RAMSEY ALGEBRAS AND STRONGLY REDUCTIBLE ULTRAFILTERS

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ABSTRACT. Hindman's Theorem says that every finite coloring of the positive natural numbers has a monochromatic set of finite sums. A Ramsey algebra is a structure that satisfies an analogue of Hindman's Theorem. It is known that Martin's Axiom implies the existence of strongly summable ultrafilters, that is nonprincipal ultrafilters generated by sets of finite sums. Strongly reductible ultrafilters are analogues of strongly summable ultrafilters. Assuming Martin's Axiom, this paper shows the existence of nonprincipal strongly reductible ultrafilters for a nondegenerate Ramsey algebra.

1. INTRODUCTION

The set of natural numbers $\{0,1,2,\dots\}$ is denoted by ω. Suppose $\langle x_i\rangle_{i\in\omega}$ is a sequence of natural numbers. Let $\mathrm{FS}(\langle x_i\rangle_{i\in\omega})$ denote the set $\{\sum_{i\in F} x_i \mid F \in \mathcal{P}_f(\omega)\setminus\{\varnothing\}\}$, where $\mathcal{P}_f(\omega)$ is the set of all finite subsets of ω. Hindman's Theorem [6] says that for every finite partition of the set of positive natural numbers $\mathbb{N} = X_0 \cup X_1 \cup \cdots \cup X_N$, there exists a sequence $\langle x_i\rangle_{i\in\omega}$ of positive natural numbers such that for some $0 \le j \le N$, we have $\mathrm{FS}(\langle x_i\rangle_{i\in\omega}) \subseteq X_j$. According to Hindman [5], Galvin knew that Hindman's Theorem would follow from the existence of what Galvin called an almost translation invariant ultrafilter, that is, an ultrafilter U on $\mathbb{N}$ such that $\{x\in\mathbb{N} \mid X - x \in U\} \in U$ whenever $X \in U$, where $X - x = \{y \in \mathbb{N} \mid x + y \in X\}$. In 1972 Hindman [5] showed that Hindman's Theorem (then unproven) together with the continuum hypothesis implies the existence of almost translation invariant ultrafilters.

In 1975 Galvin and Glazer (see [3] or [7]) showed the existence of almost translation invariant ultrafilters without the continuum hypothesis. An almost translation invariant ultrafilter is exactly an idempotent element of the semigroup $(\beta\mathbb{N},+)$, where $+$ is the extension of addition on $\mathbb{N}$ to $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$. As $(\beta\mathbb{N},+)$ is a compact right topological semigroup, it has an idempotent element.

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However, as accounted by Hindman in [8], van Douwen pointed out in 1985 that the ultrafilter U produced in Hindman's proof has the stronger property that it is generated by sets of finite sums, which means

$$\text{for every } X \in U, \text{ there exists } \vec{a} \in {}^{\omega}\mathbb{N} \text{ such that } \mathrm{FS}(\vec{a}) \subseteq X \text{ and } \mathrm{FS}(\vec{a}) \in U,$$

and he asked whether such ultrafilters can be shown to exist in ZFC. Hindman [8] called such ultrafilters *strongly summable* and showed that Martin's Axiom implies their existence. Later, Blass and Hindman [1] showed that their existence implies the existence of P-points. It is a well known theorem of Shelah [12, VI, § 4] that the existence of P-points cannot be proven in ZFC. Therefore, the existence of strongly summable ultrafilters is independent of ZFC.

A *Ramsey* algebra [13, 14] is a structure which possesses the property analogous to that possessed by the algebra $(\mathbb{N},+)$ as in Hindman's Theorem. Assuming Martin's Axiom, we will show the existence of the analogue of strongly summable ultrafilters for a Ramsey algebra.

Ramsey algebras are also closely related to Ramsey spaces, structures that have properties analogous to those of Ellentuck's space [4]. Every algebra can be associated to a certain topological space of infinite sequences. It follows from the abstract version of Ellentuck's Theorem by Carlson [2] that such a space is Ramsey if and only if the associated underlying algebra is Ramsey.

2. Preliminaries

To us an *algebra* is a pair $(A,\mathcal{F})$, where A is a nonempty set and $\mathcal{F}$ is a collection of operations on A, none of which is nullary. Suppose B is a nonempty subset of A and suppose B is closed under f in the usual sense for each $f \in \mathcal{F}$. If f is n-ary, let $f \upharpoonright B$ denote the restriction of f to B^n with codomain B. The algebra $(B,\{f \upharpoonright B \mid f \in \mathcal{F}\})$ is called a *subalgebra*[1] of $(A,\mathcal{F})$.

The set of infinite and finite sequences in A are denoted by ${}^{\omega}A$ and ${}^{<\omega}A$ respectively. Suppose $\vec{a} = \langle a_0, a_1, a_2, \dots\rangle$. For $n \geq 1$, let $\vec{a} \upharpoonright n$ denote the initial segment of $\vec{a}$ of length n, namely $\langle a_0, a_1, \dots, a_{n-1}\rangle$. For $n \in \omega$, let $\vec{a} - n$ denote the sequence $\langle a_n, a_{n+1}, a_{n+2}, \dots\rangle$. If $\vec{s} = \langle s_0, s_1, \dots, s_n\rangle$ and $\vec{t} = \langle t_0, t_1, t_2, \dots\rangle$, then $|\vec{s}|$ is the length of $\vec{s}$ and the concatenation $\vec{s} * \vec{t}$ of $\vec{s}$ and $\vec{t}$ is $\langle s_0, s_1, \dots, s_n, t_0, t_1, t_2, \dots\rangle$.

