AN OPERATOR KARAMATA INEQUALITY

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Abstract. We present an operator version of the Karamata inequality. More precisely, we prove that if \( A \) is a selfadjoint element of a unital \( C^∗ \)-algebra \( A \), \( ρ \) is a state on \( A \), the functions \( f, g \) are continuous on the spectrum \( σ(A) \) of \( A \) such that \( 0 < m_1 ≤ f(s) ≤ M_1 \), \( 0 < m_2 ≤ g(s) ≤ M_2 \) for all \( s ∈ σ(A) \) and \( K = (\sqrt{m_1 m_2} + \sqrt{M_1 M_2}) / (\sqrt{m_1 M_2} + \sqrt{M_1 m_2}) \), then

\[
K^{-2} ≤ \frac{ρ(f(A)g(A))}{ρ(f(A))ρ(g(A))} ≤ K^2.
\]

We also give some applications.

1. INTRODUCTION

The classical Karamata inequality [5] states that if \( f, g \) are integrable real functions on \([0, 1]\) such that \( 0 < m_1 ≤ f ≤ M_1 \) and \( 0 < m_2 ≤ g ≤ M_2 \), then

\[
\left( \frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}} \right)^{-2} ≤ \frac{\int_0^1 f(t)g(t)dt}{\int_0^1 f(t)dt \int_0^1 g(t)dt} ≤ \left( \frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}} \right)^2. \tag{1.1}
\]

The right hand constant is greater than or equal to 1. This may be regarded as a multiplicative converse inequality to the integral analogue of Čebyšev’s inequality. The additive version is known as the Grüss inequality [3] asserting that if \( f \) and \( g \) are integrable real functions on \([0, 1]\) such that \( m_1 ≤ f ≤ M_1 \) and \( m_2 ≤ g ≤ M_2 \) for some real constants \( m_1, M_1, m_2, M_2 \), then

\[
\left| \int_0^1 f(t)g(t)dt - \int_0^1 f(t)dt \int_0^1 g(t)dt \right| ≤ \frac{1}{4}(M_1 - m_1)(M_2 - m_2); \tag{1.2}
\]

and that the constant \( 1/4 \) is the best possible, see [1, 8, 9, 4, 6]. The following discrete version of (1.1) was given by Lupas [7] for positive linear functionals including the integral form of Karamata’s inequality, see also [10]:

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\]
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Theorem. (Lupas). Suppose that $X$ is a real linear space of real functions defined on a bounded interval $[a, b]$ such that the constant function $e(x) = 1$ belongs to it. If $f, g \in X$ such that $0 < m_1 \leq f \leq M_1$ and $0 < m_2 \leq g \leq M_2$ for all $x \in [a, b]$ and $F : X \to \mathbb{R}$ is a positive linear functional with $F(e) = 1$, then

$$
\left( \frac{\sqrt{m_1m_2} + \sqrt{M_1M_2}}{\sqrt{m_1M_2} + \sqrt{M_1m_2}} \right)^2 \leq \frac{F(f)F(g)}{F(fg)} \leq \left( \frac{\sqrt{m_1m_2} + \sqrt{M_1M_2}}{\sqrt{m_1M_2} + \sqrt{M_1m_2}} \right)^2.
$$

In this note, we present an operator version of the Karamata inequality.

2. Main result

We start this section with the following useful lemma.

Lemma 2.1. Let $A$ be a selfadjoint element of a unital $C^*$-algebra $\mathcal{A}$ and $\rho$ be a state on $\mathcal{A}$. Let $f, g$ be continuous functions on the spectrum $\sigma(A)$ of $A$ such that $0 < m_1 \leq f(s) \leq M_1$, $0 < m_2 \leq g(s) \leq M_2$ for all $s \in \sigma(A)$ and let $D(f) = M_1 - \rho(f(A))$ $d(f) = \rho(f(A)) - m_1$. Then

$$
\frac{m_1M_2D(f) + M_1m_2d(f)}{M_2D(f) + m_2d(f)} \leq \frac{\rho(f(A)g(A))}{\rho(g(A))} \leq \frac{M_1M_2d(f) + m_1m_2D(f)}{M_2d(f) + m_2D(f)}.
$$

Proof. For all $s, t \in \sigma(A)$ we have

$$
(M_1 - f(s))(f(t) - m_1)(M_2g(t) - m_2g(s)) \geq 0, \quad (2.1)
$$

$$
(M_1 - f(s))(f(t) - m_1)(M_2g(s) - m_2g(t)) \geq 0. \quad (2.2)
$$

(2.1) is equivalent with

$$
M_1M_2f(t)g(t) - M_1m_2f(t)g(s) - m_1M_1M_2g(t) + m_1m_2M_1g(s) \quad (2.3)
$$

$$
-M_2f(s)f(t)g(t) + m_2f(s)g(s)f(t) + m_1M_2f(s)g(t) - m_1m_2f(s)g(s) \geq 0.
$$

