FIXED POINT THEOREMS FOR MULTIVALUED MAPS IN CONE METRIC SPACES

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Abstract. The aim of this paper is to extend some results of Feng and Liu , Klim and Wardowski , Ćirić and others from the context of metric spaces to cone metric spaces by using the concept of sequentially lower semicontinuous. Examples are provided to illustrate the theory.

1. introduction and Preliminaries

Banach contraction principle is widely recognized as the source of metric fixed point theory. This principle plays an important role in several branches of mathematics. A multivalued version of the Banach contraction principle was obtained by Nadler [15] using the concept of the Hausdorff metric. Recently, Feng and Liu [7] extended Nadler's result without using the concept of the Hausdorff metric as follow:

Theorem 1.1. *(Feng-Liu* [7]*, Theorem 3.1) Let* (X, d) *be a complete metric space and let* T *be a mapping from* X *into* $Cl(X)$*. If there exist constants* $b, c \in (0, 1), c < b$*, such that for any* $x \in X$ *there is* $y \in Tx$ *satisfying the following two conditions:*

$$bd(x, y) \leq d(x, Tx)$$

and

$$d(y, Ty) \leq cd(x, y)$$

Then there exists $z \in X$ *such that* $z \in Tz$ *provided a function* $f(x) = d(x, Tx)$*, for each* $x \in X$*, is lower semicontinuous.*

Moreover, Klim and Wardowski [10] generalized Theorem 1.1 of Feng and Liu and proved the following theorem:

Theorem 1.2. *(Klim and Wardowski* [10]*, Theorem 2.1) Let* (X, d) *be a complete metric space and let* T *be a mapping from* X *into* $Cl(X)$*. Assume that the following conditions hold:*

(i) the map $f : X \to R$*; defined by* $f(x) = d(x, Tx)$*, for each* $x \in X$*, is lower semicontinuous;*
(ii) there exist a constant $b \in (0, 1)$ *and a function* $\phi : [0, \infty) \to [0, b)$ *such that*

$$\limsup_{r \to t^+} \phi(r) < b, \text{ for each } t \in [0, \infty)$$

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and such that for any $x \in X$ there is $y \in Tx$ satisfying the conditions

$$bd(x,y) \leq d(x,Tx)$$

and

$$d(y,Ty) \leq \phi(d(x,y))d(x,y).$$

Then there exists $z \in X$ such that $z \in Tz$.

Recently, Ćirić [6] generalized Theorem 1.1(Theorem 3.1 of Feng and Liu) and Theorem 1.2 (Theorem 2.1 of Klim and Wardowski). He proved the following theorem:

Theorem 1.3. *(Ćirić [6], Theorem 2.1) Let (X,d) be a complete metric space and let T be a mapping from X into $Cl(X)$. Suppose that the function $f : X \to R$; defined by $f(x) = d(x,Tx)$, for each $x \in X$, is lower semicontinuous and that there exists a function $\phi : [0,\infty) \to [a,1)$, $0 < a < 1$, satisfying*

$$\limsup_{r \to t^+} \phi(r) < 1, \text{ for each } t \in [0,\infty).$$

Assume that for any $x \in X$ there is $y \in Tx$ satisfying the following two conditions:

$$\sqrt{\phi(f(x))}d(x,y) \leq f(x),$$

and

$$f(y) \leq \phi(f(x))d(x,y).$$

Then there exists $z \in X$ such that $z \in Tz$.

In 2007, Huang and Zhang [8] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. Whereas, Rezapour and Hamlbarani [16] omitted the assumption of normality in cone metric spaces, which is a milestone in developing fixed point theory in cone metric spaces. In 2009, Wardowski [17] introduced the concept of multivalued contractions in cone metric spaces and proved the following theorem:

Theorem 1.4. *(Wardowski [17], Theorem 3.1) Let (M,d) be a complete cone metric space, P be a normal cone with normal constant K, and let $T : M \to C(M)$. Assume that a function $I : M \to R$ defined by $I(x) = \inf_{y \in Tx} ||d(x,y)||, x \in M$ is lower semicontinuous. If there exist $\lambda \in (0,1), b \in (\lambda, 1]$ such that*

$$\forall_{x \in M} \exists_{y \in Tx} \exists_{v \in D(y,Ty)} \forall_{u \in D(x,Tx)} \{[bd(x,y) \preceq u] \wedge [v \preceq \lambda d(x,y)]\},$$

then $Fix(T) \neq \emptyset$.

Since then, numerous authors have started to generalize fixed point theorems in many various directions. For some recent results (see, e.g., [1, 2, 3, 11, 12, 14, 18]).

