Linear combinations of univalent harmonic mappings convex in the direction of the imaginary axis

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Abstract. In the present paper, we introduce a family of univalent harmonic mappings, which map the unit disk onto domains convex in the direction of the imaginary axis. We find conditions for the linear combinations of mappings from this family to be univalent and convex in the direction of the imaginary axis. Linear combinations of functions from this family and harmonic mappings obtained by shearing of analytic vertical strip mappings are also studied.

1 Introduction

A complex-valued continuous function $f = u + iv$ is said to be harmonic in the open unit disk $E = \{z : |z| < 1\}$ if both u and v are real-valued harmonic functions in E. Such harmonic mappings have canonical decomposition $f = h + \overline{g}$, where h is known as the analytic and g the co-analytic part of f. A harmonic mapping $f = h + \overline{g}$ defined in E, is locally univalent and sense-preserving if and only if $h' \neq 0$ in E and the dilatation function ω, defined by $\omega = g'/h'$, satisfies $|\omega| < 1$ in E. The class of all harmonic, univalent and sense-preserving mappings $f = h + \overline{g}$ in E and normalized by the conditions $f(0) = 0$ and $f_z(0) = 1$ is denoted by S_H. Therefore, a function $f = h + \overline{g}$ in the class S_H has the representation,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n, \tag{1}$$

for all z in E. The class of functions of the type (1) with $b_1 = f_{\overline{z}}(0) = 0$ is a subclass of S_H and is generally denoted by S_H^0.

A domain Ω is said to be convex in the direction ϕ, $0 \leq \phi < \pi$, if every line parallel to the line through 0 and $e^{i\phi}$ has either connected or empty intersection with Ω. The following result due to Hengartner and Schober [3] is very useful in checking the convexity of an analytic function in the direction of the imaginary axis.

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Lemma 1.1. *Suppose f is analytic and non-constant in E. Then*

$$\Re[(1-z^2)f'(z)] \geq 0,\ z \in E$$

if and only if

(i) f is univalent in E;

(ii) f is convex in the direction of the imaginary axis;

(iii) there exist sequences $\{z'_n\}$ and $\{z''_n\}$ converging to $z=1$ and $z=-1$, respectively, such that

$$lim_{n\to\infty}\ \Re(f(z'_n)) = sup_{|z|<1}\ \Re(f(z)),$$

$$lim_{n\to\infty}\ \Re(f(z''_n)) = inf_{|z|<1}\ \Re(f(z)). \tag{2}$$

Construction of univalent harmonic mappings is neither a very easy nor a straight forward task. In 1984, Clunie and Sheil-Small [1] introduced a method, known as shear construction or shearing, for constructing a univalent harmonic mapping from a related conformal mapping. The following result of Clunie and Sheil-Small [1] is fundamental for constructing a univalent harmonic mapping convex in a given direction.

Lemma 1.2. *A locally univalent harmonic mapping $f = h + \overline{g}$ in E is a univalent harmonic mapping of E onto a domain convex in the direction ϕ if and only if $h - e^{2i\phi}g$ is a univalent analytic mapping of E onto a domain convex in the direction ϕ.*

Another way of constructing desired univalent harmonic mappings is by taking the linear combination of two suitable harmonic mappings. For example in the following result Dorff [2] identified two suitable harmonic mappings f_1 and f_2 whose linear combination is univalent and convex in the direction of the imaginary axis.

Theorem 1.3. *Let $f_1 = h_1 + \overline{g}_1$ and $f_2 = h_2 + \overline{g}_2$ be two univalent harmonic mappings convex in the direction of the imaginary axis with $\omega_1 = \omega_2$, where $\omega_1 = g'_1/h'_1$ and $\omega_2 = g'_2/h'_2$ are dilatation functions of f_1 and f_2 respectively. If f_1 and f_2 satisfy the conditions (2) above, then $f_3 = tf_1 + (1-t)f_2$, $0 \leq t \leq 1$, is univalent and convex in the direction of the imaginary axis.*

In a recent paper, Wang et al.[6] derived several sufficient conditions on univalent harmonic mappings f_1 and f_2 so that their linear combination $f_3 = tf_1 + (1-t)f_2$, $0 \leq t \leq 1$, is univalent and convex in the direction of the real axis. In particular they established:

