AN OPERATOR KARAMATA INEQUALITY

MOHAMMAD SAL MOSLEHIAN [1], MAREK NIEZGODA [2] AND RAJNA RAJIĆ [3]

Abstract. We present an operator version of the Karamata inequality. More precisely, we prove that if A is a selfadjoint element of a unital C^*-algebra $\mathscr{A}$, ρ is a state on $\mathscr{A}$, the functions f, g are continuous on the spectrum $\sigma(A)$ of A such that $0 < m_1 \le f(s) \le M_1$, $0 < m_2 \le g(s) \le M_2$ for all $s \in \sigma(A)$ and $K = \left(\sqrt{m_1 m_2} + \sqrt{M_1 M_2}\right) / \left(\sqrt{m_1 M_2} + \sqrt{M_1 m_2}\right)$, then

$$K^{-2} \le \frac{\rho(f(A)g(A))}{\rho(f(A))\rho(g(A))} \le K^2.$$

We also give some applications.

1. introduction

The classical Karamata inequality [5] states that if f, g are integrable real functions on $[0,1]$ such that $0 < m_1 \le f \le M_1$ and $0 < m_2 \le g \le M_2$, then

$$\left(\frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}\right)^{-2} \le \frac{\int_0^1 f(t)g(t)dt}{\int_0^1 f(t)dt\, \int_0^1 g(t)dt} \le \left(\frac{\sqrt{m_1 m_2} + \sqrt{M_1 M_2}}{\sqrt{m_1 M_2} + \sqrt{M_1 m_2}}\right)^{2}. \tag{1.1}$$

The right hand constant is greater than or equal to 1. This may be regarded as a multiplicative converse inequality to the integral analogue of Čebyšev's inequality. The additive version is known as the Grüss inequality [3] asserting that if f and g are integrable real functions on $[0,1]$ such that $m_1 \le f \le M_1$ and $m_2 \le g \le M_2$ for some real constants m_1, M_1, m_2, M_2, then

$$\left|\int_0^1 f(t)g(t)dt - \int_0^1 f(t)dt \int_0^1 g(t)dt\right| \le \frac{1}{4}(M_1 - m_1)(M_2 - m_2)\,; \tag{1.2}$$

and that the constant 1/4 is the best possible, see [1, 8, 9, 4, 6]. The following discrete version of (1.1) was given by Lupaş [7] for positive linear functionals including the integral form of Karamata's inequality, see also [10]:

2010 *Mathematics Subject Classification.* Primary 47A63; secondary 47B25, 15A60.

Key words and phrases. Karamata inequality, operator inequality, positive operator, unital C^*-algebra, state.

Theorem. (Lupaş). Suppose that X is a real linear space of real functions defined on a bounded interval $[a,b]$ such that the constant function $e(x)=1$ belongs to it. If $f,g\in X$ such that $0<m_1\le f\le M_1$ and $0<m_2\le g\le M_2$ for all $x\in[a,b]$ and $F:X\to\mathbb{R}$ is a positive linear functional with $F(e)=1$, then

$$\left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{-2}\le\frac{F(f)F(g)}{F(fg)}\le\left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{2}.$$

In this note, we present an operator version of the Karamata inequality.

2. Main result

We start this section with the following useful lemma.

Lemma 2.1. *Let A be a selfadjoint element of a unital C^*-algebra $\mathscr{A}$ and ρ be a state on $\mathscr{A}$. Let f,g be continuous functions on the spectrum $\sigma(A)$ of A such that $0<m_1\le f(s)\le M_1$, $0<m_2\le g(s)\le M_2$ for all $s\in\sigma(A)$ and let $D(f)=M_1-\rho(f(A))$ $d(f)=\rho(f(A))-m_1$. Then*

$$\frac{m_1M_2D(f)+M_1m_2d(f)}{M_2D(f)+m_2d(f)}\le\frac{\rho(f(A)g(A))}{\rho(g(A))}\le\frac{M_1M_2d(f)+m_1m_2D(f)}{M_2d(f)+m_2D(f)}.$$

