Maximum principles for second-order impulsive integro-differential equations with integral jump conditions

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Abstract

We are concerned with two new maximum principles for second-order impulsive integro-differential equations with integral jump conditions. The impulse effects or jump conditions of this paper are involved in terms of integral of past states. As an application, we introduce a new definition of lower and upper solutions which leads to the development of the monotone iterative technique for a periodic boundary value problem related to a nonlinear second-order impulsive functional integro-differential equation with integral jump conditions.

Keywords: maximum principle; impulsive differential inequality; impulsive integro-differential equation; monotone iterative technique

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1 Introduction

Maximum principles play an important role in the study of the qualitative theory of impulsive differential equations [1]. The monotone iterative technique coupled with the method of lower and upper solutions have used maximum principles to ensure that the sequences of approximate solutions converge to the extremal solutions of nonlinear impulsive problems (see, for example, [2-8]). Recently, some excellent results have been obtained by applying this concept to several impulsive problems which include local jump conditions, see [9-13]. These local jump conditions involve discontinuities

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in the solution values or derivative of solution values at a set of discrete points. However, there have only been a few papers that have studied maximum principles for impulsive problems with nonlocal jump conditions, see [14-18].

In a recent paper [19], the authors considered the following periodic boundary value problem for second-order impulsive integro-differential equations with integral jump conditions:

$$\begin{cases} x''(t) = f(t, x(t), (Kx)(t), (Sx)(t)), \quad t \in J = [0, T], \quad t \neq t_k, \\ \Delta x(t_k) = I_k\left(\displaystyle\int_{t_k-\delta_k}^{t_k-\varepsilon_k} x'(s)ds\right), \quad k = 1, 2, \dots, m, \\ \Delta x'(t_k) = I_k^*\left(\displaystyle\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds\right), \quad k = 1, 2, \dots, m, \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times R^3 \to R$ is continuous everywhere except at $\{t_k\} \times R^3$, $f(t_k^+, x, y, z)$ and $f(t_k^-, x, y, z)$ exist, $f(t_k^-, x, y, z) = f(t_k, x, y, z)$, $I_k \in C(R, R)$, $I_k^* \in C(R, R)$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$, $0 \leq \varepsilon_k \leq \delta_k \leq t_k - t_{k-1}$, $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$, $k = 1, 2, \dots, m$,

$$(Kx)(t) = \int_0^t k(t, s)x(s)ds, \qquad (Sx)(t) = \int_0^T h(t, s)x(s)ds,$$

$k(t, s) \in C(D, R^+)$, $h(t, s) \in C(J \times J, R^+)$, $D = \{(t, s) \in R^2, 0 \leq s \leq t \leq T\}$, $R^+ = [0, +\infty)$, $k_0 = \max\{k(t, s) : (t, s) \in D\}$, $h_0 = \max\{h(t, s) : (t, s) \in J \times J\}$. They gave some maximum principles for integral jump conditions and used the monotone iterative technique to obtain two sequences which approximate the extremal solutions of (1.1) between a lower and upper solution. We note that jump conditions of problem (1.1) depend on the areas under the curve of solutions and the derivative of solutions of the past states. This means that impulse effects of such problem have memory of path history.

In this paper, we mainly investigate maximum principles related to impulsive integro-differential equation for the following integral jump conditions:

$$\Delta x(t_k) = L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds, \quad \Delta x'(t_k) = L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} x'(s)ds, \quad k = 1, 2, \dots, m, \tag{1.2}$$

where $0 \leq \sigma_k < \tau_k \leq t_k - t_{k-1}$, $0 \leq \varepsilon_k < \delta_k \leq t_k - t_{k-1}$, L_k, L_k^* are given constants, $k = 1, 2, \dots, m$. The key tool for our proof is impulsive differential inequalities with jump conditions.

The plan of this paper is as follows. In Section 2, we present two new maximum principles. In Section 3, we obtain the existence of an extreme solution of a periodic boundary value problem by using the method of upper and lower solutions and the monotone iterative technique with a comparison result.

2 Maximum Principles

Denote $l = \max\{k : t \geq t_k,\ k = 1, 2, \ldots\}$ and $J^- = J \setminus \{t_i,\ i = 1, 2, \ldots, m\}$. Let $J \subset R$ be an interval. We define $PC(J, R) = \{x : J \to R;\ x(t)$ to be continuous everywhere except at some finite points t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k), k = 1, 2, \ldots m\}$. We also define $PC^1(J, R) = \{x \in PC(J, R)$: $x'(t)$ is continuous everywhere except for some t_k at which $x'(t_k^+)$ and $x'(t_k^-)$ exist and $x'(t_k^-) = x'(t_k), k = 1, 2, \ldots m\}$ and $PC^2(J, R) = \{x \in PC^1(J, R) : x|_{(t_k, t_{k+1})} \in C^2((t_k, t_{k+1}), R), k = 1, 2, \ldots m\}$. We prove the maximum principle by using the following lemma.

Lemma 2.1. *([20]) Let $r \in \{t_0, t_1, \ldots, t_m\}$, $c_k > -1/(\tau_k - \sigma_k)$, $0 \leq \sigma_k < \tau_k \leq t_k - t_{k-1}$, γ_k, $k = 1, 2, \ldots, m$ be constants and let $q \in PC(J, R)$, $x \in PC^1(J, R)$.*
(i) If

$$\begin{cases} x'(t) \leq q(t), \quad t \in (r, T), \quad t \neq t_k, \\ \Delta x(t_k) \leq c_k \displaystyle\int_{t_k - \tau_k}^{t_k - \sigma_k} x(s)ds + \gamma_k, \quad t_k \in (r, T), \quad k = 1, 2, \ldots, m. \end{cases}$$

Then for $t \in (r, T]$,

$$\begin{aligned} x(t) \leq x(r^+) &\left(\prod_{r^+ < t_k < t} [1 + c_k(\tau_k - \sigma_k)] \right) + \sum_{r^+ < t_k < t} \left[\prod_{t_k < t_j < t} [1 + c_j(\tau_j - \sigma_j)] \right. \\ &\times \left([1 + c_k(\tau_k - \sigma_k)] \int_{t_{k-1}}^{t_k - \tau_k} q(s)ds + \int_{t_k - \tau_k}^{t_k - \sigma_k} [1 + c_k(t_k - \sigma_k - s)] q(s)ds \right. \\ &\left. \left. + \int_{t_k - \sigma_k}^{t_k} q(s)ds \right) \right] + \int_{t_l}^{t} q(s)ds. \end{aligned}$$

(ii) If

$$\begin{cases} x'(t) \geq q(t), \quad t \in (r, T), \quad t \neq t_k, \\ \Delta x(t_k) \geq c_k \displaystyle\int_{t_k - \tau_k}^{t_k - \sigma_k} x(s)ds + \gamma_k, \quad t_k \in (r, T), \quad k = 1, 2, \ldots, m. \end{cases}$$

Then for $t \in (r, T]$,

$$\begin{aligned} x(t) \geq x(r^+) &\left(\prod_{r^+ < t_k < t} [1 + c_k(\tau_k - \sigma_k)] \right) + \sum_{r^+ < t_k < t} \left[\prod_{t_k < t_j < t} [1 + c_j(\tau_j - \sigma_j)] \right. \\ &\times \left([1 + c_k(\tau_k - \sigma_k)] \int_{t_{k-1}}^{t_k - \tau_k} q(s)ds + \int_{t_k - \tau_k}^{t_k - \sigma_k} [1 + c_k(t_k - \sigma_k - s)] q(s)ds \right. \\ &\left. \left. + \int_{t_k - \sigma_k}^{t_k} q(s)ds \right) \right] + \int_{t_l}^{t} q(s)ds. \end{aligned}$$

We now present two new maximum principles.