Now we recall some known notions, definitions and results for cone metric spaces which will be used in this work. Let E be a real Banach space and P be a subset of E. P is called a cone if and only if

1. P is closed, $P \neq \emptyset$, $P \neq \{0\}$;
2. for all $x, y \in P \Rightarrow \alpha x + \beta y \in P$, where $\alpha, \beta \in \mathbb{R}^+$;
3. $P \cap -P = \{0\}$.

For a given cone $P \subset E$, we define a *partial ordering* $\preceq$ with respect to P by the following: for $x, y \in E$, we say that $x \preceq y$ if and only if $y - x \in P$. Also, we write $x \ll y$ for $y - x \in intP$, where $intP$ denotes the *interior* of P (for details see [8]).

In this paper, we always suppose that E is a real Banach space, P is a cone in E with $intP \neq \emptyset$, and $\preceq$ is a partial ordering with respect to P.

Definition 1.5. [8] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(d1) $0 \preceq d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$
(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X, and (X, d) is called a *cone metric space*.

Definition 1.6. [8] Let (X, d) be a cone metric space and $\{x_n\}$ a sequence in X. Then

(1) $\{x_n\}$ converges to $x \in X$ whenever for every $c \in E$ with $0 \ll c$, there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$; we denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$;
(2) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$;
(3) (X, d) is said to be complete if every Cauchy sequence in X is convergent in X;
(4) A set $A \subseteq X$ is said to be closed if for any sequence $\{x_n\} \subset A$ converges to x, we have $x \in A$.
(5) A map $f : X \to \mathbb{R}$ is called lower semicontinuous if for any sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$, we have $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

Lemma 1.7. [8] *Let (X, d) be a cone metric space, and P be a normal cone with normal constant K. Let $\{x_n\}$ be any sequence in X. Then*

(1) $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$, as $n \to \infty$;
(2) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$, as $n, m \to \infty$.

The following remark is often used (in particular when dealing with cone metric spaces in which the cone need not be normal):

Remark 1.8. [9]

(1) If $u \preceq v$ and $v \ll w$, then $u \ll w$.
(2) If $0 \preceq u \ll c$ for each $c \in intP$, then $u = 0$.
(3) If $u \preceq v + c$ for each $c \in intP$, then $u \preceq v$.
(4) If $0 \preceq x \preceq y$ and $0 \leq a$, then $0 \preceq ax \preceq ay$.
(5) If $0 \preceq x_n \preceq y_n$ for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then $0 \preceq x \preceq y$.
(6) If $c \in intP$, $0 \preceq a_n$ and $a_n \to 0$, then there exists n_0 such that for all $n > n_0$ we have an $a_n \ll c$.

Let (X, d) be a cone metric space. We denote 2^X as a collection of nonempty subsets of X, $Cl(X)$ as a collection of nonempty closed subsets of X and $B(X)$ as a collection of nonempty bounded subsets of X. An element $x \in X$ is called a fixed point of a multivalued map $T : X \to 2^X$ if $x \in Tx$. Denote $Fix(T) = \{x \in X : x \in Tx\}$. For $T : X \to Cl(X)$, and $x \in X$ we denote $D(x, Tx) = \{d(x, z) : z \in Tx\}$. According to [3], we denote $s(p) = \{q \in E : p \preceq q\}$ for $p \in E$, and

$s(a,B) = \cup_{b\in B} s(d(a,b))$ for $a \in X$ and $B \in 2^X$. For $A, B \in B(X)$ we denote $s(A,B) = (\cap_{a\in A} s(a,B)) \cap (\cap_{b\in B} s(b,A))$.

In 2012, Cho at el.[4] defined sequentially lower semicontinuous as follow

Definition 1.9. Let (X,d) be a cone metric space, and let $A \in 2^X$. A function $h : X \to 2^P - \{\emptyset\}$ defined by $h(x) = s(x,A)$ is called sequentially lower semicontinuous if for any $c \in intP$, there exists $n_0 \in \mathbb{N}$ such that $h(x_n) \subset h(x) - c$ for all $n \geq n_0$, whenever $\lim_{n\to\infty} x_n = x$ for any sequence $\{x_n\} \subset X$ and $x \in X$.

The aim of this paper is to present more general results which unify and generalize the corresponding results of Feng and Liu [7], Klim and Wardowski [10], Ćirić [5],[6] and Wardowski [17] by using the concept of sequentially lower semicontinuous. We support our results by examples. In this paper we do not impose the normality condition for the cones, the only assumption is that the cone P is solid, that is $intP \neq \emptyset$

2. The Main Results

Theorem 2.1. *Let (X,d) be a complete cone metric space and $T : X \to Cl(X)$. Assume that there exist functions $\phi : P \to [0,1)$ and $\psi : P \to [\gamma, 1)$, $0 < \gamma < 1$, satisfying*

(i) $\limsup_{n\to\infty} \phi(r_n)/\psi(r_n) < 1$, *for any decreasing sequence $\{r_n\}_{n\geq 0}$ in P;*
(ii) for any $x \in X, u \in D(x,Tx)$, there exist $y \in Tx$ and $v \in D(y,Ty)$ such that

$$\alpha\psi(u)d(x,y) \preceq u,$$

and

$$v \preceq \beta\phi(u)d(x,y),$$

where $\alpha, \beta \in (0,1]$ with $0 < \beta/\alpha \leq 1$.

