On Ultraspherical Matrix Polynomials and Their Properties

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Abstract

ln this paper, the Ultraspherical matrix polynomials are introduced starting from the hypergeometric matrix function. The generating matrix function, an explicit representation, three-term matrix recurrence relations and differential recurrence relations are given. We derive the Rodrigues's formula and orthogonality properties for the Ultraspherical matrix polynomials. Finally, the expansions of the Ultraspherical matrix polynomials in a series of Hermite and Laguerre matrix polynomials are established.

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1 Introduction

Orthogonal matrix polynomials comprise an emerging field whose development is reaching important results from both the theoretical and practical points of view. Some recent results in this field can be found in [6, 7, 8]. Development of other extensions, such as Rodrigues-type formula [2, 3], a second-order Sturm-Liouville differential equation [3], or three-term recurrence relations [4]. The theory of Lie algebra of 2-variable generalized Hermite and 2-variable Laguerre matrix polynomials have earlier been developed by Subuhi Khan and Hassan [31], Subuhi Khan and Nusrat Raza [32]. Important connections between orthogonal matrix polynomials and matrix differential equations appear in [10, 11]. The matrix framework of the classical families of Hermite, Jacobi, Laguerre, Legendre and Chebychev polynomials have been introduced and studied in a number of previous papers [1, 9, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. The reason of interest for this family of Ultraspherical matrix polynomials are due to their intrinsic mathematical importance.

The aim of this paper is to define and study of a new class of Ultraspherical matrix polynomials from a different point of view, starting from a generalization of the generating function. The structure of the paper is organized as follows: In Section 2, the definition of the Ultraspherical matrix polynomials are given from the hypergeometric matrix function. The study of developments the generating matrix functions for the Ultraspherical matrix polynomials are obtained in Section 3. An explicit representation, three-term matrix recurrence relations and differential recurrence relations, in particular Ultraspherical matrix differential equations are established in Section 4. Rodrigues's formula developed for Ultraspherical matrix polynomials in section 5. We prove some orthogonality properties of the Ultraspherical matrix polynomials in Section 6. Finally, the expansions for the Ultraspherical in a series of Hermite and Laguerre matrix polynomials are obtained in Section 7.

This paper, is concerned with matrix polynomials

$$P_n(x) = A_n x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + ... + A_1 x + A_0$$

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in which the coefficients A_i are members of $\mathbb{C}^{N\times N}$, the space of real or complex matrices of order N, and x is a real number. $P_n(x)$ is of degree n if A_n is not the zero matrix for orthogonal matrix polynomials, the leading coefficient, A_n, being nonsingular [4, 15].

Throughout this paper, for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum is denoted by $\sigma(A)$ where $\sigma(A)$ is the set of all eigenvalues of A. The two-norm of A, which will be denoted by $\|A\|$, is defined by

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

where, for a vector $y \in \mathbb{C}^N$, $\|y\|_2 = (y^T y)^{\frac{1}{2}}$ is the Euclidean norm of y.

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and A, B are matrices in $\mathbb{C}^{N\times N}$ with $\sigma(A) \subset \Omega$ and $\sigma(B) \subset \Omega$, such that $AB = BA$, then from the properties of the matrix functional calculus in [5], it follows that

$$f(A)g(B) = g(B)f(A). \tag{1.1}$$

The reciprocal Gamma function $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$ is an entire function of the complex variable z. Then, the image of $\Gamma^{-1}(z)$ acting on A, denoted by $\Gamma^{-1}(A)$ is a well defined matrix. Furthermore, if

$$A + nI \quad \text{is invertible for every integer} \quad n \geq 0,$$

then $\Gamma(A)$ is invertible, its inverse coincides with $\Gamma^{-1}(A)$ [12]

$$(A)_n = A(A-I)(A-2I)(A-3I)...(A-(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A);\ n \geq 1\ (A)_0 = I. \tag{1.2}$$

From (1.2), it is easy to find that

$$(A)_{n-k} = (-1)^k (A)_n [(I - nI - A)_k]^{-1}; 0 \leq k \leq n. \tag{1.3}$$

From the relation (3) of [23], one obtains

$$\frac{(-1)^k}{(n-k)!} I = \frac{(-n)_k}{n!} I = \frac{(-nI)_k}{n!}; 0 \leq k \leq n. \tag{1.4}$$

Using the results [2], one gets

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n-k), \\ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} A(k,n-2k). \end{aligned} \tag{1.5}$$

Similarly, we can write

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k), \\ \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} A(k,n-k), \\ \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+2k). \end{aligned} \tag{1.6}$$

If A, B, and C are matrices of $\mathbb{C}^{N\times N}$ for which $C+nI$ is invertible for every integer $n \geq 0$. Then the hypergeometric matrix function ${}_2F_1(A,B;C;z)$ is defined in Jódar and Cortés [12, 13] as follows

$${}_2F_1(A,B;C;z) = \sum_{k=0}^{\infty} \frac{(A)_k (B)_k [(C)_k]^{-1}}{k!} z^k \tag{1.7}$$

and the hypergeometric matrix differential equation [12, 13] in the form

$$z(1-z)\frac{d^2W(z)}{dz^2} - zA\frac{dW(z)}{dz} + (C - z(B+I))\frac{dW(z)}{dz} - ABW(z) = 0; \quad 0 < |z| < 1. \tag{1.8}$$

The following theorem and lemma, derived in [3], will be useful in the sequel.

Theorem 1.1. *Suppose that A, B and C are matrix in $\mathbb{C}^{N\times N}$ such that the matrix C satisfies the condition that $C+nI$ is invertible for all integer $n\geq 0$. Suppose further that C and $C-B$ are positive stable with $BC=CB$. Then for $|z|<1$ and $|\frac{z}{1-z}|<1$, it follows that*

$$ {}_2F_1(A,B;C;z)=(1-z)^{-A}\ {}_2F_1(A,C-B;C;-\frac{z}{1-z}). \tag{1.9}$$

Lemma 1.1. *Let P and Q be two positive stable matrices in $\mathbb{C}^{N\times N}$ such that*

$$Re(z)>-1 \text{ and } Re(w)>-1, \text{ for all } z\in\sigma(P) \text{ and all } w\in\sigma(Q), \text{ and } PQ=QP. \tag{1.10}$$

Then,

$$\int_{-1}^{1}(1+x)^{P-I}(1-x)^{Q-I}dx=2^{P+Q-I}B(P,Q) \tag{1.11}$$

where I is the identity matrix in $\mathbb{C}^{N\times N}$ and $B(P,Q)$ denotes the Beta matrix function.

If P and Q are members of $\mathbb{C}^{N\times N}$ for which $PQ=QP$, and if, for all nonnegative integers n, $P+nI$, $Q+nI$ and $P+Q+nI$ are all invertible [12], then

$$B(P,Q)=\Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q) \tag{1.12}$$

where $B(P,Q)$ denotes the Beta matrix function [12] acting on the pair P, Q. We will exploit the following relation due to [12]

$$(1-x)^{-A}=\ {}_1F_0(A;-;x)=\sum_{n=0}^{\infty}\frac{1}{n!}(A)_n x^n;\quad |x|<1. \tag{1.13}$$

The above facts, notation, definition and theorems will be used throughout the next sections. In next section, we introduce a new matrix polynomial which represents of the Ultraspherical matrix polynomials as given by the relation and an explicit representation is given.

2 Ultraspherical matrix polynomials

Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition

$$Re(\lambda)>-\frac{1}{2},\quad \forall\,\lambda\in\sigma(A). \tag{2.1}$$

For $n\geq 0$, the Ultraspherical matrix polynomials $P_n^A(x)$ is defined by the hypergeometric matrix function

$$P_n^A(x)=\frac{(A+I)_n}{n!}\ {}_2F_1(-nI,2A+(n+1)I;A+I;\frac{1-x}{2}) \tag{2.2}$$

such that $A+(n+1)I$ is invertible for all integer $n\geq -1$ and for $|\frac{1-x}{2}|<1$. From (2.2) it follows that $P_n^A(x)$ is a polynomial of degree precisely n in x.

