Trees with 2-reinforcement number three*

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Abstract

A vertex subset S of a graph G is a 2-dominating set of G if every vertex not in S is adjacent to two vertices of S. The 2-domination number $\gamma_2(G)$ is the minimum cardinality of a 2-dominating set of G. The 2-reinforcement number $r_2(G)$ is the smallest number of extra edges whose addition to G results in a graph G' with $\gamma_2(G') < \gamma_2(G)$. Let T be a tree. It is showed by Lu, Hu and Xu that $r_2(T) \leq 3$. In this paper, we will show that $r_2(T) = 3$ if and only if there is a 2-dominating set S of T such that T contains neither S-vulnerable vertices nor S-vulnerable paths.

Keywords: 2-domination, 2-reinforcement number, S-vulnerable, trees

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1 Introduction

For terminology and notation not defined here we refer the reader to [6, 13, 14]. Let $G = (V(G), E(G))$ be a simple graph and $x \in V(G)$. The *open neighborhood*, the *closed neighborhood* and the *degree* of x are denoted by $N_G(x) = \{y \mid xy \in E(G)\}$, $N_G[x] = N_G(x) \cup \{x\}$ and $d_T(x) = |N_G(x)|$, respectively. A vertex of degree one is called a *leaf* and its neighbor is called a *stem*. Let S be a subset of $V(G)$ with $x \in S$. A vertex $y \in N_G(x)$ is called a 2-*private neighbor* of x with respect to S if $y \notin S$ and $|N_G(y) \cap S| = 2$. The 2-*private neighborhood* of x with respect to S, denoted by $N_2(x, S, G)$, is defined as the set of 2-private neighbors of x with respect to S in G.

For any $S \subseteq V(G)$, the subgraph induced by $V(G) - S$ is denoted by $G - S$. For $B \subseteq E(G)$, we use $G - B$ to denote the subgraph with vertex set $V(G)$ and edge set $E(G) - B$. To simplify notation, if $S = \{v\}$ and $B = \{xy\}$, we write $G - v$ and $G - xy$ for $G - \{v\}$ and $G - \{xy\}$, respectively.

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Let p be a positive integer. In 1985, Fink and Jacobson [11] introduced the concept of p-domination. A set S of $V(G)$ is a *p-dominating set* of G if for each vertex $x \in V(G) \setminus S$, $|N_G(x) \cap S| \geq p$. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality of a p-dominating set of G. A p-dominating set with cardinality $\gamma_p(G)$ is called a *$\gamma_p(G)$-set*. Note that the $\gamma_1(G)$-set is the well-known minimum dominating set of G, and so $\gamma_1(G) = \gamma(G)$. For $S, T \subseteq V(G)$, S p-dominates T if $|N_G(x) \cap S| \geq p$ for each $x \in T \setminus S$. Up to the present, p-domination have been studied by a number of researchers (see, for example, [1, 2, 3, 4, 7, 8, 9, 10, 12, 18, 22, 24]).

In order to investigate the vulnerability of p-domination, Lu, Hu and Xu [20] recently introduce the *p-reinforcement number* $r_p(G)$ of a graph G, which is the smallest number of extra edges whose addition to G results in a graph G' with $\gamma_p(G') < \gamma_p(G)$. If $\gamma_p(G) \leq p$, they define $r_p(G) = 0$. Clearly, the p-reinforcement number is a generalization of the classical reinforcement number which was introduced by Kok and Mynhardt [19] and studied by some authors [5, 15, 16, 17, 25]. In [20], the authors presented an equivalent parameter for calculating $r_p(G)$. As applications of this parameter, they showed that the decision problem on $r_p(G)$ is NP-hard and established some upper bounds of $r_p(G)$. In particular, they obtained the following result.

Theorem 1.1 ([20]) *$r_p(T) \leq p + 1$ for any tree T and $p \geq 2$.*

In [23], Lu and Xu gave a constructive characterization of the trees attaining the upper bound in Theorem 1.1 when $p \geq 3$. However, for $p = 2$, the characterization is invalid because a key conclusion is not true. In this paper, we will present an equivalent condition for all trees with 2-reinforcement number 3. For this purpose, we introduce two additional notations.

Let S be a vertex subset of a graph G. A vertex $x \in S$ is *S-vulnerable* in G if

$$|N_2(x, S, G)| \leq \min\{2, |N_G(x) \cap S|\}. \tag{1.1}$$

Let ℓ be a positive integer. A path $P = x_0x_1 \ldots x_\ell$ is *S-vulnerable* in G if

(1) $S \cap V(P)$ is a 2-dominating set of P, and

(2) for every $x \in S \cap V(P)$,

$$|N_2(x, S, G) \setminus V(P)| \leq \begin{cases} \min\{1, |N_G(x) \cap S|\} & \text{if } x \in \{x_0, x_\ell\}; \\ 0 & \text{if } x \notin \{x_0, x_\ell\}. \end{cases} \tag{1.2}$$

We now state our main result as follows.

Theorem 1.2 *Let T be a tree. Then $r_2(T) = 3$ if and only if there exists a 2-dominating set S of T such that T contains neither S-vulnerable vertices nor S-vulnerable paths.*

In Section 2, we will give some lemmas which will be used later. The proof of Theorem 1.2 is postponed to Sections 3.

2 Lemmas

Let G be a graph and $X \subseteq V(G)$ with $|X| \geq 2$. For any $x \in V(G)$, define

$$\eta_2(x, X, G) = \begin{cases} \max\{0, 2 - |N_G(x) \cap X|\} & \text{if } x \notin X; \\ 0 & \text{if } x \in X, \end{cases} \tag{2.1}$$

and then there is a subset $B_x \subseteq E(G^c)$ with $|B_x| = \eta_2(x, X, G)$ such that x can be 2-dominated by X in $G+B_x$, where G^c is the complement of G. Hence X is a 2-dominating set of $G + (\cup_{x\in V(G)} B_x)$. By the definition of r_2,

$$r_2(G) \leq | \cup_{x\in V(G)} B_x| = \sum_{x\in V(G)} \eta_2(x, X, G).$$

Motivated by this inequality, Lu, Hu and Xu [20] define for any $X, S \subseteq V(G)$,

$$\eta_2(S, X, G) = \sum_{x\in S} \eta_2(x, X, G) \tag{2.2}$$

and give the following two lemmas.

Lemma 2.1 ([20]). *Let G be a graph with $\gamma_2(G) \geq 3$. Then*

$$r_2(G) = \min\{\eta_2(V(G), X, G) \ : \ X \subseteq V(G) \text{ with } |X| < \gamma_2(G)\}.$$

A set $X \subseteq V(G)$ is called an *$\eta_2(G)$-set* if $|X| < \gamma_2(G)$ and $r_2(G) = \eta_2(V(G), X, G)$.