Using the continuous functional calculus and the positivity of the state $\rho$ it follows from (2.3) that

$$
M_1M_2f(t)g(t) - M_1m_2f(t)\rho(g(A)) - m_1M_1M_2g(t) + m_1m_2M_1\rho(g(A))
$$

$$
-M_2\rho(f(A))f(t)g(t) + m_2\rho(f(A)g(A))f(t) + m_1M_2\rho(f(A))g(t)
$$

$$
-m_1m_2\rho(f(A)g(A)) \geq 0. \quad (2.4)
$$
By the same technique we get from (2.4) that
\[
M_1M_2\rho(f(A)g(A)) - M_1m_2\rho(f(A))\rho(g(A)) - m_1M_2\rho(g(A)) \\
+ m_1m_2M_1\rho(g(A)) - M_2\rho(f(A))\rho(f(A)g(A)) + m_2\rho(f(A)g(A))\rho(f(A)) \\
+ m_1M_2\rho(f(A))\rho(g(A)) - m_1m_2\rho(f(A)g(A)) \geq 0,
\]
(2.5)
or equivalently
\[
(M_1M_2 - m_1m_2)\rho(f(A)g(A)) + (m_1M_2 - m_2M_1)\rho(f(A))\rho(g(A)) \\
\geq (M_2 - m_2)\rho(f(A))\rho(f(A)g(A)) + m_1M_1(M_2 - m_2)\rho(g(A)),
\]
that is,
\[
\frac{m_1M_2 D(f) + M_1m_2 d(f)}{M_2D(f) + m_2d(f)} \leq \frac{\rho(f(A)g(A))}{\rho(g(A))}.
\]
Similarly, from (2.2) it follows that
\[
(M_1M_2 - m_1m_2)\rho(f(A))\rho(g(A)) + (m_1M_2 - m_2M_1)\rho(f(A)g(A)) \\
\geq (M_2 - m_2)\rho(f(A))\rho(f(A)g(A)) + m_1M_1(M_2 - m_2)\rho(g(A)),
\]
that is,
\[
\frac{\rho(f(A)g(A))}{\rho(g(A))} \leq \frac{M_1M_2 d(f) + m_1m_2 D(f)}{M_2d(f) + m_2D(f)}.
\]

\[\square\]

**Theorem 2.2.** Let \(A\) be a selfadjoint element of a unital \(C^*\)-algebra \(\mathcal{A}\) and \(\rho\) be a state on \(\mathcal{A}\). Let \(f, g\) be continuous functions on the spectrum \(\sigma(A)\) of \(A\) such that \(0 < m_1 \leq f(s) \leq M_1, 0 < m_2 \leq g(s) \leq M_2\) for all \(s \in \sigma(A)\). If
\[
K = \frac{\sqrt{m_1m_2} + \sqrt{M_1M_2}}{\sqrt{m_1M_2} + \sqrt{M_1m_2}},
\]
then
\[
K^{-2} \leq \frac{\rho(f(A)g(A))}{\rho(f(A))\rho(g(A))} \leq K^2.
\]

**Proof.** Let us define functions \(m, M : [m_1, M_1] \to [0, \infty)\) by
\[
m(t) = \frac{(M_1m_2 - m_1M_2)t + m_1M_1(M_2 - m_2)}{(M_1M_2 - m_1m_2)t - (M_2 - m_2)t^2},
\]
\[
M(t) = \frac{(M_1M_2 - m_1m_2)t - m_1M_1(M_2 - m_2)}{(M_2 - m_2)t^2 + (m_2M_1 - m_1M_2)t}.
\]
If $f$ or $g$ is a constant function, then $K = 1$. Let us assume that $m_i \neq M_i$, $i = 1, 2$. If
\[ t_1 = \frac{\sqrt{m_1 M_1 (\sqrt{m_1 m_2} + \sqrt{M_1 M_2})}}{\sqrt{M_1 m_2} + \sqrt{m_1 M_2}}, \]
\[ t_2 = \frac{\sqrt{m_1 M_1 (\sqrt{M_1 m_2} + \sqrt{m_1 M_2})}}{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}, \]
then $t_i \in [m_1, M_1]$, $i = 1, 2$, and
\begin{align*}
\min_{t \in [m_1, M_1]} m(t) &= m(t_1) = K^{-2}, \\
\max_{t \in [m_1, M_1]} M(t) &= M(t_2) = K^2.
\end{align*}
From Lemma 2.1 we have
\[ m(\rho(f(A))) \leq \frac{\rho(f(A)g(A))}{\rho(f(A))\rho(g(A))} \leq M(\rho(f(A))), \]
where $\rho(f(A)) \in [m_1, M_1]$. So, the theorem is proved. \hfill \Box

As usual, let $\mathbb{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$.