A *pre-partial ordering* on a set A is a binary relation on A which is reflexive and transitive.

Fix an algebra $(A,\mathcal{F})$ for the rest of this section.

[1] Note that $f \upharpoonright B$ and $g \upharpoonright B$ can be equal when f and g are distinct. Our notions of "algebra" and "subalgebra" are different from but compatible with that in the theory of universal algebra.

Definition 2.1. An operation f on A is an *orderly composition* of $\mathcal{F}$ iff there exist $g, h_1, \dots, h_n \in \mathcal{F}$ such that $f(\bar{x}_1, \dots, \bar{x}_n) = g(h_1(\bar{x}_1), \dots, h_n(\bar{x}_n))$.[2] We say that $\mathcal{F}$ is *closed under orderly composition* iff $f \in \mathcal{F}$ whenever f is an orderly composition of $\mathcal{F}$. The collection of *orderly terms* over $\mathcal{F}$ is the smallest collection of operations on A that contains $\mathcal{F}$ and the identity function on A and that is closed under orderly composition.

Definition 2.2. Suppose $\vec{a}, \vec{b}$ are infinite sequences in A. We say that $\vec{a}$ is a *reduction* of $\vec{b}$ with respect to $\mathcal{F}$, and write $\vec{a} \leq_{\mathcal{F}} \vec{b}$ iff there are finite sequences $\vec{b}_n$ and orderly terms f_n over $\mathcal{F}$ for $n \in \omega$ such that $\vec{b}_0 * \vec{b}_1 * \vec{b}_2 * \cdots$ is a subsequence of $\vec{b}$ and $\vec{a}(n) = f_n(\vec{b}_n)$ for all $n \in \omega$.

Definition 2.3. Suppose $\vec{a}, \vec{b}$ are finite sequences in A. We say that $\vec{a}$ is a *reduction* of $\vec{b}$, and write $\vec{a} \trianglelefteq_{\mathcal{F}} \vec{b}$ iff there are finite sequences $\vec{b}_n$ and orderly terms f_n over $\mathcal{F}$ for $n < |\vec{a}|$ such that $\vec{b}_0 * \vec{b}_1 * \cdots * \vec{b}_{|\vec{a}|-1}$ is a subsequence of $\vec{b}$ and $\vec{a} = \langle f_0(\vec{b}_0), f_1(\vec{b}_1), \dots, f_{|\vec{a}|-1}(\vec{b}_{|\vec{a}|-1}) \rangle$.

It is easy to check that $\leq_{\mathcal{F}}$ is a pre-partial ordering on ${}^{\omega}A$ and that $\trianglelefteq_{\mathcal{F}}$ is a pre-partial ordering on ${}^{<\omega}A$. Our definitions of $\leq_{\mathcal{F}}$ and $\trianglelefteq_{\mathcal{F}}$ are equivalent to a special case of the ones given in [2], where the collection of operations contains all projections.

Lemma 2.4. *Suppose $\vec{a}, \vec{b}$ are infinite sequences in A. Then $\vec{a} \leq_{\mathcal{F}} \vec{b}$ if and only if every initial segment of $\vec{a}$ is a reduction of some initial segment of $\vec{b}$.*

Proof. The only if direction is immediate from the definition of $\leq_{\mathcal{F}}$. The if direction is proved in Lemma 4.3 of [2]. □

Definition 2.5. Suppose $\vec{b}$ is an infinite sequence in A. An element a of A is a *finite reduction* of $\vec{b}$ with respect to $\mathcal{F}$ iff a is equal to $f(\vec{b}_0)$ for some orderly term f over $\mathcal{F}$ and some finite subsequence $\vec{b}_0$ of $\vec{b}$. Define $\mathrm{FR}_{\mathcal{F}}(\vec{b})$ to be the set of all finite reductions of $\vec{b}$ with respect to $\mathcal{F}$.

Note that if $\vec{a} \leq_{\mathcal{F}} b$, then $\mathrm{FR}_{\mathcal{F}}(\vec{a}) \subseteq \mathrm{FR}_{\mathcal{F}}(\vec{b})$.

Definition 2.6. We say that $(A, \mathcal{F})$ is Ramsey iff for every $\vec{a} \in {}^{\omega}A$ and every $X \subseteq A$, there exists $\vec{b} \leq_{\mathcal{F}} \vec{a}$ such that $\mathrm{FR}_{\mathcal{F}}(\vec{b})$ is either contained in or disjoint from X.

It is a consequence of Hindman's Theorem that every semigroup is a Ramsey algebra (Corollary 5.15 in [10]). There are examples of Ramsey algebras which are not semigroups (see [14]).