An application of Theorem **1.1**, to (2.2) yields

$$P_n^A(x)=\frac{(A+I)_n}{n!}\left(\frac{x+1}{2}\right)^n\ {}_2F_1(-nI,-nI-A;A+I;\frac{x-1}{x+1}). \tag{2.3}$$

Each of (2.2), (2.3) and using (1.2), (1.3) and (1.4) yields a finite series form for Ultraspherical matrix polynomials $P_n^A(x)$

$$P_n^A(x)=\sum_{k=0}^{n}\frac{(A+I)_n(2A+I)_{n+k}[(A+I)_k]^{-1}[(2A+I)_n]^{-1}}{k!(n-k)!}\left(\frac{x-1}{2}\right)^k \tag{2.4}$$

such that $A+(k+1)I$ and $2A+(n+1)I$ are invertible for all integer $k\geq -1$ and $n\geq -1$. Equation (2.4) is expanded forms of (2.2) and (2.3), respectively.

3 Generating matrix function for Ultraspherical matrix polynomials

We now give the generating matrix function for the Ultraspherical matrix polynomials.

Theorem 3.1. *Suppose that A is a matrix in $\mathbb{C}^{N\times N}$ satisfying (2.1). Then the generating matrix function for Ultraspherical matrix polynomials has the following representation*

$$(1-t)^{-2A-I}\ {}_1F_0(A+\frac{1}{2}I;-;\frac{2t(x-1)}{(1-t)^2})=\sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n \tag{3.1}$$

where the hypergeometric matrix function ${}_1F_0(...,...;...;...)$ is given as

$${}_1F_0(A+\frac{1}{2}I;-;\frac{2t(x-1)}{(1-t)^2})=\sum_{k=0}^{\infty}\frac{\left((A+\frac{1}{2}I)\right)_k}{k!}\left(\frac{2t(x-1)}{(1-t)^2}\right)^k$$

and $A+(n+1)I$ is invertible for all integer $n\geq -1$.

Proof: By using (1.6) and (2.2)

$$\begin{aligned}\sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(2A+I)_{n+k}[(A+I)_k]^{-1}}{k!(n-k)!}(\frac{x-1}{2})^k t^n\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(2A+I)_{n+2k}[(A+I)_k]^{-1}}{k!n!2^k}(x-1)^k t^{n+k}\\ &=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2A+2kI+I)_n t^n}{n!}\frac{(2A+I)_{2k}[(A+I)_k]^{-1}(x-1)^k t^k}{k!2^k}\\ &=\sum_{k=0}^{\infty}\frac{(2A+I)_{2k}[(A+I)_k]^{-1}(x-1)^k t^k}{k!2^k}(1-t)^{-(2A+(2k+1)I)}.\end{aligned}$$

Since $(2A+I)_{2k}=2^{2k}\left(A+I\right)_k\left(A+\frac{1}{2}I\right)_k$, it follows that

$$\begin{aligned}&\sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n\\ &\quad=\sum_{k=0}^{\infty}\frac{2^{2k}\left((A+\frac{1}{2}I)\right)_k(x-1)^k t^k}{k!2^k}(1-t)^{-(2A+(2k+1)I)}.\end{aligned}$$

Therefore, the representation of the generating matrix function for the Ultraspherical matrix polynomials (3.1) is established and the proof of Theorem 3.1 is completed.

Another representation of the generating matrix functions for the Ultraspherical matrix polynomials given in the following theorem.

Theorem 3.2. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying (2.1). Then we have thus derived generating matrix function*

$${}_0F_1(-;A+I;\frac{t(x-1)}{2})\ {}_0F_1(-;A+I;\frac{t(x+1)}{2})=\sum_{n=0}^{\infty}[(A+I)_n]^{-1}[(A+I)_n]^{-1}P_n^A(x)t^n \tag{3.2}$$

where the hypergeometric matrix functions ${}_0F_1(-;...;...)$ are given as

$${}_0F_1(-;A+I;\frac{t(x-1)}{2})=\sum_{k=0}^{\infty}\frac{[(A+I)_k]^{-1}}{k!}(\frac{x-1}{2})^k t^k$$

and

$${}_0F_1(-;A+I;\frac{t(x+1)}{2})=\sum_{n=0}^{\infty}\frac{[(A+I)_n]^{-1}}{n!}(\frac{x+1}{2})^n t^n$$

where $A+(k+1)I$ and $A+(n+1)I$ are invertible for all integer $k\geq -1$ and $n\geq -1$.

Proof: From (2.3) and (1.5), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}[(A+I)_n]^{-1}[(A+I)_n]^{-1}P_n^A(x)t^n\\
&\qquad=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{[(A+I)_k]^{-1}[(A+I)_{n-k}]^{-1}(\frac{x-1}{2})^k(\frac{x+1}{2})^{n-k}t^n}{k!(n-k)!}\\
&\qquad=\Big[\sum_{k=0}^{\infty}\frac{[(A+I)_k]^{-1}}{k!}(\frac{x-1}{2})^k t^k\Big]\Big[\sum_{n=0}^{\infty}\frac{[(A+I)_n]^{-1}}{n!}(\frac{x+1}{2})^n t^n\Big].
\end{aligned}$$

Hence the equation (3.2) is established and the proof of Theorem 3.2 is completed.

The following result gives another representation of the generating matrix functions for the Ultraspherical matrix polynomials.

Theorem 3.3. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying (2.1). Then a generating matrix function representation for Ultraspherical matrix polynomials has the following*

$$\begin{aligned}
F(x,t,A)=\sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n&=\big[1-2tx+t^2\big]^{-(A+\frac{1}{2}I)}\\
;\ |t|<r,\ |x|<1.&
\end{aligned}\tag{3.3}$$

If r_1 and r_2 are the roots of the quadratic equation $1-2xt+t^2=0$ and if r is the minimum of the set $\{r_1,r_2\}$, then the matrix function $F(x,t,A)$ regarded as a function of t, is analytic in the disk $|t|<r$, for every real number in $|x|<1$.

Proof: From (2.2), it follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n\\
&\qquad=\sum_{n=0}^{\infty}\frac{(2A+I)_n t^n}{n!}\ {}_2F_1(-nI,2A+(n+1)I;A+I;\frac{1-x}{2})\\
&\qquad=\sum_{n=0}^{\infty}\frac{(2A+I)_n}{n!}\sum_{k=0}^{n}\frac{(-nI)_k(2A+(n+1)I)_k[(A+I)_k]^{-1}}{k!}(\frac{1-x}{2})^k t^n\\
&\qquad=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-nI)_k(2A+I)_n(2A+(n+1)I)_k[(A+I)_k]^{-1}}{k!n!}(\frac{1-x}{2})^k t^n.
\end{aligned}\tag{3.4}$$

Therefore, by using (3.4) and (1.2) , we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n\\
&\qquad=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-nI)_k(2A+I)_{n+k}[(A+I)_k]^{-1}}{k!n!}(\frac{1-x}{2})^k t^n\\
&\qquad=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k(-nI)_k(2A+I)_{n+k}[(A+I)_k]^{-1}}{k!n!}2^{-k}(x-1)^k t^n.
\end{aligned}$$

Using (1.4) and applying (1.6), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n\\
&\qquad=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(2A+I)_{n+k}[(A+I)_k]^{-1}}{k!(n-k)!}\left(\frac{x-1}{2}\right)^k t^n\\
&\qquad=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(2A+I)_{n+2k}[(A+I)_k]^{-1}}{k!n!}\left(\frac{x-1}{2}\right)^k t^{n+k}.
\end{aligned}$$