Theorem 2.1. *Assume that* $x \in PC^2(J,R)$ *satisfies*

$$\begin{aligned} x''(t) &\leq -Mx(t) - Wx(\theta(t)) - N\int_0^t k(t,s)x(s)ds \\ &\quad -L\int_0^T h(t,s)x(s)ds, \quad t \in J^-, \qquad (2.1) \\ \Delta x(t_k) &\leq -L_k\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds, \quad k=1,2,...,m, \qquad (2.2) \\ \Delta x'(t_k) &\leq -L_k^*\int_{t_k-\delta_k}^{t_k-\varepsilon_k} x'(s)ds, \quad k=1,2,...,m, \qquad (2.3) \\ x(0) &\leq x(T), \quad x'(0) \leq x'(T), \qquad (2.4) \end{aligned}$$

where constants $M > 0$, $W \geq 0$, $N \geq 0$, $L \geq 0$, $0 \leq L_k < 1/(\tau_k - \sigma_k)$, $0 \leq L_k^* < 1/(\delta_k - \varepsilon_k)$, $0 \leq \sigma_k < \tau_k \leq t_k - t_{k-1}$, $0 \leq \varepsilon_k < \delta_k \leq t_k - t_{k-1}$, $k = 1, 2, \ldots, m$ *and* $\theta \in C(J, J)$.
If

$$\prod_{k=1}^m A_k \geq \hat{L} + \int_0^T U(s)ds \qquad (2.5)$$

where

$$\begin{aligned} A_k &= 1 - L_k(\tau_k - \sigma_k), \\ \hat{L} &= \max\{L_k(\tau_k - \sigma_k)\}, \\ U(t) &= \frac{\prod_{0<t_k<t} A_k^*}{1 - \prod_{k=1}^m A_k^*}\left[\sum_{k=1}^m \prod_{j=k+1}^m A_j^* B_k^* + \int_{t_m}^T r(s)ds\right] \\ &\quad + \left[\sum_{0<t_k<t}\prod_{t_k<t_j<t} A_j^* B_k^* + \int_{t_l}^t r(s)ds\right], \\ A_k^* &= 1 - L_k^*(\delta_k - \varepsilon_k), \quad \textit{which} \quad \prod_{k=1}^m A_k^* < 1, \\ B_k^* &= A_k^*\int_{t_{k-1}}^{t_k-\delta_k} r(s)ds + \int_{t_k-\delta_k}^{t_k-\varepsilon_k}[1 - L_k^*(t_k - \varepsilon_k - s)]r(s)ds + \int_{t_k-\varepsilon_k}^{t_k} r(s)ds, \\ r(t) &= M + W + N\int_0^t k(t,s)ds + L\int_0^T h(t,s)ds, \end{aligned}$$

then $x(t) \leq 0$, $t \in J$.

Proof. Suppose, to the contrary, that $x(t) > 0$ for some $t \in J$. Assume that there exists $t^* \in J$ such that $x(t^*) > 0$. We now consider the following two cases:

Case (i). If $x(t) \geq 0$ for all $t \in J$ and $x \not\equiv 0$. Then, from (2.1), we have

$$x''(t) \leq -Mx(t) - Wx(\theta(t)) - N\int_0^t k(t,s)x(s)ds - L\int_0^T h(t,s)x(s)ds \leq 0, \qquad t \in J^-. \tag{2.6}$$

Applying Lemma 2.1 for (2.3) and (2.6), we obtain

$$x'(t) \leq x'(0) \prod_{0<t_k<t} \left[1 - L_k^*(\delta_k - \varepsilon_k)\right].$$

For $t = T$, we get $x'(0) \leq x'(T) \leq x'(0) \prod_{0<t_k<T} [1 - L_k^*(\delta_k - \varepsilon_k)]$ and therefore $x'(0) \leq 0$. Hence $x'(t) \leq 0$. Also, we have

$$x(t_k^+) \leq x(t_k^-) - L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds \leq x(t_k^-).$$

Thus $x(t)$ is non-increasing for $t \in J$, and therefore $x(T) \leq x(0)$. From condition (2.4), we have $x(0) \leq x(T)$ and therefore $x(0) = x(T) = x$ is a constant. Therefore $x(t) \equiv C > 0$, which implies that

$$0 = x''(t) \leq -MC - WC - N\int_0^t k(t,s)Cds - L\int_0^T h(t,s)Cds \leq -MC < 0,$$

which is a contradiction.

Case (ii). If $x(t) < 0$ for some $t \in J$. Let $\inf_{t\in J} x(t) = -\lambda < 0$, then there exists $t_* \in (t_u, t_{u+1}]$, for some u such that $x(t_*) = -\lambda$ or $x(t_u^+) = -\lambda$. Without loss of generality, we only consider $x(t_*) = -\lambda$. For the case $x(t_u^+) = -\lambda$ the proof is similar. From (2.1), we get

$$x''(t) \leq \lambda\left(M + W + N\int_0^t k(t,s)ds + L\int_0^T h(t,s)ds\right) = \lambda r(t), \qquad t \in J^-. \tag{2.7}$$

Using Lemma 2.1 part (i) for (2.3), (2.7), we obtain

$$\begin{aligned} x'(t) \leq x'(0)&\left(\prod_{0<t_k<t}[1 - L_k^*(\delta_k - \varepsilon_k)]\right) + \lambda \sum_{0<t_k<t}\left[\prod_{t_k<t_j<t}[1 - L_j^*(\delta_j - \varepsilon_j)]\right. \\ &\times \left([1 - L_k^*(\delta_k - \varepsilon_k)]\int_{t_{k-1}}^{t_k-\delta_k} r(s)ds + \int_{t_k-\delta_k}^{t_k-\varepsilon_k}[1 - L_k^*(t_k - \varepsilon_k - s)]r(s)ds\right. \\ &\left.\left. + \int_{t_k-\varepsilon_k}^{t_k} r(s)ds\right)\right] + \lambda\int_{t_l}^t r(s)ds. \end{aligned}$$