Lemma 2.2 ([20]). *Let G be a graph. If X is an $\eta_2(G)$-set, then $|X| = \gamma_2(G) - 1$.*

Lemma 2.3 *Let G be a graph containing a path $P = xy_1y_2y_3z$ with $d_G(y_i) = 2$ for $i = 1, 2, 3$. Denote by G' the graph obtained from G by replacing $\{y_1, y_2, y_3\}$ with a single vertex y adjacent to x and z. If $\gamma_2(G') \geq 3$, then $r_2(G) \geq \min\{3, r_2(G')\}$.*

Proof. Notice that $N_{G'}(y) = \{x, z\}$. Let D be a $\gamma_2(G')$-set. If $y \notin D$, to 2-dominate y, $x \in D$ and $z \in D$, and hence $D \cup \{y_2\}$ is a 2-dominating set of G, which means that

$$\gamma_2(G) \leq |D \cup \{y_2\}| = |D| + 1 = \gamma_2(G') + 1. \tag{2.3}$$

If $y \in D$, then let $S = (D \setminus \{y\}) \cup \{y_1, y_3\}$. Clearly, $|N_G(x) \cap S| = |N_{G'}(x) \cap D|$ and $|N_G(z) \cap S| = |N_{G'}(z) \cap D|$. It follows that $\{x, z\}$ is 2-dominated by S. Since $D \setminus \{y\}$ 2-dominates $V(G') \setminus \{x, y, z\}$ $(= V(G) \setminus V(P))$ and $N_G(y_2) = \{y_1, y_3\}$, S is a 2-dominating set of G, and thus

$$\gamma_2(G) \leq |S| = |(D \setminus \{y\}) \cup \{y_1, y_3\}| = |D| + 1 = \gamma_2(G') + 1. \tag{2.4}$$

Summing up (2.3) and (2.4), we obtain that $\gamma_2(G) \leq \gamma_2(G') + 1$.

In the following, we show that $r_2(G) \geq \min\{3, r_2(G')\}$. Suppose, to be contrary, that $r_2(G) < \min\{3, r_2(G')\}$, and let X be an $\eta_2(G)$-set such that $|X \cap \{y_1, y_2, y_3\}|$ is as small as possible. Then Lemmas 2.1 and 2.2 yield that

$$\eta_2(V(G), X, G) = r_2(G) < \min\{3, r_2(G')\} \tag{2.5}$$

and $|X| = \gamma_2(G) - 1$, respectively.

If $X \cap \{y_1, y_2, y_3\} = \emptyset$, we obtain from (2.1) that $\eta_2(y_1, X, G) \geq 1$ and $\eta_2(y_2, X, G) = 2$. By (2.2),

$$\eta_2(V(G), X, G) \geq \eta_2(y_1, X, G) + \eta_2(y_2, X, G) \geq 3.$$

This contradicts (2.5).

If $X\cap\{y_1,y_2,y_3\}=\{y_1\}$ or $\{y_3\}$, without loss of generality, say $X\cap\{y_1,y_2,y_3\}=\{y_1\}$, then (2.1) implies that $\eta_2(y_2,X,G)=1$ and $\eta_2(y_3,X,G)\geq 1$. By (2.2) and (2.5),

$$2\leq \eta_2(y_2,X,G)+\eta_2(y_3,X,G)\leq \eta_2(V(G),X,G)<3,$$

which implies that $\eta_2(y_3,X,G)=1$ (and thus $z\in X$ since $N_G(y_3)=\{y_2,z\}$ and $y_2\notin X$) and $\eta_2(V(G)\setminus\{y_2,y_3\},X,G)=0$. Let $X'=X\setminus\{y_1\}$. Then it follows from (2.2) that

$$\eta_2(x,X,G)\leq \eta_2(V(G)\setminus\{y_2,y_3\},X,G)=0 \tag{2.6}$$

and

$$\begin{aligned}\eta_2(V(G')\setminus\{x,y\},X',G') &= \eta_2(V(G)\setminus\{x,y_1,y_2,y_3\},X,G)\\ &\leq \eta_2(V(G)\setminus\{y_2,y_3\},X,G)\\ &= 0.\end{aligned}$$

By (2.1), (2.6) implies that either $x\in X$ or $|N_G(x)\cap X|\geq 2$, and hence

$$\eta_2(x,X',G')\leq 1$$

because $y_1\in N_G(x)\cap X$ but $y_1\notin X'$. Since $z\in X$, $z\in X'$ and so $\eta_2(y,X',G')\leq 1$. Recalling the facts that $|X|=\gamma_2(G)-1$ and $\gamma_2(G)\leq\gamma_2(G')+1$, we obtain that

$$|X'|=|X|-1=\gamma_2(G)-2\leq\gamma_2(G')-1.$$

Therefore,

$$\begin{aligned}r_2(G') &\leq \eta_2(V(G'),X',G') \quad \text{(by Lemma 2.1)}\\ &= \eta_2(y,X',G')+\eta_2(x,X',G')+\eta_2(V(G')\setminus\{x,y\},X',G') \quad \text{(by (2.2))}\\ &\leq 1+1+0\\ &= \eta_2(y_2,X,G)+\eta_2(y_3,X,G)+\eta_2(V(G)\setminus\{y_2,y_3\},X,G)\\ &= \eta_2(V(G),X,G) \quad \text{(by (2.2))}\\ &= r_2(G). \quad \text{(since } X \text{ is an } \eta_2(G)\text{-set)}\end{aligned}$$

This also contradicts (2.5).

If $X\cap\{y_1,y_2,y_3\}=\{y_2\}$, let $X'=X\setminus\{y_2\}$, then $|X'|=|X|-1=\gamma_2(G)-2\leq\gamma_2(G')-1$. Since $d_{G'}(y)=2$ and $d_G(y_i)=2$ for $i\in\{1,2,3\}$, it follows from (2.1) and (2.2) that

$$\begin{aligned}\eta_2(y,X',G') &= \eta_2(\{y_1,y_2,y_3\},X,G)\\ \eta_2(V(G)\setminus\{y\},X',G') &= \eta_2(V(G)\setminus\{y_1,y_2,y_3\},X,G).\end{aligned}$$

Hence by Lemma 2.1 and (2.2),

$$\begin{aligned}r_2(G') &\leq \eta_2(V(G'),X',G')\\ &= \eta_2(\{y,X',G'\})+\eta_2(V(G')\setminus\{y\},X',G')\\ &= \eta_2(\{y_1,y_2,y_3\},X,G)+\eta_2(V(G)\setminus\{y_1,y_2,y_3\},X,G)\\ &= \eta_2(V(G),X,G)\\ &= r_2(G),\end{aligned}$$

which contradicts (2.5).