Theorem 1.4. *Let $f_j = h_j + \overline{g}_j \in S_H$ with $h_j(z) + g_j(z) = z/(1-z)$ for $j = 1, 2$. Then $f_3 = tf_1 + (1-t)f_2$, $0 \leq t \leq 1$, is univalent and convex in the direction of the real axis.*

From the above two papers it is observed that dilatation functions of f_1 and f_2 play an important role in deciding the behavior of their linear combinations. In the present paper, our aim is to study linear combinations of mappings from a family of locally univalent and sense-preserving harmonic mappings $f_\alpha = h_\alpha + \overline{g}_\alpha$, obtained by shearing of $F_\alpha(z) = h_\alpha(z) + g_\alpha(z) = z(1-\alpha z)/(1-z^2)$, $\alpha \in [-1,1]$ when suitable dilatations $\omega = g'_\alpha/h'_\alpha$ are given. Linear combinations of f_α and f_θ are also studied, where $f_\theta = h_\theta + \overline{g}_\theta$ is the harmonic mapping obtained by shearing of analytic vertical strip mapping

$$h_\theta(z) + g_\theta(z) = \frac{1}{2i\,\sin\theta}\log\left(\frac{1+ze^{i\theta}}{1+ze^{-i\theta}}\right), \theta \in (0,\pi), \tag{3}$$

when a dilatation $\omega = g'_\theta/h'_\theta$, with $|\omega| < 1$, is given.

2 Main Results

Let

$$f_\alpha = h_\alpha + \overline{g}_\alpha, \text{ where } F_\alpha(z) = h_\alpha(z) + g_\alpha(z) = \frac{z(1-\alpha z)}{1-z^2},\ \alpha \in [-1,1] \ \textit{and} \ \left|\frac{g'_\alpha}{h'_\alpha}\right| < 1,$$

be a normalized , locally univalent and sense-preserving harmonic mapping in E. We first prove that f_α is in S_H and convex in the direction of the imaginary axis. Since

$$\Re[(1-z^2)F'_\alpha(z)] = \Re\left[\frac{1+z^2-2\alpha z}{(1-z^2)}\right] = \frac{(1-|z|^2)(1+|z|^2-2\alpha\Re(z))}{|1-z^2|^2} > 0 \text{ for all } z \in E, \tag{4}$$

therefore, in view of Lemma 1.1, the analytic function $F_\alpha = h_\alpha + g_\alpha$ is univalent in E and convex in the direction of the imaginary axis. Consequently, by Lemma 1.2, the harmonic mapping $f_\alpha = h_\alpha + \overline{g}_\alpha$ is in S_H and also convex in the direction of the imaginary axis. However the harmonic mappings $f_\alpha = h_\alpha + \overline{g}_\alpha$, $\alpha \in [-1,1]$, may not be convex in the direction of the real axis, in general (e.g. take $\alpha = -0.5$ and the dilatation $\omega(z) = g'(z)/h'(z) = -z^2$).

In the following result we show that for the linear combination of f_{α_1} and f_{α_2} to be in S_H and convex in the direction of the imaginary axis it is sufficient that the linear combination is locally univalent and sense-preserving.

Theorem 2.1. *Let $f_{\alpha_i} = h_{\alpha_i} + \overline{g}_{\alpha_i}$, where $h_{\alpha_i}(z) + g_{\alpha_i}(z) = z(1-\alpha_i z)/(1-z^2)$, $\alpha_i \in [-1,1]$ and $|g'_{\alpha_i}/h_{\alpha'_i}| < 1$ in E , $i = 1,2$, be two normalized harmonic mappings. Then the mapping $f = tf_{\alpha_1} + (1-t)f_{\alpha_2}$,$0 \le t \le 1$, is in S_H and is convex in the direction of the imaginary axis, provided f is locally univalent and sense-preserving.*

Proof. Let $f = h + \overline{g}$ and $h + g = F$. Then it is easy to verify that $F = tF_{\alpha_1} + (1-t)F_{\alpha_2}$, where $F_{\alpha_i} = h_{\alpha_i} + g_{\alpha_i}$ $(i = 1,2)$. Using (4), we immediately get

$$\Re\big[(1-z^2)F'(z)\big] = t\Re\big[(1-z^2)F'_{\alpha_1}(z)\big] + (1-t)\Re\big[(1-z^2)F'_{\alpha_2}(z)\big] > 0, \text{ for all } z \in E.$$