Taking into account the definition of constants A_k^* and B_k^*, the above inequality can be written as

$$x'(t) \leq x'(0)\prod_{0<t_k<t} A_k^* + \lambda \sum_{0<t_k<t}\prod_{t_k<t_j<t} A_j^* B_k^* + \lambda\int_{t_l}^t r(s)ds, \quad t \in J. \tag{2.8}$$

Substituting $t = T$ in (2.8), we get

$$x'(0) \le x'(T) \le x'(0) \prod_{0<t_k<T} A_k^* + \lambda \sum_{0<t_k<T} \prod_{t_k<t_j<T} A_j^* B_k^* + \lambda \int_{t_m}^{T} r(s)ds,$$

which implies

$$x'(0) \le \frac{\lambda}{1 - \prod_{k=1}^{m} A_k^*} \left[\sum_{k=1}^{m} \prod_{j=k+1}^{m} A_j^* B_k^* + \int_{t_m}^{T} r(s)ds \right]. \tag{2.9}$$

From (2.8) and (2.9), we obtain

$$\begin{aligned} x'(t) \le & \frac{\lambda \prod_{0<t_k<t} A_k^*}{1 - \prod_{k=1}^{m} A_k^*} \left[\sum_{k=1}^{m} \prod_{j=k+1}^{m} A_j^* B_k^* + \int_{t_m}^{T} r(s)ds \right] \\ & + \lambda \left[\sum_{0<t_k<t} \prod_{t_k<t_j<t} A_j^* B_k^* + \int_{t_l}^{t} r(s)ds \right]. \end{aligned}$$

Therefore,

$$x'(t) \le \lambda U(t), \quad t \in J. \tag{2.10}$$

The above inequality together with Lemma 2.1 part (i) and (2.2) imply that

$$\begin{aligned} x(t) \le & x(t_{u+1}^+) \left(\prod_{t_{u+1}<t_k<t} [1 - L_k(\tau_k - \sigma_k)] \right) + \lambda \sum_{t_{u+1}<t_k<t} \left[\prod_{t_k<t_j<t} [1 - L_j(\tau_j - \sigma_j)] \right. \\ & \times \left([1 - L_k(\tau_k - \sigma_k)] \int_{t_{k-1}}^{t_k-\tau_k} U(s)ds + \int_{t_k-\tau_k}^{t_k-\sigma_k} [1 - L_k(t_k - \sigma_k - s)] U(s)ds \right. \\ & \left. \left. + \int_{t_k-\sigma_k}^{t_k} U(s)ds \right) \right] + \lambda \int_{t_l}^{t} U(s)ds. \end{aligned} \tag{2.11}$$

Let

$$B_k = A_k \int_{t_{k-1}}^{t_k-\tau_k} U(s)ds + \int_{t_k-\tau_k}^{t_k-\sigma_k} [1 - L_k(t_k - \sigma_k - s)] U(s)ds + \int_{t_k-\sigma_k}^{t_k} U(s)ds.$$

Then, (2.11) can be written as

$$x(t) \le x(t_{u+1}^+) \prod_{t_{u+1}<t_k<t} A_k + \lambda \left[\sum_{t_{u+1}<t_k<t} \prod_{t_k<t_j<t} A_j B_k + \int_{t_l}^{t} U(s)ds \right], \quad t \in J. \tag{2.12}$$

Since

$$x(t_{u+1}^+) \le x(t_{u+1}) - L_{u+1} \int_{t_{u+1}-\tau_{u+1}}^{t_{u+1}-\sigma_{u+1}} x(s)ds \le x(t_{u+1}) + \lambda L_{u+1}(\tau_{u+1} - \sigma_{u+1}),$$

the equation (2.12) can be expressed as

$$x(t) \leq x(t_{u+1}) \left(\prod_{t_{u+1}<t_k<t} A_k \right) + \lambda L_{u+1}(\tau_{u+1} - \sigma_{u+1}) \left(\prod_{t_{u+1}<t_k<t} A_k \right) + \lambda \left[\sum_{t_{u+1}<t_k<t} \prod_{t_k<t_j<t} A_j B_k + \int_{t_l}^{t} U(s)ds \right], \quad t \in J. \tag{2.13}$$

Integrating (2.10) from t_* to t_{u+1}, we obtain

$$x(t_{u+1}) \leq x(t_*) + \lambda \int_{t_*}^{t_{u+1}} U(s)ds.$$

Hence,

$$x(t) \leq x(t_*) \left(\prod_{t_{u+1}<t_k<t} A_k \right) + \lambda L_{u+1}(\tau_{u+1} - \sigma_{u+1}) \left(\prod_{t_{u+1}<t_k<t} A_k \right) + \lambda \left(\prod_{t_{u+1}<t_k<t} A_k \right) \int_{t_*}^{t_{u+1}} U(s)ds + \lambda \left[\sum_{t_{u+1}<t_k<t} \prod_{t_k<t_j<t} A_j B_k + \int_{t_l}^{t} U(s)ds \right].$$

This yields

$$x(t) \leq x(t_*) \left(\prod_{t_{u+1}<t_k<t} A_k \right) + \lambda \Bigg[L_{u+1}(\tau_{u+1} - \sigma_{u+1}) \left(\prod_{t_{u+1}<t_k<t} A_k \right) + \sum_{t_u<t_k<t} \prod_{t_k<t_j<t} A_j \int_{t_{k-1}}^{t_k} U(s)ds + \int_{t_l}^{t} U(s)ds \Bigg]. \tag{2.14}$$

If $t^* > t_*$ for $t^* \in [t_v, t_{v+1})$, then

$$0 < x(t^*) \leq x(t_*) \left(\prod_{t_{u+1}<t_k<t^*} A_k \right) + \lambda \Bigg[L_{u+1}(\tau_{u+1} - \sigma_{u+1}) \left(\prod_{t_{u+1}<t_k<t^*} A_k \right) + \sum_{t_u<t_k<t^*} \prod_{t_k<t_j<t^*} A_j \int_{t_{k-1}}^{t_k} U(s)ds + \int_{t_v}^{t^*} U(s)ds \Bigg],$$

which gives

$$\prod_{t_{u+1}<t_k<t^*} A_k < L_{u+1}(\tau_{u+1} - \sigma_{u+1}) \prod_{t_{u+1}<t_k<t^*} A_k + \sum_{t_u<t_k<t^*} \prod_{t_k<t_j<t^*} A_j \int_{t_{k-1}}^{t_k} U(s)ds + \int_{t_v}^{t^*} U(s)ds. \tag{2.15}$$

Therefore

$$\prod_{k=1}^{m} A_k < \hat{L} + \int_0^T U(s)ds,$$

contradicting condition (2.5).