If $|X \cap \{y_1, y_2, y_3\}| \geq 2$, then we may assume that $X \cap \{y_1, y_2, y_3\} = \{y_1, y_3\}$ by the choice of X. Let $X' = (X \setminus \{y_1, y_3\}) \cup \{y\}$. Clearly, $|X'| = |X| - 1 = \gamma_2(G) - 2 = \gamma_2(G') - 1$ and

$$\eta_2(V(G') \setminus \{x, y, z\}, X', G') = \eta_2(V(G) \setminus V(P), X, G).$$

Since $y \in X'$ and $\{y_1, y_3\} \subseteq X$, we obtain from (2.1) and (2.2) that $\eta_2(\{x, z\}, X', G') = \eta_2(\{x, z\}, X, G)$ and $\eta_2(y, X', G') = 0 = \eta_2(\{y_1, y_2, y_3\}, X, G)$. By Lemma 2.1 and (2.2),

$$\begin{aligned}
r_2(G') &\leq \eta_2(V(G'), X', G') \\
&= \eta_2(V(G') \setminus \{x, y, z\}, X', G') + \eta_2(\{x, z\}, X', G') + \eta_2(y, X', G') \\
&= \eta_2(V(G) \setminus V(P), X, G) + \eta_2(\{x, z\}, X, G) + \eta_2(\{y_1, y_2, y_3\}, X, G) \\
&= \eta_2(V(G), X, G) \\
&= r_2(G),
\end{aligned}$$

which contradicts (2.5) again. ∎

3 Proof of Theorem 1.2

In this section we present a proof of Theorem 1.2. Recall the statement of the theorem as follows: a tree T satisfies $r_2(T) = 3$ if and only if there exists a 2-dominating set S of T such that T contains neither S-vulnerable vertices nor S-vulnerable paths. Let us begin with two simple observations.

Observation 3.1 *Every 2-dominating set of a graph G contains all leaves of G.*

Observation 3.2 *Let P be a path with length ℓ. Then $\gamma_2(P) = \lfloor (\ell + 1)/2 \rfloor + 1$.*

Lemma 3.3 *Let T be a tree with a $\gamma_2(T)$-set S. If T contains S-vulnerable vertices, then $r_2(T) \leq 2$.*

Proof. Let x be an S-vulnerable vertex in T. Then $x \in S$. Since S is a $\gamma_2(T)$-set, $|S \setminus \{x\}| < |S| = \gamma_2(T)$ and for $y \in N_T(x) \setminus S$, $|N_T(y) \cap S| \geq 2$ with equality if and only if $y \in N_2(x, S, T)$. Thus for $y \in N_T(x)$, we can directly calculate by (2.1) that

$$\eta_2(y, S \setminus \{x\}, T) = \begin{cases} 1 & \text{if } y \in N_2(x, S, T); \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Hence

$$\begin{aligned}
r_2(T) &\leq \eta_2(V(T), S \setminus \{x\}, T) \quad \text{(by Lemma 2.1)} \\
&= \eta_2(x, S \setminus \{x\}, T) + \eta_2(N_T(x), S \setminus \{x\}, T) \quad \text{(since } S \text{ is a } \gamma_2(T)\text{-set)} \\
&= \eta_2(x, S \setminus \{x\}, T) + \sum_{y \in N_T(x)} \eta_2(y, S \setminus \{x\}, T) \quad \text{(by (2.2))} \\
&= \max\{0, 2 - |N_T(x) \cap (S \setminus \{x\})|\} + |N_2(x, S, T)| \quad \text{(by (2.1) and (3.1))} \\
&\leq \max\{0, 2 - |N_T(x) \cap S|\} + \min\{2, |N_T(x) \cap S|\} \quad \text{(by (1.1))} \\
&= 2.
\end{aligned}$$

The proof of the lemma is completed. ∎

Lemma 3.4 *Let T be a tree with a $\gamma_2(T)$-set S. If T contains S-vulnerable paths, then $r_2(T) \le 2$.*

Proof. It is sufficient to consider the case that T has no S-vulnerable vertices by Lemma 3.3. Let $P = x_0x_1 \dots x_\ell$ be a shortest S-vulnerable path in T. From the definition of S-vulnerable path, we know that $\ell \ge 1$, $S \cap V(P)$ 2-dominates $V(P)$ and every vertex in $S \cap V(P)$ satisfies (1.2).

Suppose that there is some $i \in \{0, 1, \dots, \ell - 1\}$ such that $x_i \in S$ and $x_{i+1} \in S$. Then $|N_T(x_i) \cap S| \ge |\{x_{i+1}\}| = 1$ and $|N_2(x_i, S, T)| \le 1$ by (1.2). It follows that

$$|N_2(x_i, S, T)| \le 1 \le \min\{2, |N_T(x_i) \cap S|\},$$

which means that x_i is S-vulnerable in T. This contradicts the assumption that T has no S-vulnerable vertices. So arbitrary two vertices in $S \cap V(P)$ are nonadjacent in T.

If ℓ is odd, since $S \cap V(P)$ 2-dominates $V(P)$, $|S \cap V(P)| \ge \gamma_2(P) = (\ell + 1)/2 + 1$ by Observation 3.2. This implies that there are two adjacent vertices in $S \cap V(P)$, a contradiction. Assume that ℓ is even below.

Because $S \cap V(P)$ 2-dominates $V(P)$, $|S \cap V(P)| \ge \ell/2 + 1$ by Observation 3.2. Let

$$X = (S \setminus V(P)) \cup \{x_1, x_3, \dots, x_{\ell-1}\}.$$

Since S is a $\gamma_2(T)$-set and every vertex in $(S \cap V(P)) \setminus \{x_0, x_\ell\}$ has no 2-private neighbor in $V(T) \setminus V(P)$ by (1.2), X has cardinality $|X| = |S| - |S \cap V(P)| + \ell/2 \le |S| - 1 < \gamma_2(T)$, and 2-dominates $V(T) \setminus (N_T[x_0] \cup N_T[x_\ell])$. Hence by Lemma 2.1 and (2.2),

$$\begin{aligned} r_2(T) \le \eta_2(V(T), X, T) &= \eta_2(N_T[x_0] \cup N_T[x_\ell], X, T) \\ &\le \eta_2(N_T[x_0], X, T) + \eta_2(N_T[x_\ell], X, T). \end{aligned} \tag{3.2}$$

We claim that $\eta_2(N_T[x_0], X, T) \le 1$. Note that $x_0 \in S \cap V(P)$ by Observation 3.1 since $S \cap V(P)$ is a 2-dominating set of P. Because $S \cap V(P)$ has no two adjacent vertices, $x_1 \notin S$ and thus

$$|N_T(x_0) \cap X| = |(N_T(x_0) \cap S) \cup \{x_1\}| = |N_T(x_0) \cap S| + 1. \tag{3.3}$$

Since $x_0 \in S \setminus X$, $x_1 \in X$ and S is a $\gamma_2(T)$-set, we can obtain from (2.1) that for any $y \in N_T(x_0)$,

$$\eta_2(y, X, T) = \begin{cases} 1 & \text{if } y \in N_2(x_0, S, T) \setminus \{x_1\}; \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Therefore,

$$\begin{aligned} \eta_2(N_T[x_0], X, T) &= \eta_2(x_0, X, T) + \sum_{y \in N_T(x_0)} \eta_2(y, X, T) \quad \text{(by (2.2))} \\ \text{(by (2.1) and (3.4))} &= \max\{0, 2 - |N_T(x_0) \cap X|\} + |N_2(x_0, S, T) \setminus \{x_1\}| \\ \text{(by (3.3) and (1.2))} &\le \max\{0, 1 - |N_T(x_0) \cap S|\} + \min\{1, |N_T(x_0) \cap S|\} \\ &= 1. \end{aligned}$$

The claim is true.