Moreover, assume that a function h defined by $h(x) = s(x,Tx)$ is sequentially lower semicontinuous. Then T has a fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary and fixed. Take any $u_0 \in D(x_0, Tx_0)$ then from (ii) there exist $x_1 \in Tx_0$ and $u_1 \in D(x_1, Tx_1)$ such that

$$\alpha\psi(u_0)d(x_0,x_1) \preceq u_0,$$

and

$$u_1 \preceq \beta\phi(u_0)d(x_0,x_1);$$

If $x_1 = x_0$, then x_0 is a fixed point of T. Let $x_1 \neq x_0$. Now by induction, we can construct a sequence $\{x_n\}_{n\geq 0}$ such that for $x_n \in X, u_n \in D(x_n, Tx_n)$ there exist $x_{n+1} \in Tx_n$ with $x_{n+1} \neq x_n$ for $n \in \mathbb{N} \cup \{0\}$ and $u_{n+1} \in D(x_{n+1}, Tx_{n+1})$ such that

$$\alpha\psi(u_n)d(x_n,x_{n+1}) \preceq u_n, \tag{2.1}$$

and

$$u_{n+1} \preceq \beta\phi(u_n)d(x_n,x_{n+1}), \tag{2.2}$$

from (2.1) and (2.2) we get that

$$u_{n+1} \preceq (\beta/\alpha)\frac{\phi(u_n)}{\psi(u_n)}u_n. \tag{2.3}$$

From (2.3) it is not difficult to show that $\{u_n\}_{n\geq 0}$ is a decreasing sequence. From (i) there exist $b \in (0,1)$ and $n_0 \in \mathbb{N} \cup \{0\}$ such that

$$\frac{\phi(u_n)}{\psi(u_n)} < b, \text{ for all } n \geq n_0.$$

Without loss of generality, we assume $n_0 = 0$. Then, by (2.3) we get the following

$$u_n \preceq \frac{\phi(u_{n-1})}{\psi(u_{n-1})} u_{n-1} \preceq \frac{\phi(u_{n-1})}{\psi(u_{n-1})} \frac{\phi(u_{n-2})}{\psi(u_{n-2})} u_{n-2} \preceq u_0 \prod_{i=0}^{n-1} \frac{\phi(u_i)}{\psi(u_i)}.$$

Hence,

$$u_n \preceq b^n u_0. \tag{2.4}$$

Now, since $\gamma \leq \psi(u_n)$ for all $n \geq 0$, we obtain from (2.1) that $d(x_n, x_{n+1}) \preceq (1/\alpha\gamma)u_n$. By (2.4) we get

$$d(x_n, x_{n+1}) \preceq \frac{b^n}{\alpha\gamma} u_0 \text{ for all } n \geq n_0. \tag{2.5}$$

For $n > m$ we have

$$\begin{aligned} d(x_m, x_n) &\preceq \sum_{i=m}^{n-1} d(x_i, x_{i+1}) \\ &\preceq \frac{1}{\alpha\gamma} u_0 \sum_{i=m}^{n-1} b^i \\ &= \frac{b^m}{\alpha\gamma} u_0 \sum_{i=0}^{n-m-1} b^i \\ &\preceq \frac{b^m}{\alpha\gamma(1-b)} u_0. \end{aligned}$$

For $c \in intP$ and by remark 1.8(1) and (6), we deduce that $d(x_m, x_n) \ll c$ for $n > m \geq N_1$ which means $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in (X, d). Thus, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

Now, we want to show that $x^* \in Tx^*$. h is sequentially lower semicontinuous, so for any $c \in intP$, there exists $N_2 \in \mathbb{N}$ such that $s(x_n, Tx_n) \subset s(x^*, Tx^*) - c/2$ and $d(x_n, x_{n+1}) \ll c/2$ for each $n \geq N_2$. Thus, there exists $z_n \in Tx^*$ such that $d(x^*, z_n) - c/2 \preceq d(x_n, x_{n+1})$. By Remark 1.8 (1) we obtain that $d(x^*, z_n) - c/2 \ll c/2$ which implies $d(x^*, z_n) \ll c$. Then $z_n \to x^*$. As Tx^* is closed, then $x^* \in Tx^*$, hence x is a fixed point of T. $\square$

If $\alpha = \beta = 1$, we get the following theorem which is a generalization of Theorem 3.1 of Feng and Liu [7], Theorem 2.1 of Klim and Wardowski [10] and Theorem 6 of Ćirić [5] from metric space to cone metric space. Moreover, it is an extension of Theorem 3.1 of Wardowski [17] without using the normality of P.