Thus F is analytic univalent and convex in the direction of the imaginary axis, by Lemma 1.1. Therefore if $f = h + \overline{g}$ is locally univalent and sense-preserving in E, then, in view of Lemma 1.2, $f \in S_H$ and maps E onto a domain convex in the direction of the imaginary axis. $\square$

We know that $f = h + \overline{g}$ will be locally univalent and sense-preserving if and only if $h' \neq 0$ in E and its dilatation function ω satisfies $|\omega| < 1$, in E. So, we first find expression for ω.

Theorem 2.2. *Let $f_{\alpha_i} = h_{\alpha_i} + \overline{g}_{\alpha_i}, i = 1,2,$ be two normalized harmonic mappings such that $h_{\alpha_i}(z) + g_{\alpha_i}(z) = z(1-\alpha_i z)/(1-z^2)$, $\alpha_i \in [-1,1]$ and $\omega_i = g'_{\alpha_i}/h'_{\alpha_i}(|\omega_i| < 1\, in\, E), for\ i = 1,2.$ Then the dilatation function ω of $f = tf_{\alpha_1} + (1-t)f_{\alpha_2}$, $0 \leq t \leq 1$, is given by*

$$\omega(z) = \left[\frac{(1+z^2)(t\omega_1 + (1-t)\omega_2 + \omega_1\omega_2) - 2z(\alpha_1 t\omega_1 + \alpha_1 t\omega_1\omega_2 + (1-t)\omega_2\alpha_2 + (1-t)\omega_1\omega_2\alpha_2)}{(1+z^2)(1+t\omega_2 + (1-t)\omega_1) - 2z(\alpha_2 + \alpha_1 t\omega_2 + (1-t)\alpha_2\omega_1 + \alpha_1 t - \alpha_2 t)}\right]. \tag{5}$$

Proof. As $f = tf_{\alpha_1} + (1-t)f_{\alpha_2} = th_{\alpha_1} + (1-t)h_{\alpha_2} + t\overline{g}_{\alpha_1} + (1-t)\overline{g}_{\alpha_2}$ so,

$$\omega = \frac{tg'_{\alpha_1} + (1-t)g'_{\alpha_2}}{th'_{\alpha_1} + (1-t)h'_{\alpha_2}} = \frac{t\omega_1 h'_{\alpha_1} + (1-t)\omega_2 h'_{\alpha_2}}{th'_{\alpha_1} + (1-t)h'_{\alpha_2}}.$$

From $h_{\alpha_i}(z) + g_{\alpha_i}(z) = \dfrac{z(1-\alpha_i z)}{1-z^2}$ and $\omega_i = \dfrac{g'_{\alpha_i}}{h'_{\alpha_i}}$, $i = 1,2$, we get

$$h'_{\alpha_1}(z) = \frac{1+z^2-2\alpha_1 z}{(1+\omega_1)(1-z^2)^2} \quad \text{and} \quad h'_{\alpha_2}(z) = \frac{1+z^2-2\alpha_2 z}{(1+\omega_2)(1-z^2)^2}.$$

Substituting these values of h'_{α_1} and h'_{α_2} into the above expression for ω we get

$$\omega(z) = \frac{t\omega_1(1+z^2-2\alpha_1 z)(1+\omega_2) + (1-t)\omega_2(1+z^2-2\alpha_2 z)(1+\omega_1)}{t(1+\omega_2)(1+z^2-2\alpha_1 z) + (1-t)(1+\omega_1)(1+z^2-2\alpha_2 z)},$$

which reduces to (5) after rearrangement of terms in the numerator and denominator. $\square$

Theorem 2.3. *For $i = 1,2$, let $f_i = h_i + \overline{g}_i$ be two normalized locally univalent harmonic mappings such that $h_i(z) + g_i(z) = z(1-\alpha z)/(1-z^2)$, $\alpha \in [-1,1]$. Then $f = tf_1 + (1-t)f_2$, $0 \leq t \leq 1$, is in S_H and is convex in the direction of the imaginary axis.*