Every subalgebra of a Ramsey algebra is Ramsey. In fact, it is easy to see that assuming $\mathcal{F}$ is countable, $(A, \mathcal{F})$ is a Ramsey algebra if and only if every countable subalgebra is Ramsey.

[2] For notational convenience, we will use a symbol with a bar over it to indicate a list.

Definition 2.7. Suppose U is an ultrafilter on A. We say that U is *strongly reductible* for $\mathcal{F}$ iff for every $X \in U$, there exists $\vec{a} \in {}^{\omega}A$ such that $\mathrm{FR}_{\mathcal{F}}(\vec{a}) \subseteq X$ and $\mathrm{FR}_{\mathcal{F}}(\vec{a} - n) \in U$ for all $n \in \omega$.

Remark 2.8. By Lemma 2.2 in [8], U is a strongly summable ultrafilter if and only if for every $X \in U$, there exists $\vec{a} \in {}^{\omega}\mathbb{N}$ such that $\mathrm{FS}(\vec{a}) \subseteq X$ and $\mathrm{FS}(\vec{a} - n) \in U$ for all $n \in \omega$. Hence, strongly reductible ultrafilters are indeed generalizations of strongly summable ultrafilters.

Suppose $(A, \mathcal{F})$ is an algebra such that for every $\vec{a} \in {}^{\omega}A$, there exists $\vec{b} \leq_{\mathcal{F}} \vec{a}$ such that $|\mathrm{FR}_{\mathcal{F}}(\vec{b})| = 1$. Then $(A, \mathcal{F})$ is trivially Ramsey, and we say that it is a *degenerate Ramsey algebra*.

Lemma 2.9. *Suppose $(A, \mathcal{F})$ is a nondegenerate Ramsey algebra. Then there exists $\vec{a} \in {}^{\omega}A$ such that $\mathrm{FR}_{\mathcal{F}}(\vec{b})$ is infinite whenever $\vec{b} \leq_{\mathcal{F}} \vec{a}$.*

Proof. First, we will show the following: if $\mathrm{FR}_{\mathcal{F}}(\vec{a})$ is finite, then there exists $\vec{b} \leq_{\mathcal{F}} \vec{a}$ such that $|\mathrm{FR}_{\mathcal{F}}(\vec{b})| = 1$. Choose $\vec{b} \leq_{\mathcal{F}} \vec{a}$ such that $|\mathrm{FR}_{\mathcal{F}}(\vec{b})| \leq |\mathrm{FR}_{\mathcal{F}}(\vec{c})|$ for all $\vec{c} \leq_{\mathcal{F}} \vec{a}$. We claim that $|\mathrm{FR}_{\mathcal{F}}(\vec{b})| = 1$. Suppose not. Let X be a nonempty proper subset of $\mathrm{FR}_{\mathcal{F}}(\vec{b})$. Since $(A, \mathcal{F})$ is a Ramsey algebra, choose $\vec{c} \leq_{\mathcal{F}} \vec{b}$ such that $\mathrm{FR}_{\mathcal{F}}(\vec{c})$ is either contained in or disjoint from X. In either case, $\mathrm{FR}_{\mathcal{F}}(\vec{c})$ is a proper subset of $\mathrm{FR}_{\mathcal{F}}(\vec{b})$, contradicting the minimality of $\vec{b}$, as $\vec{c}$ is also a reduction of $\vec{a}$.

To prove the lemma, we argue by contradiction. Fix $\vec{a} \in {}^{\omega}A$. Then there exists $\vec{b} \leq_{\mathcal{F}} \vec{a}$ such that $\mathrm{FR}_{\mathcal{F}}(\vec{b})$ is finite. By the previous claim, there exists a reduction $\vec{c}$ of $\vec{b}$ (and thus of $\vec{a}$) such that $|\mathrm{FR}_{\mathcal{F}}(\vec{c})| = 1$. Since $\vec{a}$ is arbitrary, $(A, \mathcal{F})$ is degenerate, contradicting the hypothesis. $\square$

To say that $a \in A$ is an *idempotent* element for an algebra $(A, \mathcal{F})$ means that $f(a, \ldots, a) = a$ for every $f \in \mathcal{F}$. Note that if $\mathrm{FR}_{\mathcal{F}}(\vec{a}) = \{c\}$, then c is an *idempotent* element for $(A, \mathcal{F})$. Suppose $(A, \mathcal{F})$ is a degenerate Ramsey algebra. Then clearly every principal ultrafilter on A generated by an idempotent element of $(A, \mathcal{F})$ is strongly reductible for $\mathcal{F}$. The following example shows that a nonprincipal ultrafilter strongly reductible for $\mathcal{F}$ need not exist.

Example 2.10. Suppose f is a constant binary operation on ω, say $f(x, y) = c$ for all $x, y \in \omega$. Trivially, $(\omega, \{f\})$ is a degenerate Ramsey algebra. Nevertheless, there is no nonprincipal ultrafilter on ω strongly reductible for $\{f\}$. To see this, simply consider the cofinite set $\omega \setminus \{c\}$.

3. The Main Results

Suppose $(A, \mathcal{F})$ is an algebra. We will restrict our attention to the case where the underlying set A is countable. We will address the general case at the end of this section. Without loss of generality, we may assume A is equal to ω.