By the symmetry, $\eta_2(N_T[x_\ell], X, T) \le 1$. It follows from (3.2) that $r_2(T) \le 2$. We complete the proof of the lemma. ∎

Summing up Lemmas 3.3 and 3.4, the necessity of Theorem 1.2 follows. For the sufficiency, we need the following four lemmas.

Lemma 3.5 *Let T be a tree, $xy \in E(T)$ and T_x the component of $T - xy$ containing x. Let $S \subseteq V(T)$ and P be a path in T_x. If $S \cap V(T_x)$ is a 2-dominating set of T_x and P is $S \cap V(T_x)$-vulnerable in T_x but not S-vulnerable in T, then $x \in S \cap V(P)$ and $y \in N_2(x, S, T)$.*

Proof. Because P is $S \cap V(T_x)$-vulnerable in T_x but not S-vulnerable in T, (1.2) implies that there is a vertex $z \in S \cap V(P)$ such that

$$|N_2(z, S, G) \setminus V(P)| > |N_2(z, S \cap V(T_x), T_x) \setminus V(P)|. \tag{3.5}$$

In order to prove $x \in S \cap V(P)$ and $y \in N_2(x, S, T)$, it suffices to show that $z = x$.

Assume, to be contrary, that $z \neq x$. Then $N_T(z) \subseteq V(T_x)$ and so it follows from (3.5) that $x \in N_2(z, S, T) \setminus V(P)$ but $x \notin N_2(z, S \cap V(T_x), T_x) \setminus V(P)$. By the definition of 2-private neighbor, we obtain that $x \notin S$, $|N_T(x) \cap S| = 2$ and $|N_{T_x}(x) \cap (S \cap V(T_x))| \neq 2$. Furthermore, $|N_{T_x}(x) \cap (S \cap V(T_x))| \geq 3$ since $S \cap V(T_x)$ is a 2-dominating set of T_x. Hence we obtain a contradiction that

$$2 = |N_T(x) \cap S| \geq |N_T(x) \cap (S \cap V(T_x)| = |N_{T_x}(x) \cap (S \cap V(T_x))| \geq 3.$$

∎

Lemma 3.6 ([21]) *Let S be a 2-dominating set of a tree T. Then S is the unique $\gamma_2(T)$-set if and only if, for each $x \in S$ with $d_T(x) \geq 2$, $N_T(x) \cap S = \emptyset$ or $|N_2(x, S, T)| \geq 2$.*

Lemma 3.7 ([23]) *Let T be a tree with $r_2(T) = 3$ and S the unique $\gamma_2(T)$-set. For $x \in S$ and $y \in N_2(x, S, T)$, denote by T_y the component of $T - x$ containing y. If T_y is not the complete graph K_2, then $r_2(T_y) = 1$ and $S \cap V(T_y)$ is an $\eta_2(T_y)$-set.*

For $t \geq 2$, a *spider* S_t is a tree obtained from a star $K_{1,t}$ by attaching one leaf at each leaf of $K_{1,t}$.

Lemma 3.8 *Let S be a 2-dominating set of a tree T. If T contains neither S-vulnerable vertices nor S-vulnerable paths, then we have the following statements.*
(a) *S is the unique $\gamma_2(T)$-set.*
(b) *$|V(T)| \geq 7$ with equality if and only if $T = S_3$.*
(c) *$r_2(T) \geq 3$.*

Proof. (a) Let S' be the set of vertices in T with degree at least 2. If $S' = \emptyset$, then S is the unique $\gamma_p(T)$-set by Observation 3.1, and so the conclusion (a) follows. Assume now that $S' \neq \emptyset$, and let $x \in S'$. If x doesn't satisfy the second condition in Lemma 3.6, that is, $|N_2(x, S, T)| \leq 1$, since x is not S-vulnerable in T, it follows from (1.1) that

$$|N_T(x) \cap S| < |N_2(x, S, T)| \leq 1.$$

This fact implies that x satisfies the first condition in Lemma 3.6. By Lemma 3.6, S is the unique $\gamma_p(T)$-set. The conclusion (a) is true.

(b) Since T is a tree without S-vulnerable vertices or paths, we can directly check the validity of the conclusion (b), and omit the proof.

(c) Let $|V(T)| = n$. Then $n \geq 7$ by (b). We prove $r_2(T) \geq 3$ by induction on n.

If $n = 7$, then (b) implies that T is the spider S_3. It is not hard to determine that $r_2(T) = 3$ by (2.1) and (2.2). This establishes the base case.

Let $n \geq 8$. For any tree T' with order $n' < n$, assume that $r_2(T') \geq 3$ if there exists a 2-dominating set S' of T' such that T' has neither S'-vulnerable vertices nor S'-vulnerable paths.

We will now prove the following claims.

Claim 1 *If T has a path $xy_1y_2y_3z$ with $d_T(y_i) = 2$ for $i \in \{1, 2, 3\}$, then $r_2(T) \geq 3$.*

Proof. Replacing the path $xy_1y_2y_3z$ by a path xyz, we obtain a tree T' with order less than n. Note that S is the unique $\gamma_2(T)$-set by (a). Since $d_T(y_i) = 2$ for $i \in \{1, 2, 3\}$, $S \cap \{y_1, y_2, y_3\} = \{y_2\}$ or $\{y_1, y_3\}$, and thus let

$$S' = \begin{cases} S \setminus \{y_2\} & \text{if } S \cap \{y_1, y_2, y_3\} = \{y_2\}; \\ (S \setminus \{y_1, y_3\}) \cup \{y\} & \text{if } S \cap \{y_1, y_2, y_3\} = \{y_1, y_3\}. \end{cases}$$

It is clear that S' is a 2-dominating set of T' because S is a $\gamma_2(T)$-set by (a). Moreover, for each $v \in V(T') \setminus \{y\}$ $(= V(T) \setminus \{y_1, y_2, y_3\})$,

$$|N_{T'}(v) \cap S'| = |N_T(v) \cap S|, \text{ and} \tag{3.6}$$

$$|N_2(v, S', T')| = |N_2(v, S, T)| \text{ if } v \in S'. \tag{3.7}$$

Using the condition of Lemma 3.8, we deduce from (1.1), (1.2), (3.6) and (3.7) that T' has neither S'-vulnerable vertices nor S'-vulnerable paths. By induction, $r_2(T') \geq 3$. It follows from Lemma 2.3 that $r_2(T) \geq \min\{3, r_2(T')\} = 3$. □

Claim 2 *If T has an edge xy satisfying $x \notin S$ and $y \notin S$, then $r_2(T) \geq 3$.*

Proof. Let T_x and T_y to denote the two components of $T - xy$ containing x and y, respectively. Since $S \cap \{x, y\} = \emptyset$, Lemma 3.5 yields that T_x (resp. T_y) contains no $S \cap V(T_x)$-vulnerable (resp. $S \cap V(T_y)$-vulnerable) vertices or paths. Using (a), we obtain that $S \cap V(T_x)$ and $S \cap V(T_y)$ are respectively the unique $\gamma_2(T_x)$-set and $\gamma_2(T_y)$-set, and so

$$\gamma_2(T_x) + \gamma_2(T_y) = |S \cap V(T_x)| + |S \cap V(T_y)| = |S| = \gamma_2(T). \tag{3.8}$$

Moreover, $r_2(T_x) \geq 3$ and $r_2(T_y) \geq 3$ by induction.