Theorem 2.2. *Let (X, d) be a complete cone metric space and $T : X \to Cl(X)$. Assume that there exist functions $\phi : P \to [0, 1)$ and $\psi : P \to [\gamma, 1)$, $0 < \gamma < 1$, satisfying*

(i) $\phi(t) \leq \psi(t)$ for each $t \in P$ and $\limsup_{n\to\infty} \phi(r_n)/\psi(r_n) < 1$, for any decreasing sequence $\{r_n\}_{n\geq 0}$ in P;

(ii) for any $x \in X, u \in D(x, Tx)$, there exist $y \in Tx$ and $v \in D(y, Ty)$ such that

$$\psi(u)d(x, y) \preceq u,$$

and

$$v \preceq \phi(u)d(x, y).$$

Furthermore, assume that a function h defined by $h(x) = s(x, Tx)$ is sequentially lower semicontinuous. Then T has a fixed point in X.

Proof. As in the proof of Theorem 2.1, we have a sequence $\{x_n\}_{n\geq 0}$ in X with $x_{n+1} \neq x_n$, $u_n \in D(x_n, Tx_n)$ and $x_{n+1} \in Tx_n$ for all $n \geq 0$ such that

$$\psi(u_n)d(x_n, x_{n+1}) \preceq u_n, \tag{2.6}$$

and

$$u_{n+1} \preceq \phi(u_n)d(x_n, x_{n+1}). \tag{2.7}$$

Since $\phi(u_n) \leq \psi(u_n)$ for each $n \geq 0$, we get that $\{u_n\}_{n\geq 0}$ is a decreasing sequence. From (2.6) and (2.7) we obtain that

$$u_{n+1} \preceq \frac{\phi(u_n)}{\psi(u_n)} u_n.$$

Then we use a similar argument to that given in the proof of Theorem 2.1 to complete the proof. □

The following result is a generalization of Theorem 2.1 of Ćirić [6] to the setting of cone metric spaces.

Corollary 2.3. *Let (X, d) be a complete cone metric space and $T : X \to Cl(X)$. Assume that there exists a function $\phi : P \to [\gamma, 1)$, $0 < \gamma < 1$, satisfying*

(i) $\limsup_{n\to\infty} \phi(r_n) < 1$, for any decreasing sequence $\{r_n\}_{n\geq 0}$ in P,
(ii) for any $x \in X, u \in D(x, Tx)$, there exist $y \in Tx$ and $v \in D(y, Ty)$ such that

$$\alpha\sqrt{\phi(u)}d(x, y) \preceq u,$$

and

$$v \preceq \beta\phi(u)d(x, y),$$

where $0 < \beta \leq \alpha \leq 1$.

Moreover, assume that a function h defined by $h(x) = s(x, Tx)$ is sequentially lower semicontinuous. Then T has a fixed point in X.

Proof. Let $\psi : P \to [\sqrt{\gamma}, 1)$ be defined as $\psi(t) = \sqrt{\phi(t)}$ for each $t \in P$. Then by applying Theorem 2.1 we get the desired result. □

Theorem 2.4. *Let (X, d) be a complete cone metric space and $T : X \to Cl(X)$. Assume that there exist functions $\phi : P \to [0, 1)$ and $\psi : P \to [\gamma, 1)$, $0 < \gamma < 1$, satisfying*

(i) $\limsup_{n\to\infty} \phi(r_n)/\psi(r_n) < 1$, for any decreasing sequence $\{r_n\}_{n\geq 0}$ in P,
(ii) for any $x \in X, u \in D(x, Tx)$, there exist $y \in Tx$ and $v \in D(y, Ty)$ such that

$$\alpha\psi(d(x, y))d(x, y) \preceq u,$$

and

$$v \preceq \beta\phi(d(x, y))d(x, y),$$

where $0 < \beta/\alpha \leq \gamma$.