**Corollary 2.3.** Let $A \in \mathbb{B}(\mathcal{H})$ be a selfadjoint operator on a Hilbert space $\mathcal{H}$, $x \in \mathcal{H}$ be a unit vector and $f, g$ be continuous functions on the spectrum $\sigma(A)$ of $A$ such that $0 < m_1 \leq f(s) \leq M_1$, $0 < m_2 \leq g(s) \leq M_2$ for all $s \in \sigma(A)$. Then
\[
\left( \frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{M_1 m_2} + \sqrt{m_1 M_2}} \right)^2 \leq \frac{\langle f(A)g(A)x, x \rangle}{\langle f(A)x, x \rangle \langle g(A)x, x \rangle} \leq \left( \frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{m_1 m_2}} \right)^2.
\]

**Corollary 2.4.** Let $A \in M_n(\mathbb{C})$ be a Hermitian matrix, $f, g$ be continuous real functions on the spectrum $\sigma(A)$ of $A$ such that $0 < m_1 \leq f(s) \leq M_1$, $0 < m_2 \leq g(s) \leq M_2$ for all $s \in \sigma(A)$. Then
\[
\left( \frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{m_1 m_2}} \right)^2 \leq \frac{n \text{Tr}(f(A)g(A))}{\text{Tr}(f(A))\text{Tr}(g(A))} \leq \left( \frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{m_1 m_2}} \right)^2,
\]
where Tr denotes the usual matrix trace.

**Example 2.5.** As consequences of Corollary 2.3, we demonstrate some reverse inequalities of those presented in [2, Examples 1,2,3].

Let $A \in \mathbb{B}(\mathcal{H})$ be a selfadjoint operator on a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$ be a unit vector.
If $A$ is positive definite, $p, q > 0$ and $0 < m_1 \leq s^p \leq M_1$, $0 < m_2 \leq s^q \leq M_2$ for all $s \in \sigma(A)$ then
\[
\left(\frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}\right)^2 \leq \frac{\langle A^{p+q}x, x \rangle}{\langle A^p x, x \rangle \langle A^q x, x \rangle} \leq \left(\frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}\right)^2.
\]

If $\alpha, \beta > 0$ and $0 < m_1 \leq \exp(\alpha s) \leq M_1$, $0 < m_2 \leq \exp(\beta s) \leq M_2$ for all $s \in \sigma(A)$ then
\[
\left(\frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}\right)^2 \leq \frac{\langle \exp[(\alpha + \beta)A]x, x \rangle}{\langle \exp(\alpha A)x, x \rangle \langle \exp(\beta A)x, x \rangle} \leq \left(\frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}\right)^2.
\]

If $A$ is positive definite, $p > 0$ and $0 < m_1 \leq s^p \leq M_1$, $0 < m_2 \leq \log s \leq M_2$ for all $s \in \sigma(A)$ then
\[
\left(\frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}\right)^2 \leq \frac{\langle A^p \log Ax, x \rangle}{\langle A^p x, x \rangle \langle \log Ax, x \rangle} \leq \left(\frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}\right)^2.
\]

3. Applications for multiple elements

In this section we give a version of Theorem 2.2 for multiple elements, according to Dragomir’s technique [2].

As usual, we denote by $M_n(\mathcal{A})$ the $C^*$-algebra of $n \times n$ matrices with entries in $\mathcal{A}$.

**Theorem 3.1.** For $j = 1, 2, \ldots, n$, let $A_j$ be a selfadjoint element of a unital $C^*$-algebra $\mathcal{A}$ with unit $I$, and $\rho_j$ be a bounded positive linear functional on $\mathcal{A}$ such that $\sum_{j=1}^n \rho_j(I) = 1$, and $f, g$ be continuous functions on the spectrum $\sigma(A_j)$ of $A_j$ such that $0 < m_1 \leq f(s) \leq M_1$, $0 < m_2 \leq g(s) \leq M_2$ for all $s \in \sigma(A_j)$. Then
\[
\left(\frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}\right)^2 \leq \frac{\sum_{j=1}^n \rho_j(f(A_j)g(A_j))}{\sum_{j=1}^n \rho_j(f(A_j)) \sum_{j=1}^n \rho_j(g(A_j))} \leq \left(\frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}\right)^2.
\]