Next, assume that $t^* < t_*$. For $t = T$ in (2.14), we get

$$x(T) \leq x(t_*)\left(\prod_{t_{u+1}<t_k<T} A_k\right) + \lambda\Bigg[L_{u+1}(\tau_{u+1} - \sigma_{u+1})\left(\prod_{t_{u+1}<t_k<T} A_k\right) + \sum_{t_u<t_k<T}\prod_{t_k<t_j<T} A_j \int_{t_{k-1}}^{t_k} U(s)ds + \int_{t_m}^{T} U(s)ds\Bigg]. \tag{2.16}$$

From (2.10) and (2.2), we have by Lemma 2.1 part (i) that

$$x(t) \leq x(0)\prod_{0<t_k<t} A_k + \lambda\left[\sum_{0<t_k<t}\prod_{t_k<t_j<t} A_j B_k + \int_{t_l}^{t} U(s)ds\right].$$

In particular, for $t = t^*$,

$$x(t^*) \leq x(0)\prod_{0<t_k<t^*} A_k + \lambda\left[\sum_{0<t_k<t^*}\prod_{t_k<t_j<t^*} A_j B_k + \int_{t_v}^{t^*} U(s)ds\right]. \tag{2.17}$$

From (2.4), (2.16) and (2.17), we obtain

$$\begin{aligned} 0 < x(t^*) \leq{}& \lambda\left[\sum_{0<t_k<t^*}\prod_{t_k<t_j<t^*} A_j B_k + \int_{t_v}^{t^*} U(s)ds\right] + x(t_*)\prod_{t_{u+1}<t_k<T} A_k \prod_{0<t_k<t^*} A_k \\ &+ \lambda L_{u+1}(\tau_{u+1} - \sigma_{u+1})\prod_{t_{u+1}<t_k<T} A_k \prod_{0<t_k<t^*} A_k \\ &+ \lambda \prod_{0<t_k<t^*} A_k \left[\sum_{t_u<t_k<T}\prod_{t_k<t_j<T} A_j \int_{t_{k-1}}^{t_k} U(s)ds + \int_{t_m}^{T} U(s)ds\right], \end{aligned}$$

which gives

$$\begin{aligned} \prod_{t_{u+1}<t_k<T} A_k \prod_{0<t_k<t^*} A_k <{}& \left[\sum_{0<t_k<t^*}\prod_{t_k<t_j<t^*} A_j B_k + \int_{t_v}^{t^*} U(s)ds\right] \\ &+ L_{u+1}(\tau_{u+1} - \sigma_{u+1})\prod_{t_{u+1}<t_k<T} A_k \prod_{0<t_k<t^*} A_k \\ &+ \prod_{0<t_k<t^*} A_k \left[\sum_{t_u<t_k<T}\prod_{t_k<t_j<T} A_j \int_{t_{k-1}}^{t_k} U(s)ds + \int_{t_m}^{T} U(s)ds\right]. \end{aligned}$$

Hence

$$\prod_{k=1}^{m} A_k < \hat{L} + \int_0^T U(s)ds,$$

which is a contradiction. This completes the proof. □

Theorem 2.2. *Assume that $x \in PC^2(J, R)$ satisfies*

$$x''(t) \geq Mx(t) + Wx(\theta(t)) + N\int_0^t k(t,s)x(s)ds + L\int_0^T h(t,s)x(s)ds, \quad t \in J^-, \tag{2.18}$$

$$\Delta x(t_k) \geq L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds, \quad k = 1, 2, ..., m, \tag{2.19}$$

$$\Delta x'(t_k) \geq -L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} x'(s)ds, \quad k = 1, 2, ..., m, \tag{2.20}$$

$$x(0) \geq x(T), \quad x'(0) \geq x'(T), \tag{2.21}$$

where constants $M > 0$, $W \geq 0$, $N \geq 0$, $L \geq 0$, $L_k \geq 0$, $0 \leq L_k^* < 1/(\delta_k - \varepsilon_k)$, $0 \leq \sigma_k < \tau_k \leq t_k - t_{k-1}$, $0 \leq \varepsilon_k < \delta_k \leq t_k - t_{k-1}$, $k = 1, 2, \ldots, m$ and $\theta \in C(J, J)$. If

$$\prod_{k=1}^{m} C_k \left[\hat{L} + \sum_{k=1}^{m} D_k + \int_0^T U(s)ds\right] \leq 1 \tag{2.22}$$

where

$$C_k = 1 + L_k(\tau_k - \sigma_k),$$

$$D_k = C_k \int_{t_{k-1}}^{t_k-\tau_k} U(s)ds + \int_{t_k-\tau_k}^{t_k-\sigma_k} [1 + L_k(t_k - \sigma_k - s)]U(s)ds + \int_{t_k-\sigma_k}^{t_k} U(s)ds,$$

and $\widehat{L}$, L_k, A_k, $U(t)$ are defined in Theorem 2.1. Then $x(t) \leq 0$, $t \in J$.

Proof. The proof of this Theorem is similar to the proof of Theorem 2.1. By applying Lemma 2.1 part (*ii*) and using the definition of constants C_k and D_k, we obtain the conclusion as desired. □

Now, we show two examples to illustrate an application of the new results.

Example 2.1. Consider the following impulsive problem:

$$\begin{cases} x''(t) \leq -\frac{1}{4}x(t) - \frac{1}{5}x\left(\frac{t}{2}\right) - \frac{1}{5}\int_0^t tsx(s)ds - \frac{1}{4}\int_0^1 t^2 sx(s)ds, \quad t \in [0,1], \quad t \neq \frac{1}{2}, \\ \Delta x\left(\frac{1}{2}\right) \leq -\frac{1}{3}\int_{\frac{1}{6}}^{\frac{5}{14}} x(s)ds, \quad k = 1, \\ \Delta x'\left(\frac{1}{2}\right) \leq -8\int_{\frac{1}{6}}^{\frac{1}{4}} x'(s)ds, \quad k = 1, \\ x(0) \leq x(1), \quad x'(0) \leq x'(1), \end{cases} \tag{2.23}$$

where $k(t,s)=ts$, $h(t,s)=t^2s$, $\theta(t)=t/2$, $m=1$, $t_1=1/2$, $M=1/4$, $W=1/5$, $N=1/5$, $L=1/4$, $L_k=1/3$, $L_k^*=8$, $\sigma_k=1/7$, $\tau_k=1/3$, $\varepsilon_k=1/4$ and $\delta_k=1/3$. It is easy to see that

$$A_1 = 1-\frac{1}{3}\left(\frac{1}{3}-\frac{1}{7}\right)=\frac{59}{63}$$
$$\hat{L}=\frac{1}{3}\left(\frac{1}{3}-\frac{1}{7}\right)=\frac{4}{63}$$
$$U(t)=\frac{\prod_{0<t_k<t}A_k^*}{1-\prod_{k=1}^{m}A_k^*}\left[\sum_{k=1}^{m}\prod_{j=k+1}^{m}A_j^*B_k^*+\int_{\frac{1}{2}}^{1}r(s)ds\right]+\left[\sum_{0<t_k<t}\prod_{t_k<t_j<t}A_j^*B_k^*+\int_{t_l}^{t}r(s)ds\right]$$
$$A_1^*=1-8\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{1}{3}$$
$$B_1^*=A_1^*\int_0^{\frac{1}{6}}r(s)ds+\int_{\frac{1}{6}}^{\frac{1}{4}}\left[1-8\left(\frac{1}{2}-\frac{1}{4}-s\right)\right]r(s)ds+\int_{\frac{1}{4}}^{\frac{1}{2}}r(s)ds$$
$$r(t)=\frac{1}{4}+\frac{1}{5}+\frac{1}{5}\int_0^t tsds+\frac{1}{4}\int_0^1 t^2sds.$$

Through a simple calculation we can get

$$\prod_{k=1}^{m}A_k\approx 0.9365079365\geq \hat{L}+\int_0^T U(s)ds\approx 0.7454373117.$$

Applying Theorem 2.1, we get that $x(t)\leq 0$ for $t\in[0,1]$.