Furthermore, assume that a function h defined by $h(x) = s(x, Tx)$ is sequentially lower semicontinuous. Then T has a fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary and fixed. Take any $u_0 \in D(x_0, Tx_0)$ then from (ii) there exist $x_1 \in Tx_0$ and $u_1 \in D(x_1, Tx_1)$ such that

$$\alpha\psi(d(x_0, x_1))d(x_0, x_1) \preceq u_0, \tag{2.8}$$

and

$$u_1 \preceq \beta\phi(d(x_0, x_1))d(x_0, x_1); \tag{2.9}$$

If $x_1 = x_0$, then x_0 is a fixed point of T. Let $x_1 \neq x_0$. From (2.8) and (2.9) we get that

$$u_1 \preceq (\beta/\alpha)\frac{\phi(d(x_0, x_1))}{\psi(d(x_0, x_1))}u_0.$$

Now, we choose $x_2 \in Tx_1$ and $u_2 \in D(x_2, Tx_2)$ such that

$$\alpha\psi(d(x_1, x_2))d(x_1, x_2) \preceq u_1, \tag{2.10}$$

and

$$u_2 \preceq \beta\phi(d(x_1, x_2))d(x_1, x_2),$$

from (2.9) and (2.10) we obtain

$$\alpha\psi(d(x_1, x_2))d(x_1, x_2) \preceq \beta\phi(d(x_0, x_1))d(x_0, x_1).$$

Since $\gamma \leq \psi(d(x_1, x_2))$ and $\phi(d(x_0, x_1)) < 1$ we have

$$\alpha\gamma d(x_1, x_2) \preceq \beta d(x_0, x_1).$$

Thus,

$$d(x_1, x_2) \preceq (\beta/\alpha\gamma)d(x_0, x_1) \tag{2.11}$$

but $\beta/\alpha \leq \gamma$, so that $d(x_1, x_2) \preceq d(x_0, x_1)$. By continuing this process, there exists an iterative sequence $\{x_n\}_{n\geq 0}$ such that for $x_n \in X, u_n \in D(x_n, Tx_n)$ there exist $x_{n+1} \in Tx_n$ with $x_{n+1} \neq x_n$ for $n \in \mathbb{N} \cup \{0\}$ and $u_{n+1} \in D(x_{n+1}, Tx_{n+1})$ such that

$$\alpha\psi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \preceq u_n, \tag{2.12}$$

and

$$u_{n+1} \preceq \beta\phi(d(x_n, x_{n+1}))d(x_n, x_{n+1}), \tag{2.13}$$

from (2.12) and (2.13) we get that

$$u_{n+1} \preceq (\beta/\alpha)\frac{\phi(d(x_n, x_{n+1}))}{\psi(d(x_n, x_{n+1}))}u_n. \tag{2.14}$$

From (2.13), we have

$$u_n \preceq \beta\phi(d(x_{n-1}, x_n))d(x_{n-1}, x_n). \tag{2.15}$$

From (2.12), (2.15) and $\psi(d(x_n, x_{n+1})) \geq \gamma$ we conclude that $\{d(x_n, x_{n+1})\}_{n\geq 0}$ is decreasing. Now, from (i) there exist $b \in (0, 1)$ and $n_0 \in \mathbb{N} \cup \{0\}$ such that

$$\frac{\phi(d(x_n, x_{n+1}))}{\psi(d(x_n, x_{n+1}))} < b, \text{ for all } n \geq n_0. \tag{2.16}$$

Then by using a similar argument to that given in the proof of Theorem 2.1, we have that $x^* \in Tx^*$. $\square$

Remark 2.5. In Theorem 2.4, if $\alpha = \beta = 1$ then the condition $\phi(t) \leq \psi(t)$ for each $t \in P$ will be essentially. Moreover, if $\phi(t)$ and $\psi(t)$ are constant functions in both Theorem 2.4 and Theorem 2.1, then these theorems will be the same.

Remark 2.6. If we take $\phi(t) \leq (\psi(t))^2$ for each $t \in P$ in Theorem 2.4, we will get the following corollary which is analogue of Theorem 3.2 of Lin and Chuang [13].

Corollary 2.7. *Let (X, d) be a complete cone metric space and $T : X \to Cl(X)$. Assume that there exist functions $\phi : P \to [0, 1)$ and $\psi : P \to [\gamma, 1)$, $0 < \gamma < 1$, satisfying*

(i) $\limsup_{n\to\infty} \psi(r_n) < 1$, *for any decreasing sequence $\{r_n\}_{n\geq 0}$ in P;*

(ii) for any $x \in X, u \in D(x, Tx)$, there exist $y \in Tx$ and $v \in D(y, Ty)$ such that

$$\alpha\psi(d(x, y))d(x, y) \preceq u,$$

and

$$v \preceq \beta\phi(d(x, y))d(x, y),$$

where $\alpha, \beta \in (0, 1]$ with $0 < \beta/\alpha \leq 1$ and $\phi(t) \leq (\psi(t))^2$ for each $t \in P$.

Moreover, assume that a function h defined by $h(x) = s(x, Tx)$ is sequentially lower semicontinuous. Then T has a fixed point in X.

The following corollary is a generalization of Theorem 2.2 of Ćirić [6].

