which include w_i, $i = 1,2$, where without loss of generality, we assume that C_1^j includes directed path (u_j, w_1, u_{j+1}), and C_2^j includes directed path (v_j, w_2, v_{j+1}), where subscripts are reduced modulo 2, $j = 1,2$. Let $P_i^j = C_i^j \backslash w_i$, $i = 1,2$, $j = 1,2$. Define $C^1 = P_1^1 \cup P_2^2 \cup \{u_1v_1, v_2u_2\}$, $C^2 = P_1^2 \cup \{u_2v_2, v_2u_1\}$, and $C^3 = P_2^1 \cup \{u_1v_2, v_1u_1\}$. Let $\mathcal{C}' = \{C^1, C^2, C^3\}$, and $\mathcal{C}'' = \{C_1^1, C_1^2, C_2^1, C_2^2\}$. Thus, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}' \backslash \mathcal{C}''$ is an OCDC of G, where $|\mathcal{C}| = |\mathcal{C}_1| + |\mathcal{C}_2| - 1$. Therefore, $|\mathcal{C}| \leq |V(G)| - 1$, which is a contradiction. ■

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