From now on, fix $\mathcal{F}$ to be a collection of operations on ω until we say otherwise. Hence, we will say "a strongly reductible ultrafilter" to mean an ultrafilter on ω strongly reductible for $\mathcal{F}$. We will show under special set theoretic axioms that if $(\omega, \mathcal{F})$ is a nontrivial Ramsey algebra, then there exists a strongly reductible ultrafilter. We will follow Hindman's footsteps by first proving this result under the Continuum Hypothesis and then under Martin's Axiom. In doing so, we can highlight how Martin's axiom is used to prove a main lemma when the Continuum Hypothesis is absent.

Definition 3.1. Suppose $\vec{a}, \vec{b} \in {}^{\omega}\omega$. We say that $\vec{a}$ is *eventually a reduction* of $\vec{b}$, and write $\vec{a} \leq^*_{\mathcal{F}} \vec{b}$ iff $\vec{a} - n \leq_{\mathcal{F}} \vec{b}$ for some $n \in \omega$.

Since $\mathcal{F}$ is fixed, to improve readability, we will simply write $\leq^*$, $\leq$ and $\mathrm{FR}(\vec{a})$ for $\leq^*_{\mathcal{F}}$, $\leq_{\mathcal{F}}$ and $\mathrm{FR}_{\mathcal{F}}(\vec{a})$ respectively.

Note that $\leq^*$ is a pre-partial ordering on ${}^{\omega}\omega$. Transitivity of $\leq^*$ follows easily from the transitivity of $\leq$.

For convenience, when we say that a finite sequence $\vec{a}$ is a reduction of an infinite sequence $\vec{b}$, it is understood that $\vec{a}$ is a reduction of some initial segment of $\vec{b}$.

Lemma 3.2. *Suppose $\langle \vec{a}_n \rangle_{n\in\omega}$ is a sequence in ${}^{\omega}\omega$ such that $\vec{a}_{n+1} \leq^* \vec{a}_n$ for all $n \in \omega$. Then there exists $\vec{b} \in {}^{\omega}\omega$ such that $\vec{b} - n \leq \vec{a}_n$ for all $n \in \omega$.*

Proof. Suppose $\langle \vec{a}_n \rangle_{n\in\omega}$ is as stated. We will define a sequence $\langle M_n \rangle_{n\in\omega}$ of natural numbers inductively as follows. Let $M_0 = 0$. Suppose M_n has been chosen. Since $\vec{a}_{n+1} \leq^* \vec{a}_n$, we can choose $M_{n+1} > M_n$ such that $\vec{a}_{n+1} - M_{n+1} \leq \vec{a}_n - (M_n + 1)$. Take $\vec{b}$ to be the sequence $\langle \vec{a}_n(M_n) \rangle_{n\in\omega}$. We will show that $\vec{b} - n$ is in fact a reduction of $\vec{a}_n - M_n$ for all $n \in \omega$. By Lemma 2.4, it suffices to show that every initial segment of $\vec{b} - n$ is a reduction of $\vec{a}_n - M_n$. This follows from the following claim.

Claim. *Suppose $m \in \omega$. For each $0 \leq n \leq m$, the sequence $\langle \vec{a}_n(M_n), \dots, \vec{a}_m(M_m) \rangle$ is a reduction of $\vec{a}_n - M_n$.*

Fix $m \in \omega$. We will proceed by inverse induction on n. For the base step $n = m$, it is clear that $\langle \vec{a}_m(M_m) \rangle$ is a reduction of $\vec{a}_m - M_m$. For the inductive step, suppose $\langle \vec{a}_n(M_n), \dots, \vec{a}_m(M_m) \rangle$ is a reduction of $\vec{a}_n - M_n$. Since $\vec{a}_n - M_n \leq \vec{a}_{n-1} - (M_{n-1} + 1)$ by our construction, $\langle \vec{a}_n(M_n), \dots, \vec{a}_m(M_m) \rangle$ is a reduction of $\vec{a}_{n-1} - (M_{n-1} + 1)$. It follows that $\langle \vec{a}_{n-1}(M_{n-1}), \dots, \vec{a}_m(M_m) \rangle$ is a reduction of $\vec{a}_{n-1} - M_{n-1}$. The claim is proved. □

Lemma 3.3. *Assume λ is a limit ordinal such that $cof(\lambda) = \omega$. Suppose $\langle \vec{a}_\alpha \rangle_{\alpha\in\lambda}$ is a transfinite sequence in ${}^{\omega}\omega$ such that $\vec{a}_\alpha \leq^* \vec{a}_\beta$ whenever $\beta < \alpha$. Then there exists $\vec{b} \in {}^{\omega}\omega$ such that $\vec{b} \leq^* \vec{a}_\alpha$ for all $\alpha < \lambda$.*