Corollary 2.8. *Let (X, d) be a complete cone metric space and $T : X \to Cl(X)$. Assume that there exists a function $\phi : P \to [\gamma, 1)$, $0 < \gamma < 1$, satisfying*

(i) $\limsup_{n\to\infty} \phi(r_n) < 1$, *for any decreasing sequence $\{r_n\}_{n\geq 0}$ in P,*

(ii) for any $x \in X, u \in D(x, Tx)$, there exist $y \in Tx$ and $v \in D(y, Ty)$ such that

$$\alpha\sqrt{\phi(d(x, y))}d(x, y) \preceq u,$$

and

$$v \preceq \beta\phi(d(x, y))d(x, y),$$

where $0 < \beta/\alpha \leq \sqrt{\gamma}$.

Furthermore, assume that a function h defined by $h(x) = s(x, Tx)$ is sequentially lower semicontinuous. Then T has a fixed point in X.

The following example illustrates Theorem 2.2

Example 2.9. *Let $X = [0, 1]$, $E = \mathbb{R}^2$ be a Banach space with the maximum norm and $P = \{(x, 0) \in E : x \geq 0\}$. Let $d : X \times X \longrightarrow E$ be of the form $d(x, y) = (|x - y|, 0)$ and let $T : X \to Cl(X)$ be such that*

$$Tx = \begin{cases} \{x/3\}, & \text{if} \quad x \in [0, 1), \\ \\ \{0, 1/2, 2/7\}, & \text{if} \quad x = 1. \end{cases}$$

Let ϕ and ψ be constant functions defined as $\phi(u) = 2/3$ and $\psi(u) = 3/4$. Note that

$$h(x) = \begin{cases} \cup_{x\in[0,1)}\{q \in \mathbb{R}^2 : (\frac{2x}{3}, 0) \leq q\}, \\ \\ \{q \in \mathbb{R}^2 : (\frac{1}{2}, 0) \leq q\} \text{ if } x = 1. \end{cases}$$

Then h is sequentially lower semicontinuous. Further, for any $x \in [0, 1)$, we have

$$Tx = \{x/3\} \quad \text{and} \quad D(x, Tx) = \{(2x/3, 0)\},$$

and there exists $y = x/3 \in Tx$ *with*

$$Ty = \{x/9\} \quad \textit{and} \quad D(y, Ty) = \{(2x/9, 0)\}.$$

Now,

$$\psi(u)d(x, y) = 3/4(2x/3, 0) \leq u \textit{ where } u \in D(x, Tx)$$

and

$$v = (2x/9, 0) \leq 2/3(2x/3, 0) = \phi(u)d(x, y);$$

Now, for $x = 1$*, we have* $Tx = \{0, 1/2, 2/7\}$ *and* $D(x, Tx) = \{(1, 0), (1/2, 0), (5/7, 0)\}$*, so we can choose* $y = 1/2$ *and then* $Ty = \{1/6\}$ *and* $v = (1/3, 0) \in D(y, Ty)$*. Thus*

$$\psi(u)d(x, y) = 3/4(1/2, 0) \leq u \textit{ for each } u \in D(x, Tx)$$

and

$$v \leq 2/3(1/2, 0) = \phi(u)d(x, y);$$

Hence, all the hypotheses of Theorem 2.2 are satisfied. Therefore, 0 *is a fixed points of* T*.*

Now we shall prove a fixed point theorem for multivalued nonlinear contractions, which is a generalization of Theorem 5 of Ćirić [5] in the setting of cone metric space.

Theorem 2.10. *Let* (X, d) *be a complete cone metric space and* $T : X \to Cl(X)$*. Assume that there exist* $\phi : P \to [0, 1)$ *and* $\eta : X \to [0, 1]$ *satisfy the following*

(i) $\limsup_{n\to\infty} t_n\phi(r_n/t_n) < 1$*, for any decreasing sequence* $\{r_n\}_{n\geq 0}$ *in* P *and for any sequence* $\{t_n\}$ *in* $[0, 1]$*,*

(ii) for any $x \in X, u \in D(x, Tx)$*, there exist* $y \in Tx$ *and* $v \in D(y, Ty)$ *such that*

$$d(x, y) \preceq (2 - \eta(x)\phi(d(x, y)))u,$$

and

$$v \preceq \eta(x)\phi(d(x, y))d(x, y),$$

(iii) for any $x \in X$ *there exist* $y \in Tx$ *and* $z \in Ty$ *such that* $\eta(y)d(y, z) \preceq \eta(x)d(x, y)$*.*

Moreover, assume that a function h *defined by* $h(x) = s(x, Tx)$ *is sequentially lower semicontinuous. Then* T *has a fixed point in* X*.*