Proof. In view of Theorem 2.1, it suffices to show that f is locally univalent and sense-preserving. Let $\omega_i = g'_i/h'_i, i = 1,2$ be the dilatation functions of $f_i, i = 1,2$, respectively and ω be the dilatation function of f. If $\omega_1 = \omega_2$ in E, then there is nothing to prove. So we take $\omega_1 \neq \omega_2$ in E. By setting $\alpha_1 = \alpha_2 = \alpha$ in (5), we get

$$\omega = \frac{t\omega_1 + (1-t)\omega_2 + \omega_1\omega_2}{1 + t\omega_2 + (1-t)\omega_1}.$$

From the proof of Theorem 3 in [6], we get $|\omega| < 1$. Hence f is locally univalent and sense-preserving. $\square$

By taking $\alpha = -1$ in Theorem 2.3, we get the following result.

Corollary 2.4. *For $i = 1, 2$, let $f_i = h_i + \overline{g}_i$ be two normalized locally univalent harmonic mappings such that $h_i(z) + g_i(z) = z/(1 - z)$. Then $f = tf_1 + (1 - t)f_2$, $0 \le t \le 1$, is in S_H and is convex in the direction of the imaginary axis.*

Michalski [4], defined the class $COD_H(\theta)$ consisting of functions $f \in S_H$, which map the unit disk E onto domains convex in directions of the lines $z = te^{i\theta}, t \in \mathbb{R}$ and $z = te^{i(\theta+\frac{\pi}{2})}, t \in \mathbb{R}$ for each $\theta \in [0, \pi/2)$. Combining results of Theorem 1.4 and Corollary 2.4, we immediately get the following result.

Theorem 2.5. *Let $f_i = h_i + \overline{g}_i, i = 1, 2$, be two normalized harmonic mappings, where $h_i(z) + g_i(z) = z/(1 - z)$ and $|g_i'/h_i'| < 1$ in E for $i = 1, 2$. Then, $f = tf_1 + (1 - t)f_2$, $0 \le t \le 1$, is in $COD_H(0)$.*

The following lemma, popularly known as Cohn's Rule, will be required in proving our next result.

Lemma 2.6. *([5, p.375]) Given a polynomial $p(z) = a_0 + a_1z + a_2z^2 + ... + a_nz^n$ of degree n, let*

$$p^*(z) = z^n\overline{p\left(\frac{1}{\overline{z}}\right)} = \overline{a}_n + \overline{a}_{n-1}z + \overline{a}_{n-2}z^2 + ... + \overline{a}_0z^n.$$

Denote by r and s the number of zeros of p inside and on the unit circle $|z| = 1$, respectively. If $|a_0| < |a_n|$, then

$$p_1(z) = \frac{\overline{a}_n p(z) - a_0 p^*(z)}{z}$$

is of degree $n - 1$ and has $r_1 = r - 1$ and $s_1 = s$ number of zeros inside and on the unit circle $|z| = 1$, respectively.

We now prove the following.

Theorem 2.7. *Let $f_{\alpha_i} = h_{\alpha_i} + \overline{g}_{\alpha_i}, i = 1, 2$, be two normalized harmonic mappings such that $h_{\alpha_i}(z) + g_{\alpha_i}(z) = z(1 - \alpha_i z)/(1 - z^2)$, $\alpha_i \in [-1, 1]$, for $i = 1, 2$. If $\omega_1(z) = -z$ and $\omega_2(z) = z$ are dilatations of f_{α_1} and f_{α_2} respectively, then $f = tf_{\alpha_1} + (1 - t)f_{\alpha_2}$, $0 \le t \le 1$, is in S_H and is convex in the direction of the imaginary axis provided $\alpha_1 \ge \alpha_2$.*