Proof. For all $s,t\in\sigma(A)$ we have

$$(M_1-f(s))(f(t)-m_1)(M_2g(t)-m_2g(s))\ge 0,\tag{2.1}$$

$$(M_1-f(s))(f(t)-m_1)(M_2g(s)-m_2g(t))\ge 0.\tag{2.2}$$

(2.1) is equivalent with

$$M_1M_2f(t)g(t)-M_1m_2f(t)g(s)-m_1M_1M_2g(t)+m_1m_2M_1g(s)\tag{2.3}$$

$$-M_2f(s)f(t)g(t)+m_2f(s)g(s)f(t)+m_1M_2f(s)g(t)-m_1m_2f(s)g(s)\ge 0.$$

Using the continuous functional calculus and the positivity of the state ρ it follows from (2.3) that

$$\begin{aligned}&M_1M_2f(t)g(t)-M_1m_2f(t)\rho(g(A))-m_1M_1M_2g(t)+m_1m_2M_1\rho(g(A))\\&-M_2\rho(f(A))f(t)g(t)+m_2\rho(f(A)g(A))f(t)+m_1M_2\rho(f(A))g(t)\\&-m_1m_2\rho(f(A)g(A))\ge 0.\end{aligned}\tag{2.4}$$

By the same technique we get from (2.4) that

$$\begin{aligned}&M_1M_2\rho(f(A)g(A)) - M_1m_2\rho(f(A))\rho(g(A)) - m_1M_1M_2\rho(g(A))\\&+m_1m_2M_1\rho(g(A)) - M_2\rho(f(A))\rho(f(A)g(A)) + m_2\rho(f(A)g(A))\rho(f(A))\\&+m_1M_2\rho(f(A))\rho(g(A)) - m_1m_2\rho(f(A)g(A)) \geq 0, \qquad (2.5)\end{aligned}$$

or equivalently

$$(M_1M_2 - m_1m_2)\rho(f(A)g(A)) + (m_1M_2 - m_2M_1)\rho(f(A))\rho(g(A))$$

$$\geq (M_2 - m_2)\rho(f(A))\rho(f(A)g(A)) + m_1M_1(M_2 - m_2)\rho(g(A)),$$

that is,

$$\frac{m_1M_2D(f) + M_1m_2d(f)}{M_2D(f) + m_2d(f)} \leq \frac{\rho(f(A)g(A))}{\rho(g(A))}.$$

Similarly, from (2.2) it follows that

$$(M_1M_2 - m_1m_2)\rho(f(A))\rho(g(A)) + (m_1M_2 - m_2M_1)\rho(f(A)g(A))$$

$$\geq (M_2 - m_2)\rho(f(A))\rho(f(A)g(A)) + m_1M_1(M_2 - m_2)\rho(g(A)),$$

that is,

$$\frac{\rho(f(A)g(A))}{\rho(g(A))} \leq \frac{M_1M_2d(f) + m_1m_2D(f)}{M_2d(f) + m_2D(f)}.$$

□

Theorem 2.2. *Let A be a selfadjoint element of a unital C^*-algebra $\mathscr{A}$ and ρ be a state on $\mathscr{A}$. Let f, g be continuous functions on the spectrum $\sigma(A)$ of A such that $0 < m_1 \leq f(s) \leq M_1$, $0 < m_2 \leq g(s) \leq M_2$ for all $s \in \sigma(A)$. If*

$$K = \frac{\sqrt{m_1m_2} + \sqrt{M_1M_2}}{\sqrt{m_1M_2} + \sqrt{M_1m_2}},$$

then

$$K^{-2} \leq \frac{\rho(f(A)g(A))}{\rho(f(A))\rho(g(A))} \leq K^2.$$

Proof. Let us define functions $m, M : [m_1, M_1] \to [0, \infty)$ by

$$m(t) = \frac{(M_1m_2 - m_1M_2)t + m_1M_1(M_2 - m_2)}{(M_1M_2 - m_1m_2)t - (M_2 - m_2)t^2},$$

$$M(t) = \frac{(M_1M_2 - m_1m_2)t - m_1M_1(M_2 - m_2)}{(M_2 - m_2)t^2 + (m_2M_1 - m_1M_2)t}.$$