From the relation (1.2), we obtain that

$$\begin{aligned}
&(2A+I+2kI)_n = (2A+I)_{n+2k}[(2A+I)_{2k}]^{-1},\\
&[(2A+I)_{2k}]^{-1} = 2^{-2k}[(A+I)_k]^{-1}[(A+\frac{1}{2}I)_k]^{-1}\\
&\text{and}\\
&(2A+I+2kI)_n = 2^{-2k}(2A+I)_{n+2k}[(A+I)_k]^{-1}[(A+\frac{1}{2}I)_k]^{-1}
\end{aligned} \tag{3.5}$$

which by inserting (3.5) with the help of (1.13) yields

$$\begin{aligned}
&\sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{2^{2k}(A+I)_k(A+\frac{1}{2}I)_k[(A+I)_k]^{-1}(2A+I+2kI)_n}{k!n!}(\frac{x-1}{2})^k t^{n+k}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(A+\frac{1}{2}I)_k(2A+2kI+I)_n}{k!n!}2^k(x-1)^k t^{n+k}\\
&= \sum_{k=0}^{\infty}\frac{(A+\frac{1}{2}I)_k}{k!}2^k(x-1)^k t^k\sum_{n=0}^{\infty}\frac{(2A+2kI+I)_n}{n!}t^n\\
&= \sum_{k=0}^{\infty}\frac{(A+\frac{1}{2}I)_k}{k!}2^k(x-1)^k t^k(1-t)^{-(2A+2kI+I)}\\
&= \sum_{k=0}^{\infty}\frac{(A+\frac{1}{2}I)_k}{k!}\frac{2^k(x-1)^k t^k}{(1-t)^{2k}}(1-t)^{-(2A+I)}\\
&= \left[1-\frac{2t(x-1)}{(1-t)^2}\right]^{-(A+\frac{1}{2}I)}(1-t)^{-(2A+I)} = \left[(1-t)^2-2t(x-1)\right]^{-(A+\frac{1}{2}I)}\\
&= \left[1-2tx+t^2\right]^{-(A+\frac{1}{2}I)}.
\end{aligned}$$

Thus the result is established and the proof of Theorem 3.3 is completed.

If in (3.3), we replace x by $-x$ and t by $-t$, the left side remains unchanged and we obtain

$$P_n^A(-x) = (-1)^n P_n^A(x). \tag{3.6}$$

In equation (3.3) put $x=0$ to obtain

$$(1+t^2)^{-(A+\frac{1}{2}I)} = \sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_n^A(0)t^n.$$

Using the binomial expansion of

$$(1+t^2)^{-(A+\frac{1}{2}I)} = \sum_{n=0}^{\infty}\frac{(-1)^n(A+\frac{1}{2}I)_n}{n!}t^{2n};|t|<1$$

we get

$$P_{2n}^A(0) = \frac{(-1)^n}{n!}(A+I)_{2n}(A+\frac{1}{2}I)_n[(2A+I)_{2n}]^{-1}, \quad P_{2n+1}^A(0)=0.$$

For $x=1$ we have

$$\sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_n^A(1)t^n = \left[1-t\right]^{-(2A+I)};|t|<1.$$

So that by (1.13) it follows

$$P_n^A(1) = \frac{1}{n!}(A+I)_n.$$

Theorem 3.4. *Suppose that A is a matrix in $\mathbb{C}^{N\times N}$ satisfying (2.1). Then, the Ultraspherical matrix polynomials has the following representation*

$$P_n^A(x) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k(2x)^{n-2k}}{k!(n-2k)!}(A+\frac{1}{2}I)_{n-k}(A+I)_n[(2A+I)_n]^{-1}. \tag{3.7}$$

Proof: By using (1.13) and (1.6), we have

$$\begin{aligned}(1-2xt+t^2)^{-(A+\frac{1}{2}I)} &= \sum_{n=0}^{\infty}\frac{(A+\frac{1}{2}I)_n(2x-t)^n t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(A+\frac{1}{2}I)_n(-1)^k(2x)^{n-k}}{k!(n-k)!}t^{n+k}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k(2x)^{n-2k}(A+\frac{1}{2}I)_{n-k}}{k!(n-2k)!}t^n.\end{aligned} \tag{3.8}$$

Thus by identification of the coefficients of t^n in (3.3) and (3.8), in both sides gives the explicit representation (3.7) and hence the proof of Theorem **3.4** is completed.

From (3.6), we obtain another hypergeometric matrix function form for Ultraspherical matrix polynomials $P_n^A(x)$, namely,

$$P_n^A(x) = \frac{(-1)^n(A+I)_n}{n!}\ {}_2F_1(-nI,(n+1)I+2A;I+A;\frac{1+x}{2}), \tag{3.9}$$

$$P_n^A(x) = \frac{(2A+I)_{2n}[(2A+I)_n]^{-1}}{n!}\left(\frac{x+1}{2}\right)^n {}_2F_1(-nI,-nI-A;-2A-2nI;\frac{2}{1+x}) \tag{3.10}$$

and

$$P_n^A(x) = \sum_{k=0}^{n}\frac{(-1)^{n-k}(A+I)_n(2A+I)_{n+k}[(A+I)_k]^{-1}[(2A+I)_n]^{-1}}{k!(n-k)!}\left(\frac{x+1}{2}\right)^k \tag{3.11}$$

where $A+(n+1)I$, $A+(k+1)I$, $2A+(n+1)I$ and $2A+(n+k+1)I$ are invertible.

4 Matrix recurrence relations for Ultraspherical matrix polynomials

In this section, we derive several matrix differential recurrence relations, the pure matrix recurrence relations and Ultraspherical matrix differential equations from this matrix generating functions.

By differentiating (3.3) with respect to x and t yields, respectively

$$\frac{\partial}{\partial x}F(x,t,A) = \frac{t}{1-2xt+t^2}2(A+\frac{1}{2}I)\ F(x,t,A) \tag{4.1}$$

and

$$\frac{\partial}{\partial t}F(x,t,A) = \frac{x-t}{1-2xt+t^2}2(A+\frac{1}{2}I)\ F(x,t,A). \tag{4.2}$$

So that the matrix function $F(x,t,A)$ satisfies the partial matrix differential equation

$$(x-t)\frac{\partial}{\partial x}F(x,t,A) - t\frac{\partial}{\partial t}F(x,t,A) = 0.$$

Therefore, by (3.3), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}x(2A+I)_n[(A+I)_n]^{-1}\frac{d}{dx}P_n^A(x)t^n - \sum_{n=0}^{\infty}n(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n\\ &\qquad= \sum_{n=1}^{\infty}(2A+I)_{n-1}[(A+I)_{n-1}]^{-1}\frac{d}{dx}P_{n-1}^A(x)t^n.\end{aligned}$$

Since $\frac{d}{dx}P_0^A(x)=0$, we obtain the differential recurrence relation

$$x(2A+nI)\frac{d}{dx}P_n^A(x)-n(2A+nI)P_n^A(x)=(A+nI)\frac{d}{dx}P_{n-1}^A(x);\ n\geq 1. \tag{4.3}$$

From (4.1) and (4.2) with the aid of (3.3), we get the following

$$\frac{2(A+\frac{1}{2}I)}{1-2xt+t^2}(1-2xt+t^2)^{-(A+\frac{1}{2}I)}=\sum_{n=1}^{\infty}(2A+I)_n[(A+I)_n]^{-1}\frac{\partial}{\partial x}P_n^A(x)t^{n-1} \tag{4.4}$$

and

$$\frac{2(x-t)(A+\frac{1}{2}I)}{1-2xt+t^2}(1-2xt+t^2)^{-(A+\frac{1}{2}I)}=\sum_{n=1}^{\infty}n(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^{n-1}. \tag{4.5}$$

Note that $1-t^2-2t(x-t)=1-2xt+t^2$, we multiplying the left side of (4.4) by $1-t^2$, the left side of (4.5) by $2t$, subtract and obtain the left side of (3.3). In this way, we obtain

$$\begin{aligned}2(A+(n+\frac{1}{2})I)(2A+nI)(A+nI)P_n^A(x)&=(2A+(n+1)I)\frac{d}{dx}P_{n+1}^A(x)\\&\quad-(A+nI)\frac{d}{dx}P_{n-1}^A(x).\end{aligned} \tag{4.6}$$