**Proof.** We define a positive linear functional on $M_n(\mathcal{A})$ as follows. For $B = (B_{ij}) \in M_n(\mathcal{A})$ with $B_{ij} \in \mathcal{A}$, $i, j = 1, \ldots, n$, put $\rho(B) = \sum_{j=1}^n \rho_j(B_{jj})$. In particular, for $A = A_1 \oplus \cdots \oplus A_n$ one has $\rho(A) = \sum_{j=1}^n \rho_j(A_j)$. It is easily seen that $\rho(f(A)g(A)) = \sum_{j=1}^n \rho_j(f(A_j)g(A_j))$, $\rho(f(A)) = \sum_{j=1}^n \rho_j(f(A_j))$ and $\rho(g(A)) = \sum_{j=1}^n \rho_j(g(A_j))$.

Now, the required inequalities of Theorem 3.1 follow from the inequalities of Theorem 2.2. \ importantes
Corollary 3.2. For $j = 1, 2, \ldots, n$, let $A_j \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator on a Hilbert space $\mathcal{H}$, $x_j \in \mathcal{H}$ be a vector such that $\sum_{j=1}^{n} \|x_j\|^2 = 1$, and $f, g$ be continuous functions on the spectrum $\sigma(A_j)$ of $A_j$ such that $0 < m_1 \leq f(s) \leq M_1$, $0 < m_2 \leq g(s) \leq M_2$ for all $s \in \sigma(A_j)$. Then

$$
\left(\frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}\right)^2 \leq \frac{\sum_{j=1}^{n} \langle f(A_j)g(A_j)x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f(A_j)x_j, x_j \rangle \sum_{j=1}^{n} \langle g(A_j)x_j, x_j \rangle} \leq \left(\frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}\right)^2.
$$

Proof. Apply Theorem 3.1 for the unital $C^*$-algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and positive linear functionals $\rho_j = \langle (\cdot)x_j, x_j \rangle$, $j = 1, 2, \ldots, n$. \hfill \Box

In the forthcoming result, by $\lambda_{\max}(A)$ (resp. $\lambda_{\min}(A)$) we denote the largest (resp. smallest) eigenvalue of a Hermitian matrix $A$. In addition, the symbol $\|\cdot\|$ stands for the spectral norm on $M_n(\mathbb{C})$.

Corollary 3.3. For $j = 1, 2, \ldots, n$, let $A_j \in M_n(\mathbb{C})$ be a Hermitian matrix, $x$ be a unit vector in $\mathbb{C}^n$, and $p_j \geq 0$ be a scalar with $\sum_{j=1}^{m} p_j = 1$, and $f, g$ be continuous functions on the spectrum $\sigma(A_j)$ of $A_j$ such that $0 < m_1 \leq f(s) \leq M_1$, $0 < m_2 \leq g(s) \leq M_2$ for all $s \in \sigma(A_j)$. Then

$$
\frac{\lambda_{\max}\left(\sum_{j=1}^{n} p_j f(A_j)g(A_j)\right)}{\lambda_{\max}\left(\sum_{j=1}^{n} p_j f(A_j)\right) \cdot \lambda_{\max}\left(\sum_{j=1}^{n} p_j g(A_j)\right)} \leq \left(\frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}\right)^2
$$

or equivalently

$$
\frac{\left\|\sum_{j=1}^{n} p_j f(A_j)g(A_j)\right\|}{\left\|\sum_{j=1}^{n} p_j f(A_j)\right\| \cdot \left\|\sum_{j=1}^{n} p_j g(A_j)\right\|} \leq \left(\frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}\right)^2,
$$
and

$$
\left(\frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}\right)^{-2} \leq \frac{\lambda_{\min}\left(\sum_{j=1}^{n} p_j f(A_j)g(A_j)\right)}{\lambda_{\min}\left(\sum_{j=1}^{n} p_j f(A_j)\right) \cdot \lambda_{\min}\left(\sum_{j=1}^{n} p_j g(A_j)\right)}
$$
or equivalently

\[
\frac{\left(\sqrt{m_1m_2} + \sqrt{M_1M_2}\right)^{-2}}{\left(\frac{n}{\sum_{j=1}^{n} p_j f(A_j) g(A_j)}\right)^{-1}} \leq \left\| \frac{n}{\sum_{j=1}^{n} p_j f(A_j) g(A_j)} \right\|^{-1}.
\]

**Proof.** Use Corollary 3.2 and Courant–Fischer’s min-max theorem. \qed

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