If f or g is a constant function, then $K=1$. Let us assume that $m_i \neq M_i$, $i=1,2$. If

$$t_1 = \frac{\sqrt{m_1M_1}(\sqrt{m_1m_2}+\sqrt{M_1M_2})}{\sqrt{M_1m_2}+\sqrt{m_1M_2}},$$

$$t_2 = \frac{\sqrt{m_1M_1}(\sqrt{M_1m_2}+\sqrt{m_1M_2})}{\sqrt{m_1m_2}+\sqrt{M_1M_2}},$$

then $t_i \in [m_1, M_1]$, $i=1,2$, and

$$\min_{t\in[m_1,M_1]} m(t) = m(t_1) = K^{-2},$$

$$\max_{t\in[m_1,M_1]} M(t) = M(t_2) = K^{2}.$$

From Lemma 2.1 we have

$$m(\rho(f(A))) \leq \frac{\rho(f(A)g(A))}{\rho(f(A))\rho(g(A))} \leq M(\rho(f(A))),$$

where $\rho(f(A)) \in [m_1, M_1]$. So, the theorem is proved. □

As usual, let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a Hilbert space $\mathscr{H}$.

Corollary 2.3. *Let $A \in \mathbb{B}(\mathscr{H})$ be a selfadjoint operator on a Hilbert space $\mathscr{H}$, $x \in \mathscr{H}$ be a unit vector and f, g be continuous functions on the spectrum $\sigma(A)$ of A such that $0 < m_1 \leq f(s) \leq M_1$, $0 < m_2 \leq g(s) \leq M_2$ for all $s \in \sigma(A)$. Then*

$$\left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{-2} \leq \frac{\langle f(A)g(A)x,x\rangle}{\langle f(A)x,x\rangle\langle g(A)x,x\rangle} \leq \left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{2}.$$

Corollary 2.4. *Let $A \in M_n(\mathbb{C})$ be a Hermitian matrix, f, g be continuous real functions on the spectrum $\sigma(A)$ of A such that $0 < m_1 \leq f(s) \leq M_1$, $0 < m_2 \leq g(s) \leq M_2$ for all $s \in \sigma(A)$. Then*

$$\left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{-2} \leq \frac{n\mathrm{Tr}(f(A)g(A))}{\mathrm{Tr}(f(A))\mathrm{Tr}(g(A))} \leq \left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{2},$$

where Tr *denotes the usual matrix trace.*

Example 2.5. As consequences of Corollary 2.3, we demonstrate some reverse inequalities of those presented in [2, Examples 1,2,3].

Let $A \in \mathbb{B}(\mathscr{H})$ be a selfadjoint operator on a Hilbert space $\mathscr{H}$ and $x \in \mathscr{H}$ be a unit vector.

If A is positive definite, $p,q>0$ and $0<m_1\le s^p\le M_1$, $0<m_2\le s^q\le M_2$ for all $s\in\sigma(A)$ then

$$\left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{-2}\le\frac{\langle A^{p+q}x,x\rangle}{\langle A^px,x\rangle\langle A^qx,x\rangle}\le\left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{2}.$$

If $\alpha,\beta>0$ and $0<m_1\le\exp(\alpha s)\le M_1$, $0<m_2\le\exp(\beta s)\le M_2$ for all $s\in\sigma(A)$ then

$$\left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{-2}\le\frac{\langle\exp[(\alpha+\beta)A]x,x\rangle}{\langle\exp(\alpha A)x,x\rangle\langle\exp(\beta A)x,x\rangle}\le\left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{2}.$$

If A is positive definite, $p>0$ and $0<m_1\le s^p\le M_1$, $0<m_2\le\log s\le M_2$ for all $s\in\sigma(A)$ then

$$\left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{-2}\le\frac{\langle A^p\log Ax,x\rangle}{\langle A^px,x\rangle\langle\log Ax,x\rangle}\le\left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{2}.$$

3. Applications for multiple elements

In this section we give a version of Theorem 2.2 for multiple elements, according to Dragomir's technique [2].