Proof. In view of Theorem 2.1, it is sufficient to show that dilatation ω of f satisfies $|\omega| < 1$ in E. By using the shearing technique, we explicitly get h_{α_i} and g_{α_i}, $i = 1, 2$, as follows:

$$h_{\alpha_1}(z) = \frac{(1 - \alpha_1)}{4(1 - z)^2} - \frac{(1 + \alpha_1)}{4(1 + z)} + \frac{(1 + \alpha_1)}{8} \log\left[\frac{1 + z}{1 - z}\right] + \frac{\alpha_1}{2},$$

$$g_{\alpha_1}(z) = \frac{z(1-\alpha_1 z)}{1-z^2} - \frac{(1-\alpha_1)}{4(1-z)^2} + \frac{(1+\alpha_1)}{4(1+z)} - \frac{(1+\alpha_1)}{8}\log\left[\frac{1+z}{1-z}\right] - \frac{\alpha_1}{2};$$

and

$$h_{\alpha_2}(z) = \frac{(1-\alpha_2)}{4(1-z)} - \frac{(1+\alpha_2)}{4(1+z)^2} + \frac{(1-\alpha_2)}{8}\log\left[\frac{1+z}{1-z}\right] + \frac{\alpha_2}{2},$$

$$g_{\alpha_2}(z) = \frac{z(1-\alpha_2 z)}{1-z^2} - \frac{(1-\alpha_2)}{4(1-z)} + \frac{(1+\alpha_2)}{4(1+z)^2} - \frac{(1-\alpha_2)}{8}\log\left[\frac{1+z}{1-z}\right] - \frac{\alpha_2}{2}.$$

The case when $\alpha_1 = \alpha_2$ follows from Theorem 2.3. So, we shall only consider the case when $\alpha_1 > \alpha_2$. Setting $\omega_1(z) = -z$ and $\omega_2(z) = z$ in (5) we get

$$\begin{aligned}\omega(z) &= \left[\frac{(1+z^2)(-tz+(1-t)z-z^2) - 2z(-\alpha_1 tz - \alpha_1 tz^2 + (1-t)\alpha_2 z - (1-t)\alpha_2 z^2)}{(1+z^2)(1+tz-(1-t)z) - 2z(\alpha_2 + \alpha_1 tz - (1-t)\alpha_2 z + \alpha_1 t - \alpha_2 t)}\right]\\ &= -z\frac{[z^3 + (2t-1-2\alpha_1 t - 2\alpha_2(1-t))z^2 + (1+2\alpha_2(1-t)-2\alpha_1 t)z + (2t-1)]}{[(2t-1)z^3 + (1+2\alpha_2(1-t)-2\alpha_1 t)z^2 + (2t-1-2\alpha_1 t-2\alpha_2(1-t))z+1]}. \qquad (6)\end{aligned}$$

Let

$$\begin{aligned}\gamma(z) &= z^3 + (2t-1-2\alpha_1 t-2\alpha_2(1-t))z^2 + (1+2\alpha_2(1-t)-2\alpha_1 t)z + (2t-1)\\ &= a_3 z^3 + a_2 z^2 + a_1 z + a_0\end{aligned}$$

and

$$\gamma^*(z) = (2t-1)z^3 + (1+2\alpha_2(1-t)-2\alpha_1 t)z^2 + (2t-1-2\alpha_1 t-2\alpha_2(1-t))z + 1 = z^3\overline{\gamma\left(\frac{1}{\overline{z}}\right)}$$

and notice that by (6), $\omega(z) = -z\dfrac{\gamma(z)}{\gamma^*(z)}$.
Thus if $z_0 \neq 0$ is a zero of γ then $1/\overline{z_0}$ is a zero of γ^*. Therefore, we can write

$$\omega(z) = -z\frac{(z+A)(z+B)(z+C)}{(1+\overline{A}z)(1+\overline{B}z)(1+\overline{C}z)}.$$

For $|\beta| \le 1$, the function $\phi(z) = \dfrac{z+\beta}{1+\overline{\beta}z}$ maps $\overline{E} = \{z : |z| \le 1\}$, onto $\overline{E}$. So, to prove that $|\omega| < 1$ in E, it suffices to show that $|A| \le 1$, $|B| \le 1$ and $|C| \le 1$. We take $t \in (0, 1/2)\cup(1/2, 1)$, as the cases when $t = 0$ or $t = 1$ are trivial and the case when $t = 1/2$ will be dealt separately. As $|a_0| = 2t-1 < 1 = |a_3|$, therefore, by applying Cohn's rule on γ, it is sufficient to show that all zeros of γ_1 lie inside or on $|z| = 1$, where,

$$\begin{aligned}\gamma_1(z) &= \frac{\overline{a}_3\gamma(z) - a_0\gamma^*(z)}{z}\\ &= 4t(1-t)\left[z^2 - (\alpha_1+\alpha_2)z - (\alpha_1-\alpha_2-1)\right] \qquad (7)\\ &= b_2 z^2 + b_1 z + b_0.\end{aligned}$$