Example 2.2. Consider the following impulsive problem:

$$\begin{cases} x''(t)\geq \frac{1}{4}x(t)+\frac{1}{5}x\left(\frac{t}{2}\right)+\frac{1}{5}\int_0^t tsx(s)ds+\frac{1}{4}\int_0^1 t^2sx(s)ds, \quad t\in[0,1], \quad t\neq\frac{1}{2},\\ \Delta x\left(\frac{1}{2}\right)\geq\frac{1}{3}\int_{\frac{1}{6}}^{\frac{5}{14}}x(s)ds, \quad k=1,\\ \Delta x'\left(\frac{1}{2}\right)\geq -10\int_{\frac{1}{6}}^{\frac{1}{4}}x'(s)ds, \quad k=1,\\ x(0)\geq x(1), \quad x'(0)\geq x'(1),\end{cases} \tag{2.24}$$

where $k(t,s)=ts$, $h(t,s)=t^2s$, $\theta(t)=t/2$, $m=1$, $t_1=1/2$, $M=1/4$, $W=1/5$, $N=1/5$, $L=1/4$, $L_1=1/3$, $L_1^*=10$, $\sigma_1=1/7$, $\tau_1=1/3$, $\varepsilon_1=1/4$ and $\delta_1=1/3$. It is easy to see that

$$C_1=1+\frac{1}{3}\left(\frac{1}{3}-\frac{1}{7}\right)=\frac{67}{63}$$
$$D_1=C_1\int_0^{\frac{1}{6}}U(s)ds+\int_{\frac{1}{6}}^{\frac{5}{14}}\left[1+L_1\left(\frac{1}{2}-\frac{1}{7}-s\right)\right]U(s)ds+\int_{\frac{5}{14}}^{\frac{1}{2}}U(s)ds$$

$$\hat{L} = \frac{1}{3}\left(\frac{1}{3} - \frac{1}{7}\right) = \frac{4}{63}$$

$$U(t) = \frac{\prod_{0<t_k<t} A_k^*}{1 - \prod_{k=1}^{m} A_k^*}\left[\sum_{k=1}^{m}\prod_{j=k+1}^{m} A_j^* B_k^* + \int_{\frac{1}{2}}^{1} r(s)ds\right] + \left[\sum_{0<t_k<t}\prod_{t_k<t_j<t} A_j^* B_k^* + \int_{t_l}^{t} r(s)ds\right]$$

$$A_1^* = 1 - 10\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6}$$

$$B_1^* = A_1^* \int_0^{\frac{1}{6}} r(s)ds + \int_{\frac{1}{6}}^{\frac{1}{4}} \left[1 - 8\left(\frac{1}{2} - \frac{1}{4} - s\right)\right] r(s)ds + \int_{\frac{1}{4}}^{\frac{1}{2}} r(s)ds$$

$$r(t) = \frac{1}{4} + \frac{1}{5} + \frac{1}{5}\int_0^t tsds + \frac{1}{4}\int_0^1 t^2 sds.$$

Through a simple calculation we can get

$$\prod_{k=1}^{m} C_k \left[\hat{L} + \sum_{k=1}^{m} D_k + \int_0^T U(s)ds\right] \approx 0.9836243209 \leq 1.$$

Applying Theorem 2.2, we get that $x(t) \leq 0$ for $t \in [0, 1]$.

3 Existence Results

In this section, by using the method of upper and lower solutions and the monotone iterative technique, we obtain the existence of extreme solutions for the following periodic boundary value problem (PBVP):

$$\begin{cases} x''(t) = f(t, x(t), x(\theta(t)), (Kx)(t), (Sx)(t)), \quad t \in J^-, \\ \Delta x(t_k) = I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds\right), \quad k = 1, 2, \ldots, m, \\ \Delta x'(t_k) = I_k^*\left(\int_{t_k-\delta_k}^{t_k-\varepsilon_k} x'(s)ds\right), \quad k = 1, 2, \ldots, m, \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases} \tag{3.1}$$

which satisfies all assumptions of PBVP (1.1).
Now, we give a new definition of lower and upper solutions of PBVP (3.1).

Definition 3.1. *We say that the function α_0, $\beta_0 \in E$ are lower and upper solutions of PBVP* (3.1) *where* $0 \leq \sigma_k < \tau_k \leq t_k - t_{k-1}$, $0 \leq \varepsilon_k < \delta_k \leq t_k - t_{k-1}$, *such that*

$$\begin{cases} \alpha_0''(t) \geq f(t, \alpha_0(t), \alpha_0(\theta(t)), (K\alpha_0)(t), (S\alpha_0)(t)), \quad t \in J^-, \\ \Delta \alpha_0(t_k) \geq I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_0(s)ds\right), \quad k = 1, 2, \ldots, m, \\ \Delta \alpha_0'(t_k) \geq I_k^*\left(\int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha_0'(s)ds\right), \quad k = 1, 2, \ldots, m, \\ \alpha_0(0) \geq \alpha_0(T), \quad \alpha_0'(0) \geq \alpha_0'(T), \end{cases}$$

and

$$\begin{cases} \beta_0''(t) \le f(t, \beta_0(t), \beta_0(\theta(t)), (K\beta_0)(t), (S\beta_0)(t)), \quad t \in J^-, \\ \Delta\beta_0(t_k) \le I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \beta_0(s)ds\right), \quad k = 1, 2, \dots, m, \\ \Delta\beta_0'(t_k) \le I_k^*\left(\int_{t_k-\delta_k}^{t_k-\varepsilon_k} \beta_0'(s)ds\right), \quad k = 1, 2, \dots, m, \\ \beta_0(0) \le \beta_0(T), \quad \beta_0'(0) \le \beta_0'(T). \end{cases}$$

Consider the linear PBVP

$$\begin{cases} x''(t) = Mx(t) + Wx(\theta(t)) + N(Kx)(t) + L(Sx)(t) + g(t), \ t \in J^-, \\ \Delta x(t_k) = L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds + \gamma_k, \quad k = 1, 2, \dots, m, \\ \Delta x'(t_k) = -L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} x'(s)ds + \lambda_k, \quad k = 1, 2, \dots, m, \\ x(0) = x(T), \qquad x'(0) = x'(T), \end{cases} \tag{3.2}$$

where constants $M > 0$, $W, N, L \ge 0$, $L_k \ge 0$, $L_k^* \ge 0$, $g \in PC(J, R)$, $0 \le \varepsilon_k < \delta_k \le t_k - t_{k-1}$, $0 \le \sigma_k < \tau_k \le t_k - t_{k-1}$, γ_k, λ_k are constants, $k = 1, 2, \dots, m$. $PC(J, R)$ and $PC^1(J, R)$ are Banach spaces with the norms $\|x\|_{PC} = \sup\{x(t) : t \in J\}$ and $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$. Let $E = PC^1(J, R) \cap C^2(J^-, R)$. A function $x \in E$ is called a solution of PBVP (3.2) if it satisfies (3.2).