Let X be an $\eta_2(T)$-set. By Lemma 2.2 and (3.8), $|X| = \gamma_2(T) - 1 = \gamma_2(T_x) + \gamma_2(T_y) - 1$. Thus we may assume that

$$|X \cap V(T_x)| \leq \gamma_2(T_x) - 1.$$

Noting that xy is the unique edge of T between $V(T_x)$ and $V(T_y)$, we obtain from (2.1) and (2.2) that

$$\eta_2(V(T_x), X, T) \geq \eta_2(V(T_x), X \cap V(T_x), T_x) - 1, \tag{3.9}$$

with equality if and only if $X \cap \{x, y\} = \{y\}$ and $|N_{T_x}(x) \cap X| < 2$. Therefore,

$$\begin{aligned} r_2(T) &= \eta_2(V(T), X, T) \quad \text{(by Lemma 2.1)} \\ &= \eta_2(V(T_x), X, T) + \eta_2(V(T_y), X, T) \quad \text{(by (2.2))} \\ &\geq \eta_2(V(T_x), X \cap V(T_x), T_x) - 1 \quad \text{(by (3.9))} \\ &\geq r_2(T_x) - 1 \quad \text{(by Lemma 2.1, since } |X \cap V(T_x)| \leq \gamma_2(T_x) - 1) \\ &\geq 2. \end{aligned}$$

Suppose that $r_2(T) = 2$. Then the above equalities all hold. In particular,

$$\eta_2(V(T_x), X, T) = \eta_2(V(T_x), X \cap V(T_x), T_x) - 1 = r_2(T_x) - 1 = 2, \quad (3.10)$$
$$\eta_2(V(T_y), X, T) = 0. \quad (3.11)$$

(3.10) yields that $X \cap \{x, y\} = \{y\}$ and $r_2(T_x) = \eta_2(V(T_x), X \cap V(T_x), T_x)$, which means that $X \cap V(T_x)$ is an $\eta_2(T_x)$-set. By Lemma 2.2, $|X \cap V(T_x)| = \gamma_2(T_x) - 1$, and so

$$|X \cap V(T_y)| = |X| - |X \cap V(T_x)| = \gamma_2(T_y). \quad (3.12)$$

Since $X \cap \{x, y\} = \{y\}$, by (2.1), (2.2) and (3.11),

$$\eta_2(V(T_y), X \cap V(T_y), T_y) = \eta_2(V(T_y), X, T) = 0,$$

which implies that $X \cap V(T_y)$ is a 2-dominating set of T_y, furthermore, $X \cap V(T_y)$ is a $\gamma_2(T_y)$-set by (3.12). Since $S \cap V(T_y)$ is the unique $\gamma_2(T_y)$-set, $X \cap V(T_y) = S \cap V(T_y)$, and so $y \in S$. This contradicts that $y \notin S$, and hence $r_2(T) \geq 3$. □

Claim 3 *If T contains a vertex x not in S with $d_T(x) \geq 3$, then $r_2(T) \geq 3$.*

Proof. By Claim 2, we may assume that $N_T(x) \subseteq S$ since $x \notin S$. Let $N_T(x) = \{y_1, y_2, \ldots, y_d\}$ and $I = \{1, 2, \ldots, d\}$, where $d = d_T(x) \geq 3$. For $i \in I$, denote by T_i the component of $T - x$ containing y_i, and then the order of T_i is less than n.

Let $i \in I$. Since $N_T(x) \subseteq S$ and $d \geq 3$, $x \notin N_2(y_i, S, T)$, and hence T_i contains no $S \cap V(T_i)$-vulnerable vertices or paths by Lemma 3.5. Noting that $S \cap V(T_i)$ is a 2-dominating set of T_i because $x \notin S$ and S 2-dominates $V(T)$, we obtain by the induction on T_i that

$$r_2(T_i) \geq 3,$$

and know from (a) that $S \cap V(T_i)$ is the unique $\gamma_2(T_i)$-set. Therefore by the arbitrariness of i,

$$\sum_{i \in I} \gamma_2(T_i) = \sum_{i \in I} |S \cap V(T_i)| = |S| = \gamma_2(T).$$

We now show that $r_2(T) \geq 3$. Assume, to be contrary, that $r_2(T) \leq 2$. Let X be an $\eta_2(T)$-set. Then Lemma 2.1 implies that

$$|X \cap \{x\}| + \sum_{i \in I} |X \cap V(T_i)| = |X| = \gamma_2(T) - 1 = \sum_{i \in I} \gamma_2(T_i) - 1, \quad (3.13)$$

and it follows from (2.2) and Lemma 2.1 that

$$\eta_2(x, X, T) + \sum_{i \in I} \eta_2(V(T_i), X, T) = \eta_2(V(T), X, T) = r_2(T) \leq 2. \quad (3.14)$$

By (3.13), there is some $i \in I$, without loss of generality, say $i = 1$, such that $|X \cap V(T_1)| \leq \gamma_2(T_1) - 1$. Note the fact that, for all $i \in I$, if $|X \cap V(T_i)| \leq \gamma_2(T_i) - 1$ then

$$\eta_2(V(T_i), X, T) \geq \eta_2(V(T_i), X \cap V(T_i), T_i) - 1 \geq r_2(T_i) - 1 \geq 2,$$

in which $\eta_2(V(T_i), X, T) = 2$ if and only if $X \cap \{x, y_i\} = \{x\}$, $|N_{T_i}(y_i) \cap X| < 2$ and $X \cap V(T_i)$ is an $\eta_2(T_i)$-set. From this fact and (3.14), we deduce that for any $i \in I$,

$$\eta_2(V(T_i), X, T) = \begin{cases} 2 & \text{if } i = 1; \\ 0 & \text{if } i \neq 1. \end{cases}$$

It follows that

$$\begin{aligned} &|X \cap V(T_1)| = \gamma_2(T_1) - 1, \\ &|X \cap \{x\}| = 1, \text{ and} \\ &|X \cap V(T_i)| = \gamma_2(T_i) \ \text{ for } i \in I \setminus \{1\}. \end{aligned}$$

This contradicts (3.13). Hence $r_2(T) \geq 3$. □

Claim 4 *If T has an edge xy such that $x \in S$ and $y \in S$, then $r_2(T) \geq 3$.*