Proof. Suppose $\langle \vec{a}_\alpha \rangle_{\alpha\in\lambda}$ is as stated. Choose a strictly increasing sequence of ordinals $\gamma_0, \gamma_1, \ldots$ cofinal in λ. Clearly $\vec{a}_{\gamma_{n+1}} \leq^* \vec{a}_{\gamma_n}$ whenever $n \in \omega$. By Lemma 3.2, we can find a sequence $\vec{b}$ such that $\vec{b} - n$ is a reduction of $\vec{a}_{\gamma_n}$ for all $n \in \omega$. We claim that this $\vec{b}$ works. Fix any $\alpha < \lambda$. Choose n such that $\gamma_n > \alpha$. By the hypothesis, $\vec{a}_{\gamma_n}$ is eventually a reduction of $\vec{a}_\alpha$. Since $\vec{b} - n$ is a reduction of $\vec{a}_{\gamma_n}$, it follows that $\vec{b} - n$ is eventually a reduction of $\vec{a}_\alpha$ by transitivity. □

Theorem 3.4. *Assume the continuum hypothesis. Suppose $(\omega, \mathcal{F})$ is a nondegenerate Ramsey algebra. Then there exists a nonprincipal strongly reductible ultrafilter.*

Proof. By Lemma 2.9, we can choose a sequence $\vec{a}$ such that $\mathrm{FR}(\vec{b})$ is infinite whenever $\vec{b} \leq \vec{a}$. Suppose X_α $(\alpha < \omega_1)$ enumerates all subsets of ω. By transfinite recursion, for each $\alpha < \omega_1$ we will construct $\vec{a}_\alpha \in {}^\omega\omega$ such that

- $\mathrm{FR}(\vec{a}_\alpha)$ is infinite;
- either $\mathrm{FR}(\vec{a}_\alpha) \subseteq X_\alpha$ or $\mathrm{FR}(\vec{a}_\alpha) \subseteq \omega \backslash X_\alpha$;
- $\vec{a}_\alpha \leq^* \vec{a}_\beta$ whenever $\beta < \alpha$.

We will use the Ramsey property of $(\omega, \mathcal{F})$ repeatedly in the construction. Firstly, choose $\vec{a}_0$ to be a reduction of $\vec{a}$ such that either $\mathrm{FR}(\vec{a}_0) \subseteq X_0$ or $\mathrm{FR}(\vec{a}_0) \subseteq \omega \backslash X_0$. Suppose $\lambda < \omega_1$ and $\vec{a}_\alpha$ for each $\alpha < \lambda$ have been constructed. There are two cases. If $\lambda = \beta + 1$ for some β, then we can choose $\vec{a}_\lambda$ to be any reduction of $\vec{a}_\beta$ such that $\mathrm{FR}(\vec{a}_\lambda) \subseteq X_\lambda$ or $\mathrm{FR}(\vec{a}_\lambda) \subseteq \omega \backslash X_\lambda$. Now suppose λ is a limit ordinal. Since $\lambda < \omega_1$, we have $cof(\lambda) = \omega$. Therefore, by Lemma 3.3 we can choose a sequence $\vec{b}$ such that $\vec{b} \leq^* \vec{a}_\alpha$ for every $\alpha < \lambda$. Choose $\vec{a}_\lambda$ to be any reduction of $\vec{b}$ such that $\mathrm{FR}(\vec{a}_\lambda) \subseteq X_\lambda$ or $\mathrm{FR}(\vec{a}_\lambda) \subseteq \omega \backslash X_\lambda$. By transitivity, $\vec{a}_\lambda \leq^* \vec{a}_\alpha$ for every $\alpha < \lambda$. By the transitivity of $\leq^*$ and the choice of $\vec{a}_0$, we have $\vec{a}_\lambda - N \leq \vec{a}$ for some $N \in \omega$. By the hypothesis, $\mathrm{FR}(\vec{a}_\lambda - N)$ is infinite, and so is $\mathrm{FR}(\vec{a}_\lambda)$.

Now, for each $\alpha < \omega_1$ let F_α be the filter generated by the sets $\mathrm{FR}(\vec{a}_\alpha - n)$, meaning

$$F_\alpha = \{ X \subseteq \omega \mid \mathrm{FR}(\vec{a}_\alpha - n) \subseteq X \text{ for some } n \in \omega \}.$$

Suppose $\beta < \alpha < \omega_1$. By our construction, $\vec{a}_\alpha - N \leq \vec{a}_\beta$ for some $N \in \omega$. This implies that $\vec{a}_\alpha - (N+n) \leq \vec{a}_\beta - n$ and hence $\mathrm{FR}(\vec{a}_\alpha - (N+n)) \subseteq \mathrm{FR}(\vec{a}_\beta - n)$ for each $n \in \omega$. It follows easily that $F_\beta \subseteq F_\alpha$.

Take U to be $\bigcup_{\alpha<\omega_1} F_\alpha$. It is a standard routine argument to show that U is an ultrafilter. Suppose $X \in U$, say $X = X_\alpha$ for some $\alpha < \omega_1$. Then it must be the case that $\mathrm{FR}(\vec{a}_\alpha) \subseteq X$. By our construction, $\mathrm{FR}(\vec{a}_\alpha)$ is infinite, and so is X. Furthermore, $\mathrm{FR}(\vec{a}_\alpha - n) \in F_\alpha \subseteq U$ for all $n \in \omega$. Therefore, U is nonprincipal and strongly reductible. □

Without the continuum hypothesis, $cof(\lambda)$ need not be ω for every limit ordinal $\lambda < \mathfrak{c}$. Hence, we need Lemma 3.3 to hold without the assumption that $cof(\lambda) = \omega$. Martin's axiom is sufficient for this.