Now, if $\alpha_1 = 1$ and $\alpha_2 = -1$, then both the zeros of γ_1 lie on the circle $|z| = 1$ and otherwise, if $\alpha_1 - \alpha_2 > 0$, we have $|b_0| < |b_2|$ because $\alpha_1, \alpha_2 \in (-1, 1)$ and $4t(1-t) \neq 0$ for $t \in (0, 1/2)\cup(1/2, 1)$. Again, by applying Cohn's rule on γ_1, we need to show that all zeros of γ_2 lie inside or on $|z| = 1$, where

$$\gamma_2(z) = \frac{\overline{b}_2\gamma_1(z) - b_0\gamma_1^*(z)}{z}$$

$$= (4t(1-t))^2(\alpha_1 - \alpha_2)[(2 - \alpha_1 + \alpha_2)z - (\alpha_1 + \alpha_2)]$$

and $\gamma_1^*(z) = z^2\gamma_1\left(\frac{1}{z}\right)$.

If z_2 is the zero of γ_2 then $|z_2| \le 1$ is equivalent to $(1-\alpha_1)(1+\alpha_2) \ge 0$ which is true as $|\alpha_i| \le 1$ for $i = 1, 2$. Hence zeros of γ_1 and γ both lie in or on the unit circle $|z| = 1$.

In the case $t = 1/2$ we observe that

$$\gamma(z) = z[z^2 - (\alpha_1 + \alpha_2)z - (\alpha_1 - \alpha_2 - 1)].$$

In view of (7) we can easily verify that all the zeros of γ lie in or on the unit circle $|z| = 1$. Hence the result is proved. □

The result of Theorem 2.7 is illustrated in the following figures by choosing particular values of α_1 and α_2. Images of E under f_{α_1}, f_{α_2} and f are shown in Figure 1, Figure 2 and Figure 3, respectively.

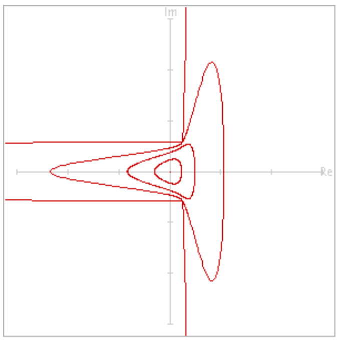

Figure 1: Image of E under f_{α_1} for $\alpha_1 = 0.5$

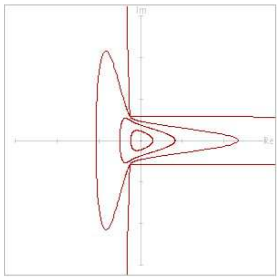

Figure 2: Image of E under f_{α_2} for $\alpha_2 = -0.5$

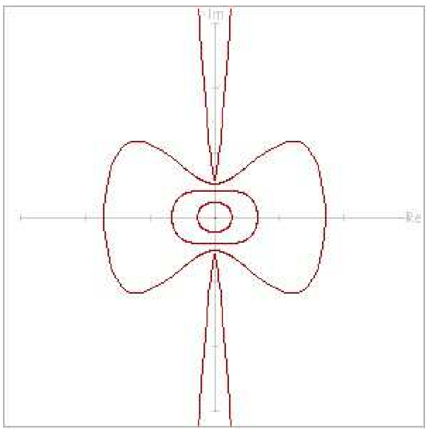

Figure 3: Image of E under f, for $\alpha_1 = 0.5$, $\alpha_2 = -0.5$ and $t = \frac{1}{4}$.

Remark 2.8. Note that in Theorem 2.7 it is not possible to take $\alpha_2 > \alpha_1$ because in that case it will then follow from (7) that the modulus of the product of zeros of γ_1 is $|1 + \alpha_2 - \alpha_1|$ which is strictly greater than 1. Hence at least one zero of γ_1 and therefore of γ shall lie outside $|z| = 1$ implying that there will exist some $z \in E$ for which $|\omega(z)| \nless 1$ i.e, linear combination of f_{α_1} and f_{α_2} shall no longer remain locally univalent and sense-preserving.