From (4.3) and (4.6), one gets

$$\begin{aligned}x(A+nI)\frac{d}{dx}P_n^A(x)&=(2A+(n+1)I)\frac{d}{dx}P_{n+1}^A(x)\\-\big[2(A+(n+\frac{1}{2})I)(A+nI)-nI\big]&(2A+nI)P_n^A(x).\end{aligned} \tag{4.7}$$

Substituting $n-1$ for n in (4.7) and putting the resulting expression for $\frac{d}{dx}P_{n-1}^A(x)$ into (4.3), gives

$$(x^2-1)\frac{d}{dx}P_n^A(x)=nxP_n^A(x)-(A+nI)P_{n-1}^A(x). \tag{4.8}$$

Now, by multiplying (4.3) by (x^2-1) and substituting for $(x^2-1)\frac{d}{dx}P_n^A(x)$ and $(x^2-1)\frac{d}{dx}P_{n-1}^A(x)$ from (4.8) to obtain the three terms matrix recurrence relations in the form

$$\begin{aligned}&n(A+nI)(A+(n+1)I)P_n^A(x)=x(A+(n+1)I)(2A+(2n-1)I)(2A+nI)\\&P_{n-1}^A(x)-(2A+(n-1)I)(2A+nI)(2A+(n+1)I)P_{n-2}^A(x);\ n\geq 2.\end{aligned} \tag{4.9}$$

Formulas (4.3), (4.6), (4.7) (4.8) and (4.9) are called the matrix recurrence formulas for Ultraspherical matrix polynomials.

We can write (4.4) and (4.5) in the form

$$\begin{aligned}2(A+\frac{1}{2}I)(1-2xt+t^2)^{-(A+\frac{3}{2}I)}&=\sum_{n=1}^{\infty}(2A+I)_n[(A+I)_n]^{-1}\frac{d}{dx}P_n^A(x)t^{n-1}\\&=\sum_{n=0}^{\infty}(2A+I)_{n+1}[(A+I)_{n+1}]^{-1}\frac{d}{dx}P_{n+1}^A(x)t^n.\end{aligned} \tag{4.10}$$

By applying (3.3), it follows

$$\begin{aligned}&2(A+\frac{1}{2}I)(1-2xt+t^2)^{-(A+\frac{3}{2}I)}\\&=\sum_{n=0}^{\infty}2(A+\frac{1}{2}I)(2A+2I)_n[(A+2I)_n]^{-1}\ P_n^{A+I}(x)t^n.\end{aligned} \tag{4.11}$$

Identification of the coefficients of t^n in (4.10) and (4.11) yields

$$\frac{d}{dx}P_{n+1}^A(x)=2(A+\frac{1}{2}I)(A+(n+1)I)(2A+(n+1)I)^{-1}\ P_n^{A+I}(x)$$

this gives

$$\frac{d}{dx}P_n^A(x) = 2(A+\frac{1}{2}I)(A+nI)(2A+nI)^{-1}\ P_{n-1}^{A+I}(x). \tag{4.12}$$

Iteration (4.12) yields, for $0 \le r \le n$;

$$\frac{d^r}{dx^r}P_n^A(x) = 2^r(A+\frac{1}{2}I)_r(A+nI)_r[(2A+nI)_r]^{-1}\ P_{n-r}^{A+rI}(x). \tag{4.13}$$

We conclude this section introducing the Ultraspherical matrix differential equation as follows corollary.

Corollary 4.1. *Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying (2.1). Then the Ultraspherical matrix polynomials are solutions of the matrix differential equations of the second order in the form*

$$(1-x^2)\frac{d^2}{dx^2}P_n^A(x) - 2x(A+I)\frac{d}{dx}P_n^A(x) + n(2A+(n+1)I)P_n^A(x) = 0. \tag{4.14}$$

Proof. In (4.7), replace n by $n-1$ and differentiate with respect to x to find

$$\begin{aligned} &x(A+(n-1)I)\frac{d^2}{dx^2}P_{n-1}^A(x) + (A+(n-1)I)\frac{d}{dx}P_{n-1}^A(x) = (2A+nI)\frac{d^2}{dx^2}P_n^A(x) \\ &-\left[2(2A+(n-\frac{1}{2})I)(A+(n-1)I)-(n-1)I\right](2A+(n-1)I)\frac{d}{dx}P_{n-1}^A(x). \end{aligned} \tag{4.15}$$

Also, by differentiating (4.3) with respect to x, we have

$$x(2A+nI)\frac{d^2}{dx^2}P_n^A(x) - n(2A+nI)\frac{d}{dx}P_n^A(x) = (A+nI)\frac{d}{dx}P_{n-1}^A(x). \tag{4.16}$$

From (4.3) and (4.16) by putting $\frac{d}{dx}P_{n-1}^A(x)$ and $\frac{d^2}{dx^2}P_{n-1}^A(x)$ into (4.15) and rearrangement terms, we obtain (4.14) and hence the proof of Corollary.

Differentiating the identity (2.2) with respect to x, it follows

$$\begin{aligned} DP_n^A(x) =& \frac{n(A+I)_n(2A+(n+1)I)}{2(A+I)n!}F(-nI+I,2A+nI+2I;A+2I;\frac{1-x}{2}) \\ =& \frac{(A+2I)_{n-1}(2A+(n+1)I)}{2(n-1)!}\ {}_2F_1(-(n-1)I,2A+2I+nI;A+2I;\frac{1-x}{2}) \end{aligned}$$

so that

$$DP_n^A(x) = \frac{(2A+(n+1)I)}{2}P_{n-1}^{A+I}(x). \tag{4.17}$$

Iteration of (4.1) yields, for $0 < k \le n$

$$D^kP_n^A(x) = \frac{(2A+(n+1)I)_k}{2^k}P_{n-k}^{A+kI}(x). \tag{4.18}$$

Now, we can get the differential equation in a different way and prove the following theorem.

Theorem 4.1. *Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying (2.1). For $n \ge 0$, the Ultraspherical matrix polynomial $p_n^A(x)$ satisfies the matrix differential equation*

$$(1-x^2)Y''(x) - 2x(A+I)Y'(x) + n(2A+(n+1)I)Y(x) = 0; \quad |x|<1. \tag{4.19}$$

Proof : Taking $z = \frac{x+1}{2}$, $A = 2A+(n+1)I$, $B = -nI$, and $C = A+I$,

$${}_2F_1(2A+(n+1)I,-nI;A+I;\frac{1+x}{2}) = (-1)^n n!\big[(A+nI)_n\big]^{-1}P_n^A(x)$$

from (3.14). Introduce the notation

$$W(\frac{1+x}{2}) = {}_2F_1(-nI,2A+(n+1)I;A+I;\frac{x+1}{2}). \tag{4.20}$$

Applying the chain rule in (4.20),

$$\begin{aligned} W'(\frac{1+x}{2}) &= 2(-1)^n n!\big[(A+nI)_n\big]^{-1}\frac{d}{dx}\Big(P_n^A(x)\Big), \quad \text{and} \\ W''(\frac{1+x}{2}) &= 4(-1)^n n!\big[(A+nI)_n\big]^{-1}\frac{d^2}{dx^2}\Big(P_n^A(x)\Big). \end{aligned} \tag{4.21}$$

Taking into account that

$$A+I-\Big(\frac{1+x}{2}\Big)(1-n)I = \frac{1}{2}(2A+(1+n-x-xn)I)$$

and substituting (4.20),(4.21) in (1.8) and postmultiplying $\frac{(-1)^n}{n!}(A+nI)_n$ yields

$$\begin{aligned} (1-x^2)\frac{d^2}{dx^2}\Big(P_n^A(x)\Big) - 2(A+x(A+I))\frac{d}{dx}\Big(P_n^A(x)\Big) + 2A\frac{d}{dx}\Big(P_n^A(x)\Big) \\ +n(2A+(n+1)I)P_n^A(x) = 0. \end{aligned} \tag{4.22}$$

Thus, $P_n^A(x)$, as given by (3.11), satisfies (4.19) in $|x|<1$ and the proof of Theorem **4.1** is completed. We have the following corollary.