Lemma 3.1. *$x \in E$ is a solution of PBVP* (3.2) *if and only if $x \in PC^1(J, R)$ is a solution of the following the impulsive integral equation:*

$$\begin{aligned} x(t) = \int_0^T G_1(t,s)D(s,x)\,ds + \sum_{k=1}^m \Bigg[- G_1(t,t_k)\left(L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds + \gamma_k\right) \\ + G_2(t,t_k)\left(-L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} x'(s)ds + \lambda_k\right)\Bigg] \end{aligned} \tag{3.3}$$

where $D(t,x) = -Wx(\theta(t)) - N(Kx)(t) - L(Sx)(t) - g(t)$,

$$G_1(t,s) = [2\sqrt{M}(e^{\sqrt{M}T} - 1)]^{-1} \begin{cases} e^{\sqrt{M}(T-t+s)} + e^{\sqrt{M}(t-s)}, & 0 \le s < t \le T, \\ e^{\sqrt{M}(T+t-s)} + e^{\sqrt{M}(s-t)}, & 0 \le s \le t \le T, \end{cases}$$

$$G_2(t,s) = [2(e^{\sqrt{M}T} - 1)]^{-1} \begin{cases} e^{\sqrt{M}(T-t+s)} - e^{\sqrt{M}(t-s)}, & 0 \le s < t \le T, \\ -e^{\sqrt{M}(T+t-s)} + e^{\sqrt{M}(s-t)}, & 0 \le s \le t \le T. \end{cases}$$

This proof is similar to proof of Lemma 2.1 in [2], here we omit it.

Theorem 3.1. *(Banach's fixed point theorem)[21] Let M be a closed nonempty subset of the Banach space X and $A : M \to M$ is a contraction mapping, i.e.,*

$$\|Au - Av\| \le k\|u - v\|,$$

for all $u, v \in M$, and fixed k, $0 \le k < 1$. Then the operator A has a unique fixed point $u^* \in M$.

Lemma 3.2. *Let constants $M > 0$, $W, L, N \ge 0$, $L_k \ge 0$, $L_k^* \ge 0$, $0 \le \varepsilon_k < \delta_k \le t_k - t_{k-1}$, $0 \le \sigma_k < \tau_k \le t_k - t_{k-1}$, $k = 1, 2, \ldots, m$. If*

$$\psi =: \frac{1 + e^{\sqrt{M}T}}{2\sqrt{M}\left(e^{\sqrt{M}T} - 1\right)} \left[\int_0^T \left(W + N \int_0^s k(s,r)dr + L \int_0^T h(s,r)dr \right) ds + \sum_{k=1}^m L_k(\tau_k - \sigma_k) \right] + \frac{1}{2} \sum_{k=1}^m L_k^*(\delta_k - \varepsilon_k) < 1, \tag{3.4}$$

and

$$\mu =: \frac{1}{2} \left[\int_0^T \left(W + N \int_0^s k(s,r)dr + L \int_0^T h(s,r)dr \right) ds + \sum_{k=1}^m L_k(\tau_k - \sigma_k) \right] + \frac{\sqrt{M}\left(1 + e^{\sqrt{M}T}\right)}{2\left(e^{\sqrt{M}T} - 1\right)} \sum_{k=1}^m L_k^*(\delta_k - \varepsilon_k) < 1, \tag{3.5}$$

then PBVP (3.2) has a unique solution x in E.

Proof. For any $x \in E$, we define an operator A by

$$(Ax)(t) = \int_0^T G_1(t,s)D(s,x)\,ds + \sum_{k=1}^m \left[-G_1(t,t_k)\left(L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds + \gamma_k \right) + G_2(t,t_k)\left(-L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} x'(s)ds + \lambda_k \right) \right],$$

where G_1, G_2 are given by Lemma 3.1. Then, $Ax \in PC^1(J, R)$, and

$$(Ax)'(t) = -\int_0^T G_2(t,s)D(s,x)\,ds + \sum_{k=1}^m \left[G_2(t,t_k)\left(L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds + \gamma_k \right) - MG_1(t,t_k)\left(-L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} x'(s)ds + \lambda_k \right) \right].$$

By computing directly, we have

$$\max_{(t,s)\in J^2} \{G_1(t,s)\} = \frac{1 + e^{\sqrt{M}T}}{2\sqrt{M}\left(e^{\sqrt{M}T} - 1\right)}$$

and

$$\max_{(t,s)\in J^2} \{G_2(t,s)\} = \frac{1}{2}.$$

For $x, y \in PC^1(J,R)$, we have

$$\begin{aligned}
\|Ax - Ay\|_{PC} &= \sup_{t\in J} |(Ax)(t) - (Ay)(t)| \\
&= \sup_{t\in J} \bigg| \int_0^T G_1(t,s)\Big[W(x(\theta(s)) - y(\theta(s))) \\
&\quad + N\int_0^s k(s,r)(x(r)-y(r))dr + L\int_0^T h(s,r)(x(r)-y(r))dr\Big] ds \\
&\quad + \sum_{k=1}^m \Big[-G_1(t,t_k)\Big(L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} (x(s)-y(s))ds\Big) \\
&\quad + G_2(t,t_k)\Big(-L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} (x'(s)-y'(s))ds\Big)\Big]\bigg| \\
&\le \frac{1+e^{\sqrt{M}T}}{2\sqrt{M}(e^{\sqrt{M}T}-1)}\Big[\int_0^T \Big(W + N\int_0^s k(s,r)dr + L\int_0^T h(s,r)dr\Big) ds \\
&\quad + \sum_{k=1}^m L_k(\tau_k-\sigma_k)\Big]\|x-y\|_{PC} + \frac{1}{2}\|x'-y'\|_{PC}\sum_{k=1}^m L_k^*(\delta_k-\varepsilon_k) \\
&\le \bigg\{\frac{1+e^{\sqrt{M}T}}{2\sqrt{M}(e^{\sqrt{M}T}-1)}\Big[\int_0^T \Big(W + N\int_0^s k(s,r)dr + L\int_0^T h(s,r)dr\Big) ds \\
&\quad + \sum_{k=1}^m L_k(\tau_k-\sigma_k)\Big] + \frac{1}{2}\sum_{k=1}^m L_k^*(\delta_k-\varepsilon_k)\bigg\}\|x-y\|_{PC^1} \\
&\le \psi\|x-y\|_{PC^1},
\end{aligned}$$