Before we prove the main lemma, let us remind the reader of the version of Martin's Axiom that we shall use. Suppose $(Q, \preccurlyeq)$ is a pre-partially ordered set. A subset D of Q is said to be an *antichain* iff for every distinct $a, b \in D$, there does not exist $c \in Q$ such that $c \preccurlyeq a$ and $c \preccurlyeq b$. We say that $(Q, \preccurlyeq)$ satisfies the *countable chain condition* iff every antichain in Q is countable. A subset D of Q is said to be *dense* in Q iff for every $a \in Q$, there exists $d \in D$ such that $d \preccurlyeq a$. A non-empty subset G of Q is called a *filter* iff

(1) for every $a \in G$ and $b \in Q$, if $a \preccurlyeq b$ then $b \in G$;
(2) for every $a, b \in G$, there exists $c \in G$ such that $c \preccurlyeq a$ and $c \preccurlyeq b$.

Suppose κ is an infinite cardinal. $\mathsf{MA}(\kappa)$ asserts that if Q is a pre-partially ordered set satisfying the countable chain condition and $\mathcal{F}$ is a family of dense subsets of Q for which $|\mathcal{F}| < \kappa$, then there exists a filter in Q which intersects every set in $\mathcal{F}$. Martin's Axiom states that $\mathsf{MA}(\kappa)$ holds for all infinite cardinals $\kappa < \mathfrak{c}$. Since $\mathsf{MA}(\omega)$ is true, Martin's Axiom is a consequence of the continuum hypothesis.

Lemma 3.5. *Suppose $\lambda < \mathfrak{c}$ is a limit ordinal and assume $\mathsf{MA}(|\lambda|)$. If $\langle \vec{a}_\alpha \rangle_{\alpha \in \lambda}$ is a transfinite sequence in ${}^\omega\omega$ such that $\vec{a}_\alpha \leq^* \vec{a}_\beta$ whenever $\beta < \alpha$, then there exists $\vec{b} \in {}^\omega\omega$ such that $\vec{b} \leq^* \vec{a}_\alpha$ for all $\alpha < \lambda$.*

Proof. In this proof, $\vec{a} =^* \vec{b}$ means there exist $m, n \in \omega$ such that $\vec{a} - m = \vec{b} - n$. (The motivation for this is that the filter generated by the sets $\mathrm{FR}(\vec{a} - n)$ is the same as the filter generated by the sets $\mathrm{FR}(\vec{b} - n)$ whenever $\vec{a} =^* \vec{b}$.)

Suppose $\langle \vec{a}_\alpha \rangle_{\alpha \in \lambda}$ is given as stated. Let $S = \{ \vec{a} \in {}^\omega\omega \mid \vec{a} =^* \vec{a}_\alpha \text{ for some } \alpha < \lambda \}$. We will need the observation that $\vec{a} \leq^* \vec{b}$ or $\vec{b} \leq^* \vec{a}$ whenever $\vec{a}, \vec{b} \in S$. Let

$$Q = \{ (n, \vec{a}) \mid n \in \omega, \vec{a} \in S \}$$

and define for every $(m, \vec{a}), (n, \vec{b}) \in Q$,

$$(m, \vec{a}) \preccurlyeq (n, \vec{b}) \text{ if and only if } m \geq n, \vec{a} \upharpoonright n = \vec{b} \upharpoonright n \text{ and } \vec{a} \leq \vec{b}.$$

Claim. *$(Q, \preccurlyeq)$ is a pre-partial ordering with the countable chain condition.*

It is easy to check that $\preccurlyeq$ is reflexive and transitive. Suppose $(n, \vec{a})$ and $(n, \vec{b})$ are elements of an antichain such that $\vec{a} \upharpoonright n = \vec{b} \upharpoonright n$. We will show that $(n, \vec{a}) = (n, \vec{b})$. The countable chain condition will then follow as this induces a one-to-one map from the antichain into the set of finite sequences in ω. We may assume that $\vec{a} \leq^* \vec{b}$. Then we can choose $N \geq n$ such that $\vec{a} - N \leq \vec{b} - n$. Let $\vec{c} = \vec{a} \upharpoonright n * (\vec{a} - N)$. It is clear that $(n, \vec{c}) \in Q$ and $(n, \vec{c}) \preccurlyeq (n, \vec{a})$. Since $\vec{a} \upharpoonright n = \vec{b} \upharpoonright n$, we have $\vec{a} \upharpoonright n * (\vec{a} - N) \leq \vec{b} \upharpoonright n * (\vec{b} - n)$ implying that $\vec{c} \leq \vec{b}$. Hence,

$(n,\vec{c}) \preccurlyeq (n,\vec{b})$ as well. Since $(n,\vec{a})$ and $(n,\vec{b})$ belong to an antichain, $(n,\vec{a})$ and $(n,\vec{b})$ cannot be distinct. The claim is proved.