Theorem 2.9. *Let f_{α_1} be thesame as in Theorem 2.7 and let $f_{\alpha_2} = h_{\alpha_2} + \overline{g}_{\alpha_2}$ be such that $h_{\alpha_2}(z) + g_{\alpha_2}(z) = z(1-\alpha_2 z)/(1-z^2)$, $\alpha_2 \in [-1,1]$ with dilatation ω_2 ($|\omega_2| < 1\, in\, E$). Let $f = tf_{\alpha_1} + (1-t)f_{\alpha_2}$, $0 \leq t \leq 1$, then we have the following:*
(i) If $\omega_2(z) = -z^2$ and $\alpha_1 \geq \alpha_2$, then f is in S_H and is convex in the direction of the imaginary axis.
(ii) If $\omega_2(z) = z^2$, $|\alpha_1| \geq |\alpha_2|$ and $\alpha_1\alpha_2 \geq 0$, then f is in S_H and is convex in the direction of the imaginary axis.

As the proof runs on the same lines as that of Theorem 2.7, it is omitted here.

Remark 2.10. If we take $\omega_2(z) = z^3$ in the above theorem, then we observe that f may not be locally univalent and sense-preserving. For $t = 3/4$ if we set $\alpha_1 = 0.4$ and $\alpha_2 = 0.3$ or $\alpha_1 = 0.3$ and $\alpha_2 = 0.6$, it can be easily verified that $|\omega_3| \nless 1$ in E, where ω_3 is the dilatation of f.

Remark 2.11. In [6], Wang et al. proved the following theorem:
Let $f_i = h_i + \overline{g}_i \in S_H (i = 1, 2)$ be univalent harmonic mappings convex in the direction of the real axis. Suppose also that $\Re\left((1 - \omega_1\overline{\omega_2})h_1'\overline{h_2'}\right) \geq 0$. Then $f = tf_1 + (1-t)f_2 \in S_H$, $0 \leq t \leq 1$ is convex in the direction of the real axis.
Proceeding on the same lines as in the above theorem of Wang et al. we obtain:
Let $f_{\alpha_i} = h_{\alpha_i} + \overline{g}_{\alpha_i}$ be such that $h_{\alpha_i}(z) + g_{\alpha_i}(z) = z(1-\alpha_i z)/(1-z^2)$, $\alpha_i \in [-1,1]$ for $i = 1,2$ and let $\omega_i = g'_{\alpha_i}/h'_{\alpha_i}$, $i = 1,2$, be dilatation functions of f_{α_i}, $i = 1,2$, respectively. Then $f = tf_{\alpha_1} + (1-t)f_{\alpha_2}$, $0 \leq t \leq 1$, is in S_H and convex in the direction of the imaginary axis if $\Re\left((1 - \omega_1\overline{\omega_2})h_1'\overline{h_2'}\right) \geq 0$.

We end this paper by considering one of the harmonic mappings involved in the linear combination obtained by shearing of analytic strip mapping (3).

Theorem 2.12. *Let $f_\theta = h_\theta + \overline{g}_\theta$, where $h_\theta(z) + g_\theta(z) = \dfrac{1}{2i\sin\theta}\log\left(\dfrac{1+ze^{i\theta}}{1+ze^{-i\theta}}\right)$, $\theta \in (0,\pi)$ with $|g_\theta'/h_\theta'| < 1$ in E and $f_\alpha = h_\alpha + \overline{g}_\alpha$, where $h_\alpha(z) + g_\alpha(z) = z(1-\alpha z)/(1-z^2)$, $\alpha \in [-1,1]$ with $|g_\alpha'/h_\alpha'| < 1$ in E, be two given normalized harmonic mappings. Then $f_{\theta,\alpha} = tf_\theta + (1-t)f_\alpha$, $0 \leq t \leq 1$, is in S_H and is convex in the direction of the imaginary axis provided $f_{\theta,\alpha}$ is locally univalent and sense-preserving.*

Proof. In view of the proof of Theorem 2.1 and (4), we need only to show that $\Re\left[(1-z^2)F_\theta'(z)\right] >$

0, where $F_\theta = h_\theta + g_\theta$. Let

$$\phi(z) = (1-z^2)F'_\theta(z) = \frac{1-z^2}{(1+ze^{i\theta})(1+ze^{-i\theta})}.$$

Since $\phi(0) = 1$ and for each $\gamma \in \mathbb{R}$, $\Re[\phi(e^{i\gamma})] = 0$, by the Minimum Principle for harmonic functions, we have $\Re[\phi(z)] = \Re\left[(1-z^2)F'_\theta(z)\right] > 0$, for $z \in E$. Hence we obtain our result. $\square$

The following example illustrates the result of the above theorem.