and

$$\begin{aligned}
\|(Ax)' - (Ay)'\|_{PC} &= \sup_{t\in J} \bigg| -\int_0^T G_2(t,s)\Big[W(x(\theta(s)) - y(\theta(s))) \\
&\quad + N\int_0^s k(s,r)(x(r)-y(r))dr + L\int_0^T h(s,r)(x(r)-y(r))dr\Big] ds \\
&\quad + \sum_{k=1}^m \Big[G_2(t,t_k)\Big(L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} (x(s)-y(s))ds\Big) \\
&\quad - MG_1(t,t_k)\Big(-L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} (x'(s)-y'(s))ds\Big)\Big]\bigg| \\
&\le \bigg\{\frac{1}{2}\Big[\int_0^T \Big(W + N\int_0^s k(s,r)dr + L\int_0^T h(s,r)dr\Big) ds \\
&\quad + \sum_{k=1}^m L_k(\tau_k-\sigma_k)\Big] + \frac{\sqrt{M}\left(1+e^{\sqrt{M}T}\right)}{2(e^{\sqrt{M}T}-1)}\sum_{k=1}^m L_k^*(\delta_k-\varepsilon_k)\bigg\}\|x-y\|_{PC^1} \\
&\le \mu\|x-y\|_{PC^1}.
\end{aligned} \tag{3.6}$$

This implies that

$$\|Ax - Ay\|_{PC^1} \le \max\{\psi, \mu\}\|x - y\|_{PC^1}.$$

By Theorem 3.1, A has a unique fixed point $x \in PC^1(J, R)$. From Lemma 3.1, x is also the unique solution of PBVP (3.2). This completes the proof. □

Now, we are in the position to establish existence criteria for solutions of the PVBP (3.1) by the method of lower and upper solutions and monotone iterative technique. For $\alpha_0, \beta_0 \in E$, we write $\alpha_0 \le \beta_0$ if $\alpha_0(t) \le \beta_0(t)$ for all $t \in J$. In such a case, we denote $[\alpha_0, \beta_0] = \{x \in E \ : \alpha_0(t) \le x(t) \le \beta_0(t),\, t \in J\}$.

Theorem 3.2. *Suppose that the following conditions hold:*
(H_1) α_0 *and* β_0 *are lower and upper solutions for PBVP* (3.1), *respectively, such that* $\alpha_0 \le \beta_0$.
(H_2) *The function* f *satisfies*

$$f(t, w_2, x_2, y_2, z_2) - f(t, w_1, x_1, y_1, z_1) \le M(w_2 - w_1) + W(x_2 - x_1) + N(y_2 - y_1) + L(z_2 - z_1),$$

for all $t \in J$, $\alpha_0(t) \le w_1 \le w_2 \le \beta_0(t)$, $\alpha_0(\theta(t)) \le x_1 \le x_2 \le \beta_0(\theta(t))$, $(K\alpha_0)(t) \le y_1 \le y_2 \le (K\beta_0)(t)$, $(S\alpha_0)(t) \le z_1 \le z_2 \le (S\beta_0)(t)$.
(H_3) *Constants* $M > 0$, $W, N, L \ge 0$, $0 \le L_k < 1/(\tau_k - \sigma_k)$, $0 \le L_k^* < 1/(\delta_k - \varepsilon_k)$, $0 \le \varepsilon_k < \delta_k \le t_k - t_{k-1}$, $0 \le \sigma_k < \tau_k \le t_k - t_{k-1}$, $k = 1, 2, \ldots, m$, *and they satisfy* (2.22), (3.4) *and* (3.5).
(H_4) *The functions* I_k, I_k^* *satisfy*

$$I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} x(s)ds\right) - I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} y(s)ds\right) \le L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} (x(s) - y(s))ds,$$

$$I_k^*\left(\int_{t_k-\delta_k}^{t_k-\varepsilon_k} x'(s)ds\right) - I_k^*\left(\int_{t_k-\delta_k}^{t_k-\varepsilon_k} y'(s)ds\right) \le -L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} (x'(s) - y'(s))ds,$$

where $\alpha_0(t) \le y(t) \le x(t) \le \beta_0(t)$, $0 \le \sigma_k < \tau_k \le t_k - t_{k-1}$, $0 \le \varepsilon_k < \delta_k \le t_k - t_{k-1}$, $k = 1, 2, \ldots, m$.
Then there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\} \subset E$ *which converge in* E *to the extreme solutions of PBVP* (3.1) *in* $[\alpha_0, \beta_0]$, *respectively.*

Proof. Firstly, we consider the following sequences $\{\alpha_i\}$, $\{\beta_i\}$, $i = 1, 2, \ldots$, such that

$$\begin{aligned}
\alpha_n''(t) &- M\alpha_n(t) - W\alpha_n(\theta(t)) - N(K\alpha_n)(t) - L(S\alpha_n)(t) \\
&= f(t, \alpha_{n-1}(t), \alpha_{n-1}(\theta(t)), (K\alpha_{n-1})(t), (S\alpha_{n-1})(t)) \\
&\quad - M\alpha_{n-1}(t) - W\alpha_{n-1}(\theta(t)) - N(S\alpha_{n-1})(t) - L(S\alpha_{n-1})(t), \quad t \in J^-, \\
\Delta\alpha_n(t_k) &= L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_n(s)ds + I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_{n-1}(s)ds\right) - L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \alpha_{n-1}(s)ds,
\end{aligned}$$

$$\begin{aligned}
&\quad k = 1,2,\ldots,m,\\
\Delta\alpha_n'(t_k) &= -L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha_n'(s)ds + I_k^*\left(\int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha_{n-1}'(s)ds\right) + L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \alpha_{n-1}'(s)ds,\\
&\quad k = 1,2,\ldots,m,\\
\alpha_n(0) &= \alpha_n(T), \qquad \alpha_n'(0) = \alpha_n'(T),
\end{aligned}$$

and

$$\begin{aligned}
\beta_n''(t) &- M\beta_n(t) - W\beta_n(\theta(t)) - N(K\beta_n)(t) - L(S\beta_n)(t)\\
&= f(t,\beta_{n-1}(t),\beta_{n-1}(\theta(t)),(K\beta_{n-1})(t),(S\beta_{n-1})(t))\\
&\quad - M\beta_{n-1}(t) - W\beta_{n-1}(\theta(t)) - N(S\beta_{n-1})(t) - L(S\beta_{n-1})(t), \quad t \in J^-,\\
\Delta\beta_n(t_k) &= L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \beta_n(s)ds + I_k\left(\int_{t_k-\tau_k}^{t_k-\sigma_k} \beta_{n-1}(s)ds\right) - L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \beta_{n-1}(s)ds,\\
&\quad k = 1,2,\ldots,m,\\
\Delta\beta_n'(t_k) &= -L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \beta_n'(s)ds + I_k^*\left(\int_{t_k-\delta_k}^{t_k-\varepsilon_k} \beta_{n-1}'(s)ds\right) + L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} \beta_{n-1}'(s)ds,\\
&\quad k = 1,2,\ldots,m,\\
\beta_n(0) &= \beta_n(T), \qquad \beta_n'(0) = \beta_n'(T).
\end{aligned}$$