As usual, we denote by $M_n(\mathscr{A})$ the C^*-algebra of $n\times n$ matrices with entries in $\mathscr{A}$.

Theorem 3.1. *For $j=1,2,\ldots,n$, let A_j be a selfadjoint element of a unital C^*-algebra $\mathscr{A}$ with unit I, and ρ_j be a bounded positive linear functional on $\mathscr{A}$ such that $\sum_{j=1}^n\rho_j(I)=1$, and f,g be continuous functions on the spectrum $\sigma(A_j)$ of A_j such that $0<m_1\le f(s)\le M_1$, $0<m_2\le g(s)\le M_2$ for all $s\in\sigma(A_j)$. Then*

$$\left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{-2}\le\frac{\sum\limits_{j=1}^n\rho_j(f(A_j)g(A_j))}{\sum\limits_{j=1}^n\rho_j(f(A_j))\sum\limits_{j=1}^n\rho_j(g(A_j))}\le\left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{2}.$$

Proof. We define a positive linear functional on $M_n(\mathscr{A})$ as follows. For $B=(B_{ij})\in M_n(\mathscr{A})$ with $B_{ij}\in\mathscr{A}$, $i,j=1,\ldots,n$, put $\rho(B)=\sum_{j=1}^n\rho_j(B_{jj})$. In particular, for $A=A_1\oplus\cdots\oplus A_n$ one has $\rho(A)=\sum_{j=1}^n\rho_j(A_j)$. It is easily seen that $\rho(f(A)g(A))=\sum_{j=1}^n\rho_j(f(A_j)g(A_j))$, $\rho(f(A))=\sum_{j=1}^n\rho_j(f(A_j))$ and $\rho(g(A))=\sum_{j=1}^n\rho_j(g(A_j))$.

Now, the required inequalities of Theorem 3.1 follow from the inequalities of Theorem 2.2. $\square$

Corollary 3.2. *For $j = 1, 2, \ldots, n$, let $A_j \in \mathbb{B}(\mathscr{H})$ be a selfadjoint operator on a Hilbert space $\mathscr{H}$, $x_j \in \mathscr{H}$ be a vector such that $\sum_{j=1}^{n} \|x_j\|^2 = 1$, and f, g be continuous functions on the spectrum $\sigma(A_j)$ of A_j such that $0 < m_1 \leq f(s) \leq M_1$, $0 < m_2 \leq g(s) \leq M_2$ for all $s \in \sigma(A_j)$. Then*

$$\left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{-2} \leq \frac{\sum_{j=1}^{n}\langle f(A_j)g(A_j)x_j, x_j\rangle}{\sum_{j=1}^{n}\langle f(A_j)x_j, x_j\rangle \sum_{j=1}^{n}\langle g(A_j)x_j, x_j\rangle} \leq \left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{2}.$$

Proof. Apply Theorem 3.1 for the unital C^*-algebra $\mathscr{A} = \mathbb{B}(\mathscr{H})$ and positive linear functionals $\rho_j = \langle (\cdot)x_j, x_j\rangle$, $j = 1, 2, \ldots, n$. □

In the forthcoming result, by $\lambda_{max}(A)$ (resp. $\lambda_{min}(A)$) we denote the largest (resp. smallest) eigenvalue of a Hermitian matrix A. In addition, the symbol $\|\cdot\|$ stands for the spectral norm on $M_n(\mathbb{C})$.