Corollary 4.2. *For $n \geq 0$ and $|x|<1$, $P_n^A(x)$ is a solution of the matrix differential equation*

$$\begin{aligned} &\frac{d}{dx}\left[(1+x)(1-x)^{2A+I}Y'(x)\Big(\frac{1+x}{1-x}\Big)^A\right] \\ &+n(2A+(n+1)I)(1-x)^{2A}Y(x)\Big(\frac{1+x}{1-x}\Big)^A = 0. \end{aligned} \tag{4.23}$$

Proof: Premultiplying (4.19) by $(1-x)^{2A}$ and postmultiplying by $(\frac{1+x}{1-x})^A$, then rearranging yields (4.23) for $|x|<1$.

5 Rodrigues's formula for Ultraspherical matrix polynomials

In this section, we provide Rodrigues's formula for the Ultraspherical matrix polynomials and prove the following theorem.

Theorem 5.1. *Let A be a matrix satisfying (2.1) and let $P_n^A(x)$ be the Ultraspherical matrix polynomial. Then the following Rodrigues's formula holds for $n \geq 0$ and $|x|<1$*

$$P_n^A(x) = \frac{(x^2-1)^{-A}}{n!2^n}D^n\Big[(x^2-1)^{A+nI}\Big]. \tag{5.1}$$

Proof: Equation (2.3), we can be written

$$P_n^A(x) = \sum_{k=0}^{n}\frac{(A+I)_n(A+I)_n[(A+I)_k]^{-1}[(A+I)_{n-k}]^{-1}(x-1)^k(x+1)^{n-k}}{k!2^n(n-k)!}. \tag{5.2}$$

The differential operator is denoted by D, with $D^k(f(x)) = \frac{d^kf(x)}{dx^k}$. It is easy to show from (1.3) that for an arbitrary matrix $\mathbb{C}^{N\times N}$ [2], then for non-negative integral s and m, one gets

$$\begin{aligned} D^s x^{A+mI} =& (A+mI)(A+(m-1)I)(A+(m-2)I)...(A+(m-s+1)I)x^{A+(m-s)I} \\ &; \quad s=0,1,2,... \end{aligned}$$

or

$$D^s x^{A+mI} = (A+I)_m[(A+I)_{m-s}]^{-1}x^{A+(m-s)I}. \tag{5.3}$$

From (5.3) we obtain

$$D^k(x+1)^{A+nI} = (A+I)_n[(A+I)_{n-k}]^{-1}(x+1)^{A+(n-k)I} \tag{5.4}$$

and

$$D^{n-k}(x-1)^{A+nI} = (A+I)_n[(A+I)_k]^{-1}(x-1)^{A+kI}. \tag{5.5}$$

Therefore (5.2) can be put in the form

$$P_n^A(x) = \frac{(x-1)^{-A}(x+1)^{-A}}{n!2^n} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \Big[D^{n-k}(x-1)^{A+nI}\Big]\Big[D^k(x+1)^{A+nI}\Big]. \tag{5.6}$$

By Leibnitz' rule for the n-derivative of a product, equation (5.6) yields the Rodrigues's formula one gets

$$P_n^A(x) = \frac{(x^2-1)^{-A}}{n!2^n} D^n\Big[(x^2-1)^{A+nI}\Big]$$

or we now give another representation of the Rodrigues's formula for the Ultraspherical matrix polynomials

$$P_n^A(x) = \frac{(-1)^n(1-x^2)^{-A}}{n!2^n} D^n(1-x^2)^{A+nI}. \tag{5.7}$$

Equation (5.7) is more desirable than (5.1) when we work in the interval $|x| < 1$. Thus the result is established.

In the following, we obtain the orthogonality for Ultraspherical matrix polynomials which satisfy (2.1).

6 Orthogonality for Ultraspherical matrix polynomials

Since $-2x(A+I) = (1-x)(I+A) - (1+x)(I+A)$, we may put (4.14) in the form

$$\begin{aligned}(1-x^2)^{A+I}D^2P_n^A(x) + [(1-x)(I+A) - (1+x)(I+A)](1-x^2)^A DP_n^A(x)\\ +n(2A+(n+1)I)(1-x^2)^A P_n^A(x) = 0\end{aligned}$$

this yield

$$D[(1-x^2)^{A+I}DP_n^A(x)] + n(2A+(n+1)I)(1-x^2)^A P_n^A(x) = 0. \tag{6.1}$$

From (6.1) and the same equation with n replaced by m, that P_n^A and $P_m^A(x)$ commute, it follows that

$$\begin{aligned}&[n(2A+(n+1)I) - m(2A+(m+1)I)](1-x^2)^A P_n^A(x)P_m^A(x)\\ &\quad = D\Big[(1-x^2)^{A+I}\Big(P_n^A(x)DP_m^A(x) - P_m^A(x)DP_n^A(x)\Big)\Big].\end{aligned}$$

Therefore, we may conclude that

$$\begin{aligned}&(n-m)(2A+(n+m+1)I)\int_{-1}^{1}(1-x^2)^A P_n^A(x)P_m^A(x)\\ &\quad = \Big[(1-x^2)^{A+I}\Big(P_n^A(x)DP_m^A(x) - P_m^A(x)DP_n^A(x)\Big)\Big]_{-1}^{1}.\end{aligned} \tag{6.2}$$

The condition of commutativity, $P_n^A(x)P_m^A(x) = P_m^A(x)P_n^A(x)$, then (6.2) leads us to the orthogonality property

$$\int_{-1}^{1}(1-x^2)^A P_n^A(x)P_m^A(x)dx = 0, \quad m \neq n. \tag{6.3}$$

That is, the Ultraspherical matrix polynomials form an orthogonal set over $(-1,1)$ with respect to the weight function $(1-x^2)^A$.

In order to evaluate

$$g_n(A) = \int_{-1}^{1} (1-x^2)^A [P_n^A(x)]^2 dx \tag{6.4}$$

we employ the Rodrigues's formula and integration by parts. This method incidentally furnishes a second derivation of the orthogonality property (6.3).

From (5.7), we obtain

$$(1-x^2)^A P_n^A(x) = \frac{(-1)^n}{2^n n!} D^n (1-x^2)^{A+nI}. \tag{6.5}$$

Therefore, if $Re(\lambda) > -\frac{1}{2}$, for all $\lambda \in \sigma(A)$,

$$\int_{-1}^{1} (1-x^2)^A P_n^A(x) P_m^A(x) dx = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} \{D^n (1-x^2)^{A+nI}\} P_m^A(x) dx. \tag{6.6}$$

On the right in (6.6), integrate by parts n times, each time differentiating $P_n^A(x)$ and integrating the quantity in curly brackets. At the k^{th} stage the integrated part

$$\{D^{n-k}(1-x^2)^{A+nI}\} D^{k-1} P_n^A(x)$$

is zero at both limits because of factors $(1-x^2)^{A+kI}$ with satisfying the spectral condition (2.1). After n such integrations by parts, we have

$$\int_{-1}^{1} (1-x^2)^A P_n^A(x) P_m^A(x) dx = \frac{(-1)^{2n}}{2^n n!} \int_{-1}^{1} (1-x^2)^{A+nI} D^n P_m^A(x) dx. \tag{6.7}$$