Proof. It is sufficient to show that $r_2(T) \geq 3$ for a tree T not satisfying the conditions of Claims 1∼3. We claim that for $v \in V(T)$,

$$d_T(v) \begin{cases} = 2 & \text{if } v \notin S; \\ \neq 2 & \text{if } v \in S. \end{cases} \tag{3.15}$$

In fact, it is clear that $d_T(x) = 2$ for all $x \notin S$ since T doesn't satisfy the condition of Claim 3. Then $d_T(v) = 2$ if $v \notin S$. On the other hand, if $v \in S$ then assume, to be contrary, that $d_T(v) = 2$ and let $N_T(v) = \{u_1, u_2\}$. If $S \cap \{u_1, u_2\} = \emptyset$, then both u_1 and u_2 have degree 2, which contradicts that T doesn't satisfy the condition of Claim 1. If $S \cap \{u_1, u_2\} \neq \emptyset$, then v is S-vulnerable in T, a contradiction. The claim holds.

Since S is a 2-dominating set of T, we obtain from (3.15) that all vertices of T not in S are 2-private neighbors with respect to S.

Let T_x and T_y denote the components of $T - xy$ containing x and y, respectively. Recall that S is a 2-dominating set of T and T has neither S-vulnerable vertices nor S-vulnerable paths. Since $x \in S$ and $y \in S$, $S \cap V(T_x)$ is a 2-dominating set of T_x, and Lemma 3.2 yields that T_x has no $S \cap V(T_x)$-vulnerable vertices or paths. By induction, $r_2(T_x) \geq 3$. Moreover, $S \cap V(T_x)$ is the unique $\gamma_2(T_x)$-set by (a). By the symmetry between x and y, we also have that $r_2(T_y) \geq 3$ and $S \cap V(T_y)$ is the unique $\gamma_2(T_y)$-set. Therefore,

$$\gamma_2(T) = |S| = |S \cap V(T_x)| + |S \cap V(T_y)| = \gamma_2(T_x) + \gamma_2(T_y). \tag{3.16}$$

We now show that $r_2(T) \geq 3$. Assume, to be contrary, that $r_2(T) \leq 2$. Let X be an $\eta_2(T)$-set. By Lemma 2.2 and (3.16),

$$|X \cap V(T_x)| + |X \cap V(T_y)| = |X| = \gamma_2(T) - 1 = \gamma_2(T_x) + \gamma_2(T_y) - 1,$$

from which, we may assume that $|X \cap V(T_x)| \leq \gamma_2(T_x) - 1$, and thus $|X \cap V(T_y)| \geq \gamma_2(T_y)$. Noting that xy is the unique edge in T joining $V(T_x)$ and $V(T_y)$, we obtain that

$$\eta_2(V(T_x), X, T) \geq \eta_2(V(T_x), X \cap V(T_x), T_x) - 1$$

with equality if and only if $X \cap \{x, y\} = \{y\}$ and $|N_{T_x}(x) \cap X| \le 1$. Hence

$$\begin{aligned} 2 \ge r_2(T) &= \eta_2(V(T), X, T) \quad \text{(since } X \text{ is an } \eta_2(T)\text{-set)} \\ &= \eta_2(V(T_x), X, T) + \eta_2(V(T_y), X, T) \quad \text{(by (2.2))} \\ &\ge \eta_2(V(T_x), X \cap V(T_x), T_x) - 1 \\ &\ge r_2(T_x) - 1 \quad \text{(by Lemma 2.1, since } |X \cap V(T_x)| \le \gamma_2(T_x) - 1) \\ &\ge 2, \end{aligned}$$

which yields the following results:

$$X \cap \{x, y\} = \{y\}; \tag{3.17}$$

$$|N_{T_x}(x) \cap X| \le 1; \tag{3.18}$$

$$r_2(T_x) = 3 \text{ and } X \cap V(T_x) \text{ is an } \eta_2(T_x)\text{-set.} \tag{3.19}$$

Let $N_{T_x}(x) \setminus S = \{w_1, \ldots, w_t\}$ and $N_{T_x}(x) \cap S = \{w_{t+1}, \ldots, w_{t+s}\}$. For $i \in \{1, \ldots, t+s\}$, denote by T_i the component of $T_x - x$ containing w_i. Because T contains no S-vulnerable vertices and $N_2(x, S, T) = N_{T_x}(x) \setminus S$, we obtain from (1.1) that

$$t \ge \min\{2, |(N_{T_x}(x) \cap S) \cup \{y\}|\} + 1 = \min\{3, 2 + s\}. \tag{3.20}$$

Since $\eta_2(x, X \cap V(T_x), T_x) \ge 1$ by (2.1), (3.17) and (3.18), it follows from (2.2) that

$$\begin{aligned} \sum_{i=1}^{t+s} \eta_2(V(T_i), X \cap V(T_x), T_x) &= \eta_2(V(T_x), X \cap V(T_x), T_x) - \eta_2(x, X \cap V(T_x), T_x) \\ &\le r_2(T_x) - 1 = 2. \quad \text{(by (3.19))} \end{aligned} \tag{3.21}$$

We claim that $\eta_2(V(T_i), X \cap V(T_x), T_x) \ge 1$ for each $i \in \{1, \ldots, t\}$. To be contrary, assume that there is some $i \in \{1, \ldots, t\}$, without loss of generality, say $i = 1$, such that $\eta_2(V(T_1), X \cap V(T_x), T_x) = 0$. Then $X \cap V(T_1)$ is a 2-dominating set of T_1 since $x \notin X$. Recall the obtained facts that $r_2(T_x) = 3$ and $S \cap V(T_x)$ is the unique $\gamma_2(T_x)$-set. If T_1 is the complete graph K_2, then $X \cap V(T_1) = V(T_1)$ and w_1 is a stem of T (which implies that $w_1 \notin S$. Otherwise the unique leaf of w_1 is S-vulnerable in T). Thus

$$|X \cap V(T_1)| \ge |\{x\} \cup (S \cap V(T_1))|.$$

If $T_1 \ne K_2$, since $x \in S \cap V(T_x)$ and $w_1 \in N_2(x, S, T) = N_2(x, S \cap V(T_x), T_x)$, Lemma 3.7 implies that $r_2(T_1) = 1$ and $S \cap V(T_1)$ is an $\eta_2(T_1)$-set. Therefore, by Lemma 2.1, we also obtain that

$$|X \cap V(T_1)| \ge \gamma_2(T_1) = 1 + (\gamma_2(T_1) - 1) = |\{x\} \cup (S \cap V(T_1))|.$$

Let $X_1 = [(X \cap V(T_x)) \setminus (X \cap V(T_1))] \cup [\{x\} \cup (S \cap V(T_1))]$. Then $|X_1| \le |X \cap V(T_x)| = \gamma_2(T_x) - 1$ by (3.19). Since $\{x\} \cup (S \cap V(T_1))$ 2-dominates $V(T_1)$, $\eta_2(V(T_1), X_1, T_x) = 0$ and so

$$\begin{aligned} r_2(T_x) \le \eta_2(V(T_x), X_1, T_x) &= \sum_{i=2}^{t+s} \eta_2(V(T_i), X_1, T_x) \quad \text{(by (2.2))} \\ &\le \sum_{i=2}^{t+s} \eta_2(V(T_i), X \cap V(T_x), T_x) \quad \text{(since } x \in X_1) \\ &\le 2, \quad \text{(by (3.21))} \end{aligned}$$

which contradicts (3.19). The claim holds.