Example 2.13. Let $f_\theta = h_\theta + \overline{g}_\theta$ be the harmonic mapping considered in Theorem 2.12 with $\theta = \pi/2$ and $\omega_1(z) = g'_\theta(z)/h'_\theta(z) = -z$. Take $f_\alpha = h_\alpha + \overline{g}_\alpha$ such that $h_\alpha(z) + g_\alpha(z) = z/(1-z)$ and $\omega_2(z) = g'_\alpha(z)/h'_\alpha(z) = z^2$. By shearing we get,

$$h_\theta(z) = \frac{1}{2}\tan^{-1} z - \frac{1}{2}\log(1-z) + \frac{1}{4}\log(1+z^2), \qquad g_\theta(z) = \frac{1}{2}\tan^{-1} z + \frac{1}{2}\log(1-z) - \frac{1}{4}\log(1+z^2);$$

and

$$h_\alpha(z) = \frac{z}{2(1-z)} - \frac{1}{2}\log(1-z) + \frac{1}{4}\log(1+z^2), \qquad g_\alpha(z) = \frac{z}{2(1-z)} + \frac{1}{2}\log(1-z) - \frac{1}{4}\log(1+z^2).$$

Now if ω is the dilatation of $f_{\theta,\alpha} = tf_\theta + (1-t)f_\alpha$, $0 \le t \le 1$, then,

$$|\omega| = \left|\frac{tg'_\theta + (1-t)g'_\alpha}{th'_\theta + (1-t)h'_\alpha}\right| = \left|\frac{t\omega_1 h'_\theta + (1-t)\omega_2 h'_\alpha}{th'_\theta + (1-t)h'_\alpha}\right| = \left|\frac{z(z-t)}{1-tz}\right| < 1.$$

This implies that $f_{\theta,\alpha}$ is locally univalent and sense-preserving in E. So, in view of Theorem 2.12, $f_{\theta,\alpha} \in S_H$ and is convex in the direction of the imaginary axis.
Images of E under f_θ, f_α and $f_{\theta,\alpha}$ are shown in Figure 4, Figure 5 and Figure 6, respectively.

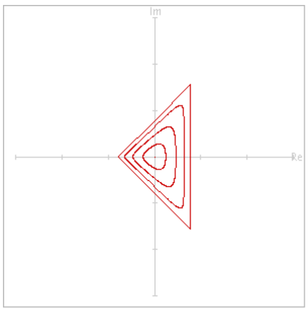

Figure 4: Image of E under f_θ for $\theta = \frac{\pi}{2}$

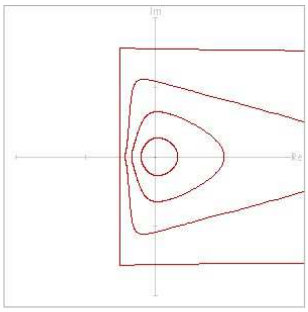

Figure 5: Image of E under f_α for $\alpha = -1$

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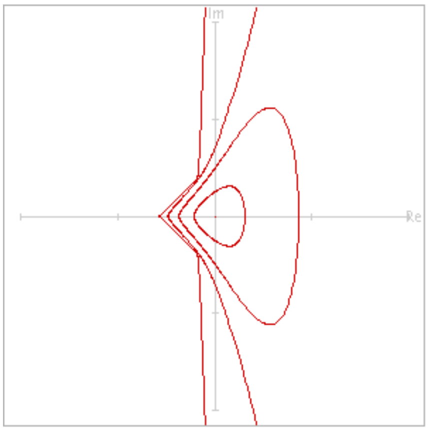

Figure 6: Image of E under $f_{\theta,\alpha}$ for $\theta = \frac{\pi}{2}$, $\alpha = -1$ and $t = \frac{3}{4}$.

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