Now, consider the following subsets of Q.

$$D(\alpha) = \{\, (n,\vec{a}) \in Q \mid \vec{a} - n \leq \vec{a}_\alpha \,\}, \quad \alpha < \lambda$$

$$E(m) = \{\, (n,\vec{a}) \in Q \mid n \geq m \,\}, \quad m \in \omega$$

Claim. *$D(\alpha)$ and $E(m)$ are dense subsets of Q.*

Suppose $(n,\vec{a}) \in Q$. Then $(\max\{m,n\},\vec{a}) \preccurlyeq (n,\vec{a})$ and $(\max\{m,n\},\vec{a}) \in E(m)$. Hence, $E(m)$ is dense in Q.

Fix $\alpha < \lambda$. To see that $D(\alpha)$ is dense in Q, suppose $(n,\vec{a}) \in Q$. Since $\vec{a}_\alpha, \vec{a} \in S$, either $\vec{a}_\alpha \leq^* \vec{a}$ or $\vec{a} \leq^* \vec{a}_\alpha$. Assume $\vec{a}_\alpha \leq^* \vec{a}$. Then we can choose $N \geq n$ such that $\vec{a}_\alpha - N \leq \vec{a} - n$. Let $\vec{c} = \vec{a} \restriction n * (\vec{a}_\alpha - N)$. Otherwise, assume $\vec{a} \leq^* \vec{a}_\alpha$. Then we can choose $N \geq n$ such that $\vec{a} - N \leq \vec{a}_\alpha$. Let $\vec{c} = \vec{a} \restriction n * (\vec{a} - N)$. In either case, we can easily verify that $(n,\vec{c}) \preccurlyeq (n,\vec{a})$ and $(n,\vec{c}) \in D(\alpha)$. The claim is proved.

Therefore, $\{\, D(\alpha) \mid \alpha < \lambda \,\} \cup \{\, E(m) \mid m \in \omega \}$ is a family of dense subsets of Q of size $|\lambda|$. By $\mathsf{MA}(|\lambda|)$, choose a filter G on Q that intersects all these subsets. Let

$$\vec{b} = \bigcup \{\, \vec{a} \restriction n \mid (n,\vec{a}) \in G \}.$$

We claim that $\vec{b}$ is a function on ω. Suppose $(n,\vec{a}), (n',\vec{a}') \in G$. Since G is a filter, choose $(n'',\vec{a}'') \in G$ such that $(n'',\vec{a}'') \preccurlyeq (n,\vec{a})$ and $(n'',\vec{a}'') \preccurlyeq (n',\vec{a}')$. This means that $\vec{a}'' \restriction n = \vec{a} \restriction n$ and $\vec{a}'' \restriction n' = \vec{a}' \restriction n'$, implying that $\vec{a} \restriction n$ and $\vec{a}' \restriction n'$ are compatible. It follows that $\vec{b}$ is a function. To see that the domain of $\vec{b}$ is ω, suppose $m \in \omega$. Since $G \cap E(m+1) \neq \varnothing$, choose $(n,\vec{a}) \in G$ such that $n \geq m+1$. Then $\vec{a} \restriction n$ is an initial segment of $\vec{b}$, so that m is in the domain of $\vec{b}$.

It remains to prove that $\vec{b} \leq^* \vec{a}_\alpha$ for all $\alpha < \lambda$. Fix $\alpha < \lambda$. Suppose $(n,\vec{a}) \in G \cap D(\alpha)$. We claim that $\vec{b} - n \leq \vec{a}_\alpha$. By Lemma 2.4, it suffices to show that every initial segment of $\vec{b} - n$ is a reduction of $\vec{a}_\alpha$. Fix $m > n$. Since $G \cap E(m) \neq \varnothing$, choose $(m',\vec{c}) \in G$ with $m' \geq m$. By the definition of $\vec{b}$, it suffices to show that $\langle \vec{c}(n), \dots, \vec{c}(m'-1) \rangle$ is a reduction of $\vec{a}_\alpha$. Choose $(m'',\vec{d}) \in G$ such that $(m'',\vec{d}) \preccurlyeq (n,\vec{a})$ and $(m'',\vec{d}) \preccurlyeq (m',\vec{c})$. Since $\vec{d} \restriction m' = \vec{c} \restriction m'$, the sequence $\langle \vec{c}(n), \dots, \vec{c}(m'-1) \rangle$ is equal to $\langle \vec{d}(n), \dots, \vec{d}(m'-1) \rangle$. Meanwhile, $\vec{d} \leq \vec{a}$ implies that $\vec{d} - n$ is a reduction of $\vec{a} - n$, which in turn is a reduction of $\vec{a}_\alpha$ because $(n,\vec{a}) \in D(\alpha)$. Therefore, $\langle \vec{d}(n), \dots, \vec{d}(m'-1) \rangle$ is a reduction of $\vec{a}_\alpha$, and so is $\langle \vec{c}(n), \dots, \vec{c}(m'-1) \rangle$. This completes the proof of the theorem. $\square$

Theorem 3.6. *Assume Martin's Axiom. Suppose $(\omega, \mathcal{F})$ is a nondegenerate Ramsey algebra. Then there exists a nonprincipal strongly reductible ultrafilter.*

Proof. This is similar to the proof of Theorem 3.4. At the limit stage, to get a sequence $\vec{b}$ such that $\vec{b} \le^* \vec{a}_\alpha$ for all $\alpha < \lambda$, use Lemma 3.5 instead of Lemma 3.3. □

Now, we will address the case where the underlying set is not assumed to be countable.