If $n \neq m$ we may choose n to the larger and therefore conclude that

$$\int_{-1}^{1} (1-x^2)^A P_n^A(x) P_m^A(x) dx = 0, \quad n \neq m. \tag{6.8}$$

In (6.7), we have a tool for the evaluation of the $g_n(A)$ of (6.4), but we need $D^n P_n^A(x)$. From

$$P_n^A(x) = \frac{(A+I)_n}{n!} \, {}_2F_1(-nI, 2A+(n+1)I; A+I; \frac{1-x}{2})$$

and repeated application of the formula for the derivative of hypergeometric matrix functions, we obtain

$$D^n P_n^A(x) = \frac{(-\frac{1}{2})^n (-nI)_n (2A+(n+1)I)_n}{n!} \, {}_2F_1(0, 2A+(2n+1)I; A+(n+1)I; \frac{1-x}{2})$$

from which

$$D^n P_n^A(x) = \frac{(2A+I)_{2n}[(2A+I)_n]^{-1}}{2^n}. \tag{6.9}$$

Now (6.7) with $n = m$ yields

$$g_n(A) = \int_{-1}^{1} (1-x^2)^A [P_n^A(x)]^2 dx = \frac{(2A+I)_{2n}[(2A+I)_n]^{-1}}{2^{2n} n!} \int_{-1}^{1} (1-x^2)^{A+nI} dx.$$

Using Lemma **1.1**, we get

$$\begin{aligned}\int_{-1}^{1} (1-x)^{A+nI}(1+x)^{A+nI} dx &= 2^{2A+(2n+1)I} B(A+(n+1)I, A+(n+1)I) \\ &= 2^{2A+(2n+1)I} \Gamma(A+(n+1)I)\Gamma(A+(n+1)I)\Gamma^{-1}(2A+2(n+1)I).\end{aligned}$$

Hence

$$\begin{aligned}g_n(A) = &\frac{2^{2A+(2n+1)I}(2A+I)_{2n}(2A+I)_{2n}\Gamma(A+(n+1)I)\Gamma(A+(n+1)I)}{2^{2n} n!} \\ &[(2A+I)_n]^{-1}\Gamma^{-1}(2A+2(n+1)I)\end{aligned}$$

or

$$g_n(A) = \frac{2^{2A+I}\Gamma(A+(n+1)I)\Gamma(A+(n+1)I)}{n!} [(2A+(2n+1)I)]^{-1}\Gamma^{-1}(2A+(n+1)I). \tag{6.10}$$

From (6.9), we conclude that

$$P_n^A(x) = \frac{(2A+I)_{2n}x^n[(2A+I)_n]^{-1}}{n!2^n} + \Pi_{n-1}(x) \tag{6.11}$$

in which $\Pi_{n-1}(x)$ is a matrix polynomial of degree $n-1$. In summary, we have obtained the following important result, orthogonality of the Ultraspherical matrix polynomials, $P_n^A(x)$, that are defined by (2.2).

Theorem 6.1. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfy the spectral conditions (2.1) and $P_n^A(x)P_m^A(x) = P_m^A(x)P_n^A(x)$. Then for any nonnegative integers n and m,*

$$\begin{aligned}&\int_{-1}^{1}(1-x^2)^A P_n^A(x)P_m^A(x)dx\\ &= \begin{cases} 0, & n \neq m;\\ \frac{2^{2A+I}}{n!}\Gamma(2A+(2n+1)I)\Gamma^{-1}(2A+(n+1)I) & \\ \Gamma(A+(n+1)I)\Gamma(A+(n+1)I)\Gamma^{-1}(2A+2(n+1)I), & n=m.\end{cases}\end{aligned} \tag{6.12}$$

Making use of the hypergeometric representation (2.2) in the familiar orthogonality property (6.12), and setting

$$x = 1-2t, \quad 0<t<1$$

we obtain

$$\begin{aligned}&\int_0^1 t^A(1-t)^A\ {}_2F_1(-mI, 2A+(m+1)I; A+I; t)\\ &{}_2F_1(-nI, 2A+(n+1)I; A+I; t)dt = n!\Gamma(A+I)\Gamma(A+I)\Gamma(A+(n+1)I)\\ &[(2A+(2n+1)I)]^{-1}[\Gamma(A+(n+1)I)]^{-1}[\Gamma(2A+(n+1)I)]^{-1}\delta_{mn}\end{aligned} \tag{6.13}$$

where

$$\delta_{mn} = \begin{cases} 0, & n\neq m;\\ 1, & n=m.\end{cases}$$

In view of the hypergeometric matrix function representation (2.4), we find from the orthogonality property (6.12) with

$$x = 1-\frac{2}{t}, \quad 1<t<\infty$$

that

$$\begin{aligned}&\int_1^\infty t^{-2A-(m+n+2)I}(t-1)^A\ {}_2F_1(-mI, -A-mI; -2A-2mI; t)\\ &{}_2F_1(-nI, -A-nI; -2A-2nI; t)dt = n!\Gamma(A+(n+1)I)\Gamma(A+(n+1)I)\\ &\Gamma(2A+(n+1)I)[\Gamma(2A+(2n+1)I)]^{-1}[\Gamma(2A+(2n+2)I)]^{-1}\delta_{mn}.\end{aligned} \tag{6.14}$$

Thus, if we employ the hypergeometric representation (2.3) on the left-hand side of the orthogonality property (6.12) and set

$$x = \frac{1-t}{1+t}, \quad 0<t<\infty$$

we get

$$\begin{aligned}&\int_0^\infty t^A(1+t)^{-2A-(m+n+2)I}\ {}_2F_1(-mI,-A-mI;A+I;-t)\\&\quad {}_2F_1(-nI,-A-nI;A+I;-t)dt=n!\Gamma(A+I)\Gamma(A+I)\Gamma(A+(n+1)I)\\&\quad [(2A+(2n+1)I)]^{-1}[\Gamma(A+(n+1)I)]^{-1}[\Gamma(2A+(n+1)I)]^{-1}\delta_{mn}.\end{aligned}\tag{6.15}$$

Finally, the Ultraspherical matrix polynomials are expanded in series of Hermite and Laguerre matrix polynomials.

7 Expanding of Ultraspherical matrix polynomials in series of Hermite and Laguerre matrix polynomials

If A is a positive stable matrix in $\mathbb{C}^{N\times N}$, then the n^{th} Hermite matrix polynomials [1, 19] was defined by

$$H_n(x,A)=n!\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k}{k!(n-2k)!}(x\sqrt{2A})^{n-2k}\tag{7.1}$$

and the expansion of x^nI in a series of Hermite matrix polynomials have been given in [1, 19]

$$(x\sqrt{2A})^n=n!\sum_{k=0}^{[\frac{1}{2}n]}\frac{1}{k!(n-2k)!}H_{n-2k}(x,A).\tag{7.2}$$

Now, let us expand the Ultraspherical matrix polynomials in series of Hermite matrix polynomials. Employing (3.7) and (1.6) with the aid of (7.2) and taking into account that each matrix commutes with itself, one gets

$$\begin{aligned}\sum_{n=0}^\infty(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n&=\sum_{n=0}^\infty\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k(A+\frac{1}{2}I)_{n-k}(2x)^{n-2k}}{k!(n-2k)!}t^n\\&=\sum_{n=0}^\infty\sum_{k=0}^\infty\frac{(-1)^k(A+\frac{1}{2}I)_{n+k}(2x)^n}{k!n!}t^{n+2k}\\&=\sum_{n=0}^\infty\sum_{k=0}^\infty\sum_{s=0}^{[\frac{1}{2}n]}\frac{(-1)^k2^n(A+\frac{1}{2}I)_{n+k}(\sqrt{2A})^{-n}}{k!s!(n-2s)!}H_{n-2s}(x,A)t^{n+2k}.\end{aligned}\tag{7.3}$$