Proof. Let $x_0 \in X$ be arbitrary and fixed. Take any $u_0 \in D(x_0, Tx_0)$ then from (ii) there exist $x_1 \in Tx_0$ and $u_1 \in D(x_1, Tx_1)$ such that

$$d(x_0, x_1) \preceq (2 - \eta(x_0)\phi(d(x_0, x_1)))u_0, \tag{2.17}$$

and

$$u_1 \preceq \eta(x_0)\phi(d(x_0, x_1))d(x_0, x_1). \tag{2.18}$$

If $x_1 = x_0$, then x_0 is a fixed point of T. Let $x_1 \neq x_0$, from (2.17) and (2.18) we get that

$$u_1 \preceq \eta(x_0)\phi(d(x_0, x_1))(2 - \eta(x_0)\phi(d(x_0, x_1)))u_0. \tag{2.19}$$

Now, we put a function ψ from P into $[0, 1)$ by

$$\psi(\eta(x)d(x, y)) = \eta(x)\phi(d(x, y))(2 - \eta(x)\phi(d(x, y))). \tag{2.20}$$

That is,

$$\psi(\eta(x)d(x, y)) = 1 - (1 - \eta(x)\phi(d(x, y)))^2.$$

And $\psi(\eta(x)d(x,y)) < 1$. By (2.19) and (2.20) we obtain

$$u_1 \preceq \psi(\eta(x_0)d(x_0,x_1))u_0.$$

Now, we choose $x_2 \in Tx_1$ and $u_2 \in D(x_2, Tx_2)$ such that

$$d(x_1,x_2) \preceq (2-\eta(x_1)\phi(d(x_1,x_2)))u_1 \tag{2.21}$$

and

$$u_2 \preceq \eta(x_1)\phi(d(x_1,x_2))d(x_1,x_2) \tag{2.22}$$

by (2.20), (2.21) and (2.22) we have

$$u_2 \preceq \psi(\eta(x_1)d(x_1,x_2))u_1,$$

and

$$\eta(x_1)d(x_1,x_2) \preceq \eta(x_0)d(x_0,x_1).$$

Now by induction, we can construct a sequence $\{x_n\}_{n\geq 0}$ such that for $x_n \in X, u_n \in D(x_n, Tx_n)$ there exist $x_{n+1} \in Tx_n$ with $x_{n+1} \neq x_n$ for $n \in \mathbb{N} \cup \{0\}$ and $u_{n+1} \in D(x_{n+1}, Tx_{n+1})$ such that

$$d(x_n,x_{n+1}) \preceq (2-\eta(x_n)\phi(d(x_n,x_{n+1})))u_n \tag{2.23}$$

and

$$u_{n+1} \preceq \eta(x_n)\phi(d(x_n,x_{n+1}))d(x_n,x_{n+1}).$$

By (2.20) we obtain

$$u_{n+1} \preceq \psi(\eta(x_n)d(x_n,x_{n+1}))u_n. \tag{2.24}$$

Furthermore,

$$\eta(x_n)d(x_n,x_{n+1}) \preceq \eta(x_{n-1})d(x_{n-1},x_n).$$

Now, we have the sequence $\{\eta(x_n)d(x_n,x_{n+1})\}_{n\geq 0}$ which is decreasing. By choosing $t_n = (1,1,1,\ldots)$ in (i) then there exist $b \in (0,1)$ and $n_0 \in \mathbb{N} \cup \{0\}$ such that

$$\psi(\eta(x_n)d(x_n,x_{n+1})) < b, \text{ for all } n \geq n_0.$$

Without loss of generality, we can take $n_0 = 0$. Then, by (2.24) we get the following

$$u_n \preceq \psi(\eta(x_{n-1})d(x_{n-1},x_n))u_{n-1} \preceq \cdots \preceq \prod_{i=0}^{n-1} \psi(\eta(x_i)d(x_i,x_{i+1}))u_0.$$

That is,

$$u_n \preceq b^n u_0. \tag{2.25}$$

From (2.23) we have

$$d(x_n,x_{n+1}) \preceq 2u_n,$$

and by (2.25) we obtain

$$d(x_n,x_{n+1}) \preceq 2b^n u_0.$$

For $n > m$, we have

$$\begin{aligned} d(x_m,x_n) &\preceq \sum_{i=m}^{n-1} d(x_i,x_{i+1}) \\ &\preceq 2b^m u_0 \sum_{i=0}^{n-m-1} b^i \\ &\preceq \frac{2b^m}{1-b} u_0 \end{aligned}$$