Corollary 3.3. *For $j = 1, 2, \ldots, n$, let $A_j \in M_n(\mathbb{C})$ be a Hermitian matrix, x be a unit vector in $\mathbb{C}^n$, and $p_j \geq 0$ be a scalar with $\sum_{j=1}^{m} p_j = 1$, and f, g be continuous functions on the spectrum $\sigma(A_j)$ of A_j such that $0 < m_1 \leq f(s) \leq M_1$, $0 < m_2 \leq g(s) \leq M_2$ for all $s \in \sigma(A_j)$. Then*

$$\frac{\lambda_{max}\left(\sum_{j=1}^{n} p_j f(A_j)g(A_j)\right)}{\lambda_{max}\left(\sum_{j=1}^{n} p_j f(A_j)\right)\cdot\lambda_{max}\left(\sum_{j=1}^{n} p_j g(A_j)\right)} \leq \left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{2}$$

or equivalently

$$\frac{\left\|\sum_{j=1}^{n} p_j f(A_j)g(A_j)\right\|}{\left\|\sum_{j=1}^{n} p_j f(A_j)\right\|\cdot\left\|\sum_{j=1}^{n} p_j g(A_j)\right\|} \leq \left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{2},$$

and

$$\left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{-2} \leq \frac{\lambda_{min}\left(\sum_{j=1}^{n} p_j f(A_j)g(A_j)\right)}{\lambda_{min}\left(\sum_{j=1}^{n} p_j f(A_j)\right)\cdot\lambda_{min}\left(\sum_{j=1}^{n} p_j g(A_j)\right)}$$

or equivalently

$$\left(\frac{\sqrt{m_1m_2}+\sqrt{M_1M_2}}{\sqrt{m_1M_2}+\sqrt{M_1m_2}}\right)^{-2} \leq \frac{\left\|\left(\sum\limits_{j=1}^{n} p_jf(A_j)g(A_j)\right)^{-1}\right\|^{-1}}{\left\|\left(\sum\limits_{j=1}^{n} p_jf(A_j)\right)^{-1}\right\|^{-1}\cdot\left\|\left(\sum\limits_{j=1}^{n} p_jg(A_j)\right)^{-1}\right\|^{-1}}.$$

Proof. Use Corollary 3.2 and Courant–Fischer's min-max theorem. □

Acknowledgement. The first author was supported by a grant from Ferdowsi University of Mashhad (No. MP90249MOS).

References

[1] S.S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces,* Nova Science Publishers, New York, 2005.

[2] S.S. Dragomir, *Čebyšev's type inequalities for functions of selfadjoint operators in Hilbert spaces*, Linear and Multilinear Algebra **58** (2010), no. 7, 805–814.

[3] G. Grüss, *Über das Maximum des absoluten Betrages von* $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$, Math. Z. **39** (1935), 215–226.

[4] D.R. Jocić, D. Krtinić and M.S. Moslehian, *Landau and Grüss type inequalities for inner product type integral transformers in norm ideals*, Math. Inequal. Appl. **16** (2013), no. 1, 109–125.

[5] J. Karamata, *Inégalités relatives aux quotients et à la difference de* $\int fg$ *et* $\int f\int g$, Acad. Serbe Sci. Publ. Inst. Math. **2** (1948), 131–145.

[6] M. Krnić, N. Lovricevic and J. Pečarić, *Jensen's operator and applications to mean inequalities for operators in Hilbert space*, Bull. Malays. Math. Sci. Soc. (2) **35** (2012), no. 1, 1–14.

[7] A. Lupaş, *On two inequalities of Karamata*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 602–633 (1978), 119–123 (1979).

[8] D.S. Mitrinović, J. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic, Dordrecht, 1993.

[9] M.S. Moslehian and R. Rajić, *A Grüss inequality for n-positive linear maps*, Linear Algebra Appl. **433** (2010), no. 8-10, 1555–1560.

[10] J. Pečarić and I. Perić, *Note on Grüss type inequality*, Math. Inequal. Appl. **8** (2005), no. 2, 233–236.

[1] Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran.

E-mail address: moslehian@um.ac.ir and moslehian@member.ams.org

[2] Department of Applied Mathematics and Computer Science, University of Life Sciences in Lublin, Akademicka 13, 20-950 Lublin, Poland

E-mail address: marek.niezgoda@up.lublin.pl

[3] Faculty of Mining, Geology and Petroleum Engineering, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia

E-mail address: rajna.rajic@zg.t-com.hr