Since the matrix A commutes with itself, then we can write (7.3) in the form

$$\begin{aligned}&\sum_{n=0}^\infty(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n\\&\quad=\sum_{n=0}^\infty\sum_{k=0}^\infty\sum_{s=0}^{[\frac{1}{2}n]}\frac{(-1)^k2^n(A+\frac{1}{2}I)_{n+k}(\sqrt{2A})^{-n}}{k!s!(n-2s)!}H_{n-2s}(x,A)t^{n+2k}.\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{n=0}^\infty 2^{-n}(\sqrt{2A})^n(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n\\&\quad=\sum_{n=0}^\infty\sum_{k=0}^\infty\sum_{s=0}^{[\frac{1}{2}n]}\frac{(-1)^k(A+\frac{1}{2}I)_{n+k}}{k!s!(n-2s)!}H_{n-2s}(x,A)t^{n+2k}.\end{aligned}\tag{7.4}$$

Using (1.6) the expression (7.4) becomes

$$\begin{aligned}&\sum_{n=0}^\infty 2^{-n}(\sqrt{2A})^n(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n\\&\quad=\sum_{n=0}^\infty\sum_{k=0}^\infty\sum_{s=0}^\infty\frac{(-1)^k(A+\frac{1}{2}I)_{n+k+2s}}{k!s!n!}H_n(x,A)t^{n+2k+2s}\end{aligned}$$

this, by using (1.5), yields,

$$\sum_{n=0}^{\infty} 2^{-n}(\sqrt{2A})^n(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{k}\frac{(-1)^{k-s}(A+\frac{1}{2}I)_{n+k+s}}{(k-s)!s!n!}H_n(x,A)t^{n+2k}.$$

Since

$$(A+\frac{1}{2}I)_{n+k+s}=(A+\frac{1}{2}I+(n+k)I)_s(A+\frac{1}{2}I)_{n+k}.$$

Using (1.4) and (1.5), it follows

$$\sum_{n=0}^{\infty} 2^{-n}(\sqrt{2A})^n(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{k}\frac{(-1)^k(-kI)_s(A+\frac{1}{2}I+(n+k)I)_s(A+\frac{1}{2}I)_{n+k}}{k!s!n!}H_n(x,A)t^{n+2k}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!n!}\ {}_2F_0(-kI,A+\frac{1}{2}I+(n+k)I;-;1)(A+\frac{1}{2}I)_{n+k}H_n(x,A)t^{n+2k}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k}{k!(n-2k)!}\ {}_2F_0(-kI,A+\frac{1}{2}I+(n+k)I;-;1)(A+\frac{1}{2}I)_{n-k}H_{n-2k}(x,A)t^n$$

where the hypergeometric matrix functions ${}_2F_0(...,...;-;...)$ are given as

$${}_2F_0(-kI,A+\frac{1}{2}I+(n+k)I;-;1)=\sum_{s=0}^{\infty}\frac{(-kI)_s(A+\frac{1}{2}I+(n+k)I)_s}{s!}.$$

Therefore, by identification of coefficient of t^n, we obtain an expansion of Ultraspherical matrix polynomials as a series of Hermite matrix polynomials in the form

$$P_n^A(x)=\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k(A+\frac{1}{2}I)_{n-k}}{k!(n-2k)!}\ {}_2F_0(-kI,A+\frac{1}{2}I+(n+k)I;-;1)$$
$$2^n(A+I)_n[(2A+I)_n]^{-1}(\sqrt{2A})^{-n}H_{n-2k}(x,A).$$

Furthermore, the n^{th} Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ is defined by

$$L_n^{(A,\lambda)}(x)=\sum_{k=0}^{n}\frac{(-1)^k\lambda^k x^k}{k!(n-k)!}(A+I)_n[(A+I)_k]^{-1} \tag{7.5}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ such that $-k$ is not an eigenvalue of A, for every integer $k>0$ and λ is a complex number such that $Re(\lambda)>0$.

In (7.5), $\lambda=1$ gives

$$L_n^{(A)}(x)=\sum_{k=0}^{n}\frac{(-1)^k x^k}{k!(n-k)!}(A+I)_n[(A+I)_k]^{-1}. \tag{7.6}$$

The expansion of x^nI in a series of Laguerre matrix polynomials [9] in the form

$$x^nI=n!\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}(A+I)_n[(A+I)_k]^{-1}L_k^{(A)}(x). \tag{7.7}$$

We use (7.7) to expand the Ultraspherical matrix polynomials in series of Laguerre matrix polynomials. We consider the series

$$
\begin{aligned}
\sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n &= \sum_{n=0}^{\infty}\sum_{s=0}^{[\frac{1}{2}n]}\frac{(-1)^s(A+\frac{1}{2}I)_{n-s}(2x)^{n-2s}}{s!(n-2s)!}t^n \\
&= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s(A+\frac{1}{2}I)_{n+s}(2x)^n}{s!n!}t^{n+2s} \\
&= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{k+s}2^n(A+\frac{1}{2}I)_{n+s}}{s!(n-k)!}(A+\frac{3}{2}I)_n[(A+\frac{3}{2}I)_k]^{-1}L_k^{(A)}(x)t^{n+2s}
\end{aligned}
\tag{7.8}
$$

which, by using (1.6), becomes

$$
\begin{aligned}
\sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n =& \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{k+s}2^{n+k}}{n!s!}(A+\frac{1}{2}I)_{n+k+s}(A+\frac{3}{2}I)_{n+k} \\
&[(A+\frac{3}{2}I)_k]^{-1}L_k^{(A)}(x)t^{n+k+2s}.
\end{aligned}
$$

From (1.5), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n =& \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{[\frac{1}{2}n]}\frac{(-1)^{k+s}2^{n+k-2s}}{s!(n-2s)!}(A+\frac{1}{2}I)_{n+k-s} \\
&(A+\frac{3}{2}I)_{n+k-2s}[(A+\frac{3}{2}I)_k]^{-1}L_k^{(A)}(x)t^{n+k}.
\end{aligned}
\tag{7.9}
$$

Form (1.2), it is easy to find that

$$(A+\frac{1}{2}I)_{2n} = 2^{2n}(\frac{1}{2}(A+\frac{3}{2}I))_n(\frac{1}{2}(A+\frac{1}{2}I))_n$$

and

$$(A+\frac{1}{2}I)_{n+k} = (A+\frac{1}{2}I)_n(A+(n+\frac{1}{2})I)_k.$$

In accordance with (1.3), one gets

$$(A+\frac{1}{2}I)_{n+k-s} = (-1)^s(A+\frac{1}{2}I)_{n+k}[((1-n-k)I-A-\frac{1}{2}I)_s]^{-1}$$

and

$$
\begin{aligned}
(A+\frac{3}{2}I)_{n+k-2s} = 2^{-2s}(A+\frac{3}{2}I)_{n+k}&[(\frac{1}{2}((1-n-k)I-A-\frac{1}{2}I))_s]^{-1} \\
&[(-\frac{1}{2}((n+k)I+A+\frac{1}{2}I))_s]^{-1}.
\end{aligned}
$$