Thus, for $c \in intP$ we have $(2b^m/1-b)u_0 \ll c$, for all $m \geq N_1$. Using Remark 1.8 (1), we deduce that $d(x_m, x_n) \ll c$ for $n > m \geq N_1$. Then $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in a complete cone metric space X, so there exists $x^* \in X$such that $\lim_{n\to\infty} x_n = x^*$. Then using the same argument that given in the proof of Theorem 2.1, we get that $x^* \in Tx^*$. □

Corollary 2.11. *Let (X, d) be a complete cone metric space and $T : X \to Cl(X)$. Assume that there exists a function $\phi : P \to [0, 1)$ satisfy the following*

(i) $\limsup_{n\to\infty} \phi(r_n) < 1$, *for any decreasing sequence $\{r_n\}_{n\geq 0}$ in P;*

(ii) for any $x \in X, u \in D(x, Tx)$, there exist $y \in Tx$ and $v \in D(y, Ty)$ such that

$$d(x, y) \preceq (2 - \phi(d(x, y)))u,$$

and

$$v \preceq \phi(d(x, y))d(x, y),$$

(iii) $d(y, z) \preceq d(x, y)$ for $x \in X$ and some $y \in Tx$ and $z \in Ty$.

Furthermore, assume that a function h defined by $h(x) = s(x, Tx)$ is sequentially lower semicontinuous. Then T has a fixed point in X.

The following example illustrate Theorem 2.10

Example 2.12. *Let $X = [0, 1/2]$, $E = \mathbb{R}^2$ be a Banach space with the maximum norm and $P = \{(x, y) \in E : x, y \geq 0\}$. Let $d : X \times X \longrightarrow E$ be of the form $d(x, y) = (|x - y|, \beta|x - y|)$, where $\beta < 1$ and let $T : X \to Cl(X)$ be such that*

$$Tx = \begin{cases} \{x^2/2\}, & \text{if} \quad x \in [0, 1/3) \cup (1/3, 1/2], \\ \{0, 1/3\}, & \text{if} \quad x = 1/3. \end{cases}$$

Let $\phi : P \to [0, 1)$ be defined as

$$\phi(s, t) = \begin{cases} s + 1/2, & \text{if} \quad s \text{ and } t \in [0, 1/2), \\ 5/8, & \text{if} \quad s \text{ or } t \notin [0, 1/2). \end{cases}$$

and $\eta : X \to [0, 1]$ such that $\eta(x) = x$. Note that

$$h(x) = \begin{cases} \cup_{x\in[0,1/2]-\{1/3\}}\{q \in \mathbb{R}^2 : (x - x^2/2, \beta(x - x^2/2)) \leq q\}, \\ \{q \in \mathbb{R}^2 : (0, 0) \leq q\} \text{ if } x = \frac{1}{3}. \end{cases}$$

Then h is sequentially lower semicontinuous. Further, for any $x \in [0, 1/3) \cup (1/3, 1/2]$, we have

$$Tx = \{x^2/2\} \quad \text{and} \quad D(x, Tx) = \{(x - x^2/2, \beta(x - x^2/2))\},$$

and there exists $y = x^2/2 \in Tx$ with

$$Ty = \{x^4/8\} \quad \text{and} \quad D(y, Ty) = \{(x^2/2 - x^4/8, \beta(x^2/2 - x^4/8))\}.$$

Now, since $\phi(d(x, y))$ and $\eta(x) \leq 1$,

$$d(x, y) = (x - x^2/2, \beta(x - x^2/2)) \leq (2 - \eta(x)\phi(d(x, y)))u \quad \text{for} \quad u \in D(x, Tx).$$

Since $x \leq 1/2$, $x - x^2/2$ and $\beta(x - x^2/2) \in [0, 1/2)$ for any $x \in X$. Therefore,

$$v \preceq \eta(x)\phi(d(x, y))d(x, y) \quad \text{for} \quad v \in D(y, Ty).$$

Now, for $x = 1/3$*, we have* $Tx = \{0, 1/3\}$ *and* $D(x, Tx) = \{(0,0), (1/3, \beta/3)\}$ *so we can choose* $y = 1/3$ *and* $v = (0,0) \in D(y, Ty) = \{(0,0), (1/3, \beta/3)\}$ *such that*

$$d(x, y) \preceq (2 - \eta(x)\phi(d(x, y)))u,$$

and

$$v \preceq \eta(x)\phi(d(x, y))d(x, y);$$

Hence, the condition (ii) is satisfy for $x \in X$*.*

For $x \in [0, 1/3) \cup (1/3, 1/2]$ *we take* $y = x^2/2 \in Tx$ *and* $z = x^4/8 \in Ty$*, for* $x = 1/3$ *we choose* $y = 1/3$ *and* $z = 1/3$*. Thus,* $\eta(y)d(y, z) \preceq \eta(x)d(x, y)$ *for any* $x \in X$*. Thus* 0 *and* $1/3$ *are fixed points of* T*.*

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