By the above claim, (3.21) and (3.20) imply that $t = 2$, $s = 0$ and

$$\eta_2(x, X \cap V(T_x), T_x) = 1, \tag{3.22}$$
$$\eta_2(V(T_1), X \cap V(T_x), T_x) = 1 \tag{3.23}$$
$$\eta_2(V(T_2), X \cap V(T_x), T_x) = 1. \tag{3.24}$$

By (3.22), exactly one of w_1 and w_2 belongs to X, without loss of generality, say

$$w_1 \in X \text{ and } w_2 \notin X.$$

Notice that $d_T(w_1) = 2$ by (3.15) since $w_1 \notin S$. Let v be the unique vertex in $N_T(w_1)\setminus\{x\}$. Since S is a 2-dominating set of T, $v \in S$ in order to 2-dominate w_1.

Furthermore, we will now show that $|N_2(v, S, T)| \geq 3$, which implies that v has at least three neighbors with degree 2 in T by (3.15). Assume, to be contrary, that $|N_2(v, S, T)| \leq 2$. Noting that all vertices of T not in S are 2-private neighbors with respect to S, we know that v has at most two neighbors not in S. If $d_T(v) = 1$, then the path xw_1v is S-vulnerable in T, a contradiction. If $d_T(v) = 2$, then let u be the unique vertex in $N_T(v) \setminus \{w_1\}$. From the assumption that T doesn't satisfy the condition of Claim 1, it follows that $d_T(u) \neq 2$ because $d_T(w_1) = d_T(v) = 2$, and thus $u \in S$ by (3.15). Hence v is S-vulnerable in T, a contradiction. If $d_T(v) \geq 3$, then

$$|N_T(v) \cap S| = |N_T(v)| - |N_2(v, S, T)| = d_T(v) - |N_2(v, S, T)| \geq 1,$$

that is, v has at least one neighbor in S, and so the path xw_1v is also S-vulnerable in T, a contradiction.

To the end, we will show that there is a vertex subset $X_2 \subseteq V(T_x)$ such that $|X_2| = |X \cap V(T_x)|$ and $\eta_2(V(T_x), X_2, T_x) < 3$. This contradicts (3.19), and hence $r_2(T) \geq 3$.

If $v \in X$, then let

$$X_2 = [(X \cap V(T_x)) \setminus \{w_1\}] \cup \{x\}.$$

Clearly, $|X_2| = |X \cap V(T_x)|$ because $w_1 \in X \cap V(T_x)$. Since $N_{T_x}(w_1) = \{x, v\} \subseteq X_2$, it follows from (2.1) that $\eta_2(x, X_2, T_x) = \eta_2(w_1, X_2, T_x) = 0$. Therefore, by (2.1) and (2.2),

$$\begin{aligned} \eta_2(V(T_x), X_2, T_x) &= \eta_2(x, X_2, T) + \eta_2(V(T_1), X_2, T_x) + \eta_2(V(T_2), X_2, T_x) \\ &\leq \eta_2(V(T_1), X \cap V(T_x), T_x) + \eta_2(V(T_2), X \cap V(T_x), T_x) \\ &= 2. \quad \text{(by (3.23) and (3.24))} \end{aligned}$$

Since $|X_2| = |X \cap V(T_x)|$, by (3.19) and Lemma 2.1, we obtain a contradiction that

$$3 = r_2(T_x) \leq \eta_2(V(T_x), X_2, T_x) \leq 2.$$

If $v \notin X$, since $|N_2(v, S, T)| \geq 3$, (3.23) implies that there is a vertex $u \in N_2(v, S, T) \setminus \{w_1\}$ such that

$$\eta_2(V(T_u), X \cap V(T_x), T_x) = 0, \tag{3.25}$$

where T_u is the component of $T - v$ containing u. Since $v \notin X$, (3.25) implies that $X \cap V(T_u)$ is a 2-dominating set of T_u and so $|X \cap V(T_u)| \geq \gamma_2(T_u)$. Let

$$X_2 = [(X \cap V(T_x)) \setminus ((X \cap V(T_u)) \cup \{w_1\})] \cup \{x, v\} \cup (S \cap V(T_u)).$$

Recall the obtained facts that $d_T(u) = 2$ (by (3.15) since $u \notin S$), $r_2(T_x) = 3$ and $S \cap V(T_x)$ is the unique $\gamma_2(T_x)$-set. If $T_u = K_2$, then the unique vertex in $V(T_u) \setminus \{u\}$ belongs to $S \cap V(T_u)$ by Observation 3.1, and so $\eta_2(V(T_u), X_2, T_x) = 0$ and

$$|X \cap V(T_u)| = |V(T_u)| = |\{v\} \cup (S \cap V(T_u))|,$$

from which we obtain that $|X_2| = |X \cap V(T_x)|$. If $T_u \neq K_2$, since $u \in N_2(v, S, T) = N_2(v, S \cap V(T_x), T_x)$, Lemma 3.7 implies that $r_2(T_u) = 1$ and $S \cap V(T_u)$ is an $\eta_2(T_u)$-set, and so

$$|X \cap V(T_u)| \geq \gamma_2(T_u) = 1 + (\gamma_2(T_u) - 1) = |\{v\}| + |S \cap V(T_u)| = |\{v\} \cup (S \cap V(T_u))|$$

and $\{v\} \cup (S \cap V(T_u))$ 2-dominates $V(T_u)$. Therefore,

$$|X_2| = |X \cap V(T_x)| \text{ and } \eta_2(V(T_u), X_2, T_x) = 0.$$

Since $N_{T_x}(w_1) = \{x, v\} \subseteq X_2$, it follows from (2.1) and (2.2) that $\eta_2(x, X_2, T_x) = \eta_2(w_1, X_2, T_x) = \eta_2(v, X_2, T_x) = 0$ and

$$\begin{aligned}
& \eta_2(V(T_x), X_2, T_x) \\
= \ & \eta_2(V(T_1) \setminus (V(T_u) \cup \{w_1, v\}), X_2, T_x) + \eta_2(V(T_2), X_2, T_x) \\
\leq \ & \eta_2(V(T_1) \setminus (V(T_u) \cup \{w_1, v\}), X \cap V(T_x), T_x) + \eta_2(V(T_2), X \cap V(T_x), T_x) \\
\leq \ & \eta_2(V(T_1), X \cap V(T_x), T_x) + \eta_2(V(T_2), X \cap V(T_x), T_x) \\
= \ & 2. \quad \text{(by (3.23) and (3.24))}
\end{aligned}$$

Since $|X_2| = |X \cap V(T_x)|$, (3.19) and Lemma 2.1 yield a contradiction that

$$3 = r_2(T_x) \leq \eta_2(V(T_x), X_2, T_x) \leq 2.$$

The proof of Claim 4 is complete. □

We now return to the proof of Lemma 3.8 (c). In the following, assume that T is a tree not satisfying the conditions of Claims 1∼4. Since S is 2-dominating set of T, for any $x \in V(T)$, we obtain from this assumption that

$$d_T(x) = 2, \qquad \text{if } x \notin S, \tag{3.26}$$

$$d_T(x) = |N_T(x)| = |N_2(x, S, T)| \neq 2, \qquad \text{if } x \in S. \tag{3.27}$$

Moreover, every stem in T has exact one leaf because T has neither S-vulnerable vertices nor S-vulnerable paths.