Therefore

$$\begin{aligned}
\sum_{n=0}^{\infty}(2A+I)_n[(A+I)_n]^{-1}P_n^A(x)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{[\frac{1}{2}n]}\frac{(-1)^{k+s}2^{n+k-2s}}{s!(n-2s)!}(-1)^s(A+\frac{1}{2}I)_{n+k}\\
&\quad [((1-n-k)I-A-\frac{1}{2}I)_s]^{-1}2^{-2s}(A+\frac{3}{2}I)_{n+k}\\
&\quad [(\frac{1}{2}((1-n-k)I-A-\frac{1}{2}I))_s]^{-1}[(-\frac{1}{2}((n+k)I+A+\frac{1}{2}I))_s]^{-1}[(A+\frac{3}{2}I)_k]^{-1}L_k^{(A)}(x)t^{n+k}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{[\frac{1}{2}n]}\frac{1}{s!}(-\frac{1}{2}nI)_s(-\frac{1}{2}(n-\frac{1}{2})I)_s\\
&\quad [((1-n-k)I-A-\frac{1}{2}I)_s]^{-1}[(\frac{1}{2}((1-n-k)I-A-\frac{1}{2}I))_s]^{-1}[(-\frac{1}{2}((n+k)I+A+\frac{1}{2}I))_s]^{-1}\\
&\quad (\frac{1}{4})^s\frac{(-1)^k2^{n+k}}{n!}(A+\frac{1}{2}I)_{n+k}(A+\frac{3}{2}I)_{n+k}[(A+\frac{3}{2}I)_k]^{-1}L_k^{(A)}(x)t^{n+k}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\ {}_2F_3(-\frac{1}{2}nI,-\frac{1}{2}(n-\frac{1}{2})I;(1-n-k)I-A-\frac{1}{2}I,\frac{1}{2}((1-n-k)I-A-\frac{1}{2}I)\\
&\quad ,-\frac{1}{2}((n+k)I+A+\frac{1}{2}I);\frac{1}{4})\frac{(-1)^k2^{n+k}}{n!}(A+\frac{1}{2}I)_{n+k}(A+\frac{3}{2}I)_{n+k}[(A+\frac{3}{2}I)_k]^{-1}L_k^{(A)}(x)t^{n+k}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\ {}_2F_3(-\frac{1}{2}(n-k)I,-\frac{1}{2}((n-k)-\frac{1}{2})I;(1-n)I-A-\frac{1}{2}I,\frac{1}{2}((1-n)I-A-\frac{1}{2}I)\\
&\quad ,-\frac{1}{2}(A+nI+\frac{1}{2}I);\frac{1}{4})\frac{(-1)^k2^n}{(n-k)!}(A+\frac{1}{2}I)_n(A+\frac{3}{2}I)_n[(A+\frac{3}{2}I)_k]^{-1}L_k^{(A)}(x)t^n
\end{aligned} \tag{7.10}$$

where the hypergeometric matrix functions ${}_2F_3(...,...;...,...,...;...)$ are given as

$$\begin{aligned}
&{}_2F_3(-\frac{1}{2}(n-k)I,-\frac{1}{2}((n-k)-\frac{1}{2})I;(1-n)I-A-\frac{1}{2}I,\frac{1}{2}((1-n)I-A-\frac{1}{2}I),\\
&\qquad -\frac{1}{2}(A+nI+\frac{1}{2}I);\frac{1}{4}) = \sum_{s=0}^{\infty}\frac{1}{4^s}(-\frac{1}{2}(n-k)I)_s(-\frac{1}{2}((n-k)-\frac{1}{2})I)_s\\
&\frac{[((1-n)I-A-\frac{1}{2}I)_s]^{-1}[(\frac{1}{2}((1-n)I-A-\frac{1}{2}I))_s]^{-1}[(-\frac{1}{2}(A+nI+\frac{1}{2}I))_s]^{-1}}{s!}.
\end{aligned}$$

Equation the coefficients of t^n gives an expansion of as a series of Ultraspherical matrix polynomials in the form:

$$\begin{aligned}
P_n^A(x) =&(A+I)_n[(2A+I)_n]^{-1}\sum_{k=0}^{n}\ {}_2F_3(-\frac{1}{2}(n-k)I,-\frac{1}{2}((n-k)-\frac{1}{2})I\\
&;(1-n)I-A-\frac{1}{2}I,\frac{1}{2}((1-n)I-A-\frac{1}{2}I)\\
&,-\frac{1}{2}(A+nI+\frac{1}{2}I);\frac{1}{4})\frac{(-1)^k2^n}{(n-k)!}(A+\frac{1}{2}I)_n(A+\frac{3}{2}I)_n[(A+\frac{3}{2}I)_k]^{-1}L_k^{(A)}(x)
\end{aligned} \tag{7.11}$$

this can be written in a convenient form as follows:

$$\begin{aligned}
P_n^A(x) =&(A+I)_n[(2A+I)_n]^{-1}\frac{2^n}{n!}(A+\frac{1}{2}I)_n(A+\frac{3}{2}I)_n\sum_{k=0}^{n}\ {}_2F_3(-\frac{1}{2}(n-k)I\\
&,-\frac{1}{2}((n-k)-\frac{1}{2})I;(1-n)I-A-\frac{1}{2}I\\
&,\frac{1}{2}((1-n)I-A-\frac{1}{2}I),-\frac{1}{2}(A+nI+\frac{1}{2}I);\frac{1}{4})(-nI)_k[(A+\frac{3}{2}I)_k]^{-1}L_k^{(A)}(x).
\end{aligned}$$

These results are summarized below.

Theorem 7.1. *Let A be matrix in $\mathbb{C}^{N\times N}$ satisfying (2.1). Expansions Ultraspherical matrix polynomials in series of Hermite and Laguerre matrix polynomials relevent to our present investigation are given by*

$$\begin{aligned}P_n^A(x) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (A+\frac{1}{2}I)_{n-k}}{k!(n-2k)!}\ {}_2F_0(-kI, A+\frac{1}{2}I+(n+k)I; -; 1)\\ 2^n (A+I)_n[(2A+I)_n]^{-1}(\sqrt{2A})^{-n} H_{n-2k}(x,A)\end{aligned} \tag{7.12}$$

and

$$\begin{aligned}P_n^A(x) =&(A+I)_n[(2A+I)_n]^{-1}\frac{2^n}{n!}(A+\frac{1}{2}I)_n(A+\frac{3}{2}I)_n\sum_{k=0}^{n}\ {}_2F_3(-\frac{1}{2}(n-k)I\\ &, -\frac{1}{2}((n-k)-\frac{1}{2})I; (1-n)I-A-\frac{1}{2}I\\ &, \frac{1}{2}((1-n)I-A-\frac{1}{2}I), -\frac{1}{2}(A+nI+\frac{1}{2}I); \frac{1}{4})(-nI)_k[(A+\frac{3}{2}I)_k]^{-1}L_k^{(A)}(x).\end{aligned} \tag{7.13}$$

In the next papers, we will be to present a systematic investigation of the matrix extension of the multivariable Ultraspherical polynomials generated function by

$$\begin{aligned}\sum_{n=0}^{\infty}(2A_r+rI)_n[(A_r+rI)_n]^{-1}P_n^{A_1,A_2,...,A_r}(\mathbf{x})t^n =& \prod_{i=1}^{r}\left[1-2tx_i+t^2\right]^{-(A_i+\frac{1}{2}iI)}\\ &; \ |2x_it-t^2|<1;\ i=1,2,...,r.\end{aligned} \tag{7.14}$$

where A_i be a matrix in $\mathbb{C}^{N\times N}$ satisfying (2.1), and $\mathbf{x}=(x_1,x_2,...,x_r)$. From (7.14) yields the following explicit representation:

$$\begin{aligned}P_n^{A_1,A_2,...,A_r}(\mathbf{x}) =&(A_r+rI)_n[(2A_r+rI)_n]^{-1}\\ &\sum_{2k_1+2k_2+...+2k_r+n_1+n_2+...+n_r=n}\ \prod_{i=1}^{r}\frac{(-1)^{k_i}(2x_i)^{n_i}}{k_i!n_i!}(A_i+\frac{1}{2}iI)_{n_i-k_i}.\end{aligned} \tag{7.15}$$

We notice that the case $r=1$ in (7.14) reduces to the matrix version of the generalized Ultraspherical polynomials introduced in equation (3.3).

The above results, though far from completing the argument, can give a notion of the usefulness of the present method for the identification of suitable generalizations of known matrix functions. Moreover, they represent a starting point for the development of a unified theory of orthogonal matrix polynomials, which will be the subject of for coming works, the matrix extension of the multivariable Ultraspherical polynomial will be introduced. Various families of linear, multilinear and multilateral generating matrix functions of these matrix polynomial will be presented, actually in preparation, then unfortunately distended by further studies. Miscellaneous applications will be also discussed. The results of this paper are original, variant, significant and so it is interesting and capable to develop its study in the future.

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