Theorem 3.7. *Assume Martin's axiom. Suppose $(A, \mathcal{F})$ is a nondegenerate Ramsey algebra and that $\mathcal{F}$ is countable. Then there exists a nonprincipal ultrafilter U on A strongly reductible for $\mathcal{F}$.*

Proof. Let $(B, \mathcal{G})$ be the smallest subalgebra of $(A, \mathcal{F})$ generated by $\{\vec{a}(i) \mid i \in \omega\}$, where $\mathcal{G} = \{f \upharpoonright B \mid f \in \mathcal{F}\}$. Since $\mathcal{F}$ is countable, $(B, \mathcal{G})$ is a countable Ramsey algebra. Applying Theorem 3.6 with B identified as ω, choose a nonprincipal ultrafilter U on B strongly reductible for $\mathcal{G}$. Let V be $\{X \subseteq A \mid X \cap B \in U\}$. Then V is a nonprincipal ultrafilter on A. Furthermore, strongly reductibility of U for $\mathcal{G}$ easily implies strongly reductibility of V for $\mathcal{F}$. □

4. Concluding Remarks and Open Problems

Suppose $(A, \mathcal{F})$ is an algebra and suppose U is an ultrafilter on A. Consider the property

$$\text{for each } X \in U, \text{ there exists } \vec{a} \in {}^{\omega}A \text{ such that } \mathrm{FR}_{\mathcal{F}}(\vec{a}) \subseteq X \text{ and } \mathrm{FR}_{\mathcal{F}}(\vec{a}) \in U. \quad (*)$$

Property $*$ is weaker than strongly reductible. It seems natural to have defined U with the property $*$ to be strongly reductible. In fact, various results for strongly summable ultrafilters have been generalized in [9] and [11] to infinite abelian groups, where the analogue of a strongly summable ultrafilter is defined to be the one having property $*$. We do not know whether the two possible definitions are equivalent in general and for abelian groups in particular. Our choice of definition is partially motivated by the following observation: an ultrafilter strongly reductible for $\mathcal{F}$ is immediately an ultrafilter idempotent for $\mathcal{F}$, analogous to the fact that a strongly summable ultrafilter is an idempotent ultrafilter. Because a general Ramsey algebra lacks a certain algebraic property, like that enjoyed by abelian groups, our choice of definition is plausible.

Meanwhile, the analogue of the independence result for strongly summable ultrafilters does not hold in general for Ramsey algebras. It follows from [9] that the existence of nonprincipal strongly reductible ultrafilters for an infinite abelian group cannot be proven in ZFC. On the other hand, suppose $P(x, y) = x$ for all $x, y \in \omega$. Then $(\omega, \{P\})$ is trivially a Ramsey algebra and every nonprincipal ultrafilter on ω is strongly reductible for $\{P\}$. Therefore, it is natural to ask for a subclass of Ramsey algebras properly containing the class of infinite abelian groups such that the existence of the corresponding strongly reductible ultrafilters is independent of ZFC.

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REFERENCES

[1] A. Blass and N. Hindman, *On strongly summable and union ultrafilters*, Trans. Amer. Math. Soc. **304** (1987), 83-99.
[2] T. J. Carlson, *Some unifying principles in Ramsey theory*, Discrete Mathematics **68** (1988), 117–169.
[3] W. Comfort, *Ultrafilters - some old and some new results*, Bull. Amer. Math. Soc. **83** (1977), 417-455.
[4] E. Ellentuck, *A new proof that analytic sets are Ramsey*, J. Symb. Logic **39** (1974), 163-165.
[5] N. Hindman, *The existence of certain ultrafilters on* $\mathbb{N}$ *and a conjecture of Graham and Rothschild*, Proc. Amer. Math. Soc. **36** (1972), 341-346.
[6] N. Hindman, *Finite sums from sequences within cells of a partition of* $\mathbb{N}$, J. Combin. Theory (Series A) **17** (1974), 1-11.
[7] N. Hindman, *Ultrafilters and combinatorial number theory*, in Number Theory Carbondale, M. Nathanson ed., Lecture Notes in Math. **751** (1979), 119-184.
[8] N. Hindman, *Summable ultrafilters and finite sums*, in Logic and Combinatorics, S. Simpson ed., Contemp. Math. **65** (1987), 263-274.
[9] N. Hindman, I. Protasov and D. Strauss, *Strongly summable ultrafilters on abelian groups*, Matem. Studii **19** (1998), 121-132.
[10] N. Hindman and D. Strauss, *Algebra in the Stone-Čech Compactification: Theory and Applications*, Second edition, Walter de Gruyter, Berlin (2012).
[11] P. Krautzberger, *On strongly summable ultrafilters*, New York J. M. **16** (2010), 629-649.
[12] S. Shelah, *Proper and Improper Forcing*, Springer-Verlag, Berlin, 1982.
[13] W.C. Teh, *Ramsey algebras* (submitted).
[14] W.C. Teh, *Ramsey algebras and formal orderly terms* (submitted).

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