Let v be a stem and u the unique leaf of v. Since S is a 2-dominating set of T, $u \in S$ by Observation 3.1 and $v \notin S$ (otherwise, u is S-vulnerable in T). By (3.26), $d_T(v) = 2$. Denote by w the unique vertex in $N_T(v) \setminus \{u\}$. Then $N_T(v) = \{u, w\}$. Since $v \notin S$, to 2-dominate v, $|S \cap N_T(v)| \geq 2$, which implies that $w \in S$.

Let $T' = T - \{u, v\}$. Then $S \cap V(T')$ is a 2-dominating set of T' because $w \in S$ and S is a 2-dominating set of T.

We claim that T' has neither $S \cap V(T')$-vulnerable vertices nor $S \cap V(T')$-vulnerable paths. It is clear that $d_T(w) \neq 1$. Since $w \in S$, it follows from (3.27) that $d_T(w) = |N_2(w, S, T)| \geq 3$, and thus

$$d_T(w) - 1 = d_{T'}(w) = |N_2(w, S \cap V(T'), T')| = |N_2(w, S, T) \setminus \{v\}| \geq 2. \tag{3.28}$$

By (1.1), (3.28) implies that w is not $S \cap V(T')$-vulnerable in T', furthermore, T' contains no $S \cap V(T')$-vulnerable vertices. Assume that T' has an $S \cap V(T')$-vulnerable path P. Then $S \cap V(P)$ 2-dominates $V(P)$ by the definition of $S \cap V(T')$-vulnerable path, and $w \in S \cap V(P)$ by Lemma 3.5. Noting that every vertex in $S \cap V(P)$ has at most two 2-private neighbors with respect to $S \cap V(T')$ by (1.2), we obtain from (3.28) that

$$d_T(w) - 1 = d_{T'}(w) = |N_2(w, S \cap V(T'), T')| = 2. \tag{3.29}$$

In addition, for each $x \in (S \cap V(P)) \setminus \{w\}$, (3.27) implies that

$$N_{T'}(x) = N_T(x) = N_2(x, S, T) = N_2(x, S \cap V(T'), T'), \text{ and} \tag{3.30}$$

$$|N_{T'}(x)| = |N_T(x)| \neq 2. \tag{3.31}$$

Let y_1 and y_2 be two end-vertices of P. Because $S \cap V(P)$ 2-dominates $V(P)$, we have $y_1 \in S \cap V(P)$ and $y_2 \in S \cap V(P)$ by Observation 3.1. Noting that every vertex in $S \cap V(P)$ has at most two 2-private neighbors with respect to $S \cap V(T')$ by (1.2), we deduce from (3.26) and (3.30) that for each $x \in V(P) \setminus \{w\}$,

$$d_T(x) = d_{T'}(x) = \begin{cases} 2 & \text{if } x \in V(P) \setminus S; \\ 1 & \text{if } x \in \{y_1, y_2\}; \\ 2 & \text{if } x \in (S \cap V(P)) \setminus \{y_1, y_2\}, \end{cases}$$

from which and (3.31) we obtain that $S \cap V(P) = \{y_1, w, y_2\}$ and $V(P) \setminus S = N_{T'}(w)$. By (3.29), T is a spider S_3 and $n = |V(T)| = |V(S_3)| = 7$, which contradicts that $n \geq 8$. The claim holds.

By (a), $S \cap V(T')$ is the unique $\gamma_2(T')$-set. By induction, $r_2(T') \geq 3$. Since S is a $\gamma_2(T)$-set by (a),

$$\gamma_2(T') = |S \cap V(T')| = |S \setminus \{u\}| = |S| - 1 = \gamma_2(T) - 1. \tag{3.32}$$

We now show that $r_2(T) \geq 3$. Let X be an $\eta_2(T)$-set such that $|X \cap \{u, v\}|$ is as small as possible. Then $|X \cap \{u, v\}| \leq 1$. If $|X \cap \{u, v\}| = 0$, by (2.1) and (2.2), we have $\eta_2(\{u, v\}, X, T) \geq 3$ and it follows from Lemma 2.1 that

$$r_2(T) = \eta_2(V(T), X, T) \geq \eta_2(\{u, v\}, X, T) \geq 3.$$

If $|X \cap \{u, v\}| = 1$, then Lemma 2.2 and (3.32) imply that

$$|X \cap V(T')| = |X| - 1 = (\gamma_2(T) - 1) - 1 = \gamma_2(T') - 1.$$

Note that the edge vw is the unique edge linking $\{u, v\}$ to $V(T')$. When $X \cap \{u, v\} = \{u\}$, we have $\eta_2(V(T'), X, T) = \eta_2(V(T'), X \cap V(T'), T')$, and thus obtain from Lemm 2.1 and (2.2) that

$$\begin{aligned} r_2(T) = \eta_2(V(T), X, T) &\geq \eta_2(V(T'), X, T) \\ &= \eta_2(V(T'), X \cap V(T'), T') \geq r_2(T') \geq 3. \end{aligned}$$

When $X \cap \{u, v\} = \{v\}$, we directly calculate by (2.1) and (2.2) that $\eta_2(\{u, v\}, X, T) = 1$ and $\eta_2(V(T'), X, T) \geq \eta_2(V(T'), X \cap V(T'), T') - 1$. Therefore,

$$\begin{aligned} r_2(T) &= \eta_2(V(T), X, T) \quad \text{(by Lemma 2.1)} \\ &= \eta_2(\{u, v\}, X, T) + \eta_2(V(T'), X, T) \quad \text{(by (2.2))} \\ &\geq 1 + [\eta_2(V(T'), X \cap V(T'), T') - 1] \\ &\geq r_2(T') \quad \text{(by Lemma 2.1, since } |X \cap V(T')| < \gamma_2(T')) \\ &\geq 3. \end{aligned}$$

This complete the proof of Lemma 3.8 (c). ∎

Applying Theorem 1.1 and Lemma 3.8 (c), the sufficiency of Theorem 1.2 is true, and so Theorem 1.2 holds.

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