Moreover, by Lemma 3.2, we have α_1 and β_1 are well defined.
Next, we shall show that

$$\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t), \qquad t \in J. \tag{3.7}$$

Let $p(t) = \alpha_0(t) - \alpha_1(t)$. By definition 3.1 of a lower solution of PBVP (3.1), we have

$$\begin{aligned}
p''(t) &= \alpha_0''(t) - \alpha_1''(t)\\
&\ge Mp(t) + Wp(\theta(t)) + N(Kp)(t) + L(Sp)(t),
\end{aligned}$$

and

$$\begin{aligned}
\Delta p(t_k) &= \Delta\alpha_0(t_k) - \Delta\alpha_1(t_k)\\
&\ge L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} p(s)ds, \ k = 1,2,\ldots,m,
\end{aligned}$$

and

$$\begin{aligned}
\Delta p'(t_k) &= \Delta\alpha_0'(t_k) - \Delta\alpha_1'(t_k)\\
&\ge -L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} p'(s)ds, \ k = 1,2,\ldots,m,
\end{aligned}$$

and

$$\begin{aligned} p(0) &= \alpha_0(0) - \alpha_1(0) \geq \alpha_0(T) - \alpha_1(T) = p(T), \\ p'(0) &= \alpha_0'(0) - \alpha_1'(0) \geq \alpha_0'(T) - \alpha_1'(T) = p'(T). \end{aligned}$$

Then by Theorem 2.2, $p(t) \leq 0$, which implies that $\alpha_0(t) \leq \alpha_1(t), t \in T$. In similar way, we can show that $\beta_0(t) \geq \beta_1(t)$, $t \in J$.
Further, we will show that $\alpha_1(t) \leq \beta_1(t)$, $t \in J$. Let $p(t) = \alpha_1(t) - \beta_1(t)$ and by (H_2), we obtain

$$\begin{aligned} &p''(t) - Mp(t) - Wp\left(\theta(t)\right) - N(Kp)(t) - L(Sp)(t) \\ &\quad = f(t, \alpha_0(t), \alpha_0\left(\theta(t)\right), (K\alpha_0)(t), (S\alpha_0)(t)) - f(t, \beta_0(t), \beta_0\left(\theta(t)\right), (K\beta_0)(t), (S\beta_0)(t)) \\ &\qquad - M\left(\alpha_0(t) - \beta_0(t)\right) - W\left[\alpha_0\left(\theta(t)\right) - \beta_0\left(\theta(t)\right)\right] \\ &\qquad - N\left[(K\alpha_0)(t) - (K\beta_0)(t)\right] - L\left[(S\alpha_0)(t) - (S\beta_0)(t)\right] \\ &\quad \geq 0. \end{aligned}$$

From (H_4), we get

$$\begin{aligned} &\Delta p(t_k) \geq L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} p(s)ds, \quad k = 1, 2, \ldots, m, \\ &\Delta p'(t_k) \geq -L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} p'(s)ds, \quad k = 1, 2, \ldots, m, \\ &p(0) = p(T), \qquad p'(0) = p'(T). \end{aligned}$$

Applying Theorem 2.2, we get $p(t) \leq 0$, which implies $\alpha_1(t) \leq \beta_1(t)$.
Using mathematical induction, we can show that

$$\alpha_0(t) \leq \alpha_1(t) \leq \cdots \leq \alpha_n(t) \leq \beta_n(t) \leq \cdots \leq \beta_1(t) \leq \beta_0(t), \qquad t \in J,$$

for $n = 1, 2, \ldots$. Employing standard argument, we have

$$\lim_{n\to\infty} \alpha_n(t) = \alpha(t), \qquad \lim_{n\to\infty} \beta_n(t) = \beta(t),$$

uniformly on $t \in J$ and the limit functions $\alpha(t)$, $\beta(t)$ satisfy problem (3.1). Moreover $\alpha(t), \beta(t) \in [\alpha_0(t), \beta_0(t)]$.

Finally, we will show that α is the minimal solution and β is the maximal solution of PBVP (3.1), respectively. To prove it we assume that x is any solution of problem PBVP (3.1) such that $x \in [\alpha_0, \beta_0]$. Let $\alpha_{n-1}(t) \leq x(t) \leq \beta_{n-1}(t)$, $t \in J$, for some positive integer n. Put $p(t) = \alpha_n(t) - x(t)$. Then

$$p''(t) - Mp(t) - Wp\left(\theta(t)\right) - N(Kp)(t) - L(Sp)(t)$$

$$
\begin{aligned}
&= f(t,\alpha_{n-1}(t),\alpha_{n-1}(\theta(t)),(K\alpha_{n-1})(t),(S\alpha_{n-1})(t)) \\
&\quad - f(t,x(t),x(\theta(t)),(Kx)(t),(Sx)(t)) \\
&\quad - M(\alpha_{n-1}(t)-x(t)) - W[\alpha_{n-1}(\theta(t)) - x(\theta(t))] \\
&\quad - N[(K\alpha_{n-1})(t)-(Kx)(t)] - L[(S\alpha_{n-1})(t)-(Sx)(t)] \\
&\geq 0.
\end{aligned}
$$

From (H_4), we have

$$
\begin{aligned}
&\Delta p(t_k) \geq L_k \int_{t_k-\tau_k}^{t_k-\sigma_k} p(s)ds, \quad k=1,2,\ldots,m, \\
&\Delta p'(t_k) \geq -L_k^* \int_{t_k-\delta_k}^{t_k-\varepsilon_k} p'(s)ds, \quad k=1,2,\ldots,m, \\
&p(0)=p(T), \qquad p'(0)=p'(T).
\end{aligned}
$$

Using Theorem 2.2, we have for all $t \in J$, $p(t) \leq 0$, i.e., $\alpha_n(t) \leq x(t)$. Similarly, we can prove that $x(t) \leq \beta_n(t)$, $t \in J$. Therefore, $\alpha_n(t) \leq x(t) \leq \beta_n(t)$, for all $t \in J$, which implies $\alpha(t) \leq x(t) \leq \beta(t)$. The proof is complete. □

Now, in order to illustrate our results, we consider an example.

Example 3.1. Consider the following impulsive periodic boundary value problem

$$
\begin{cases}
u''(t) = \dfrac{1}{12}\sin t^3(u(t)-3t) + \dfrac{t}{21}u\left(\dfrac{t}{2}\right) + \dfrac{\sin t}{40}\left[\displaystyle\int_0^t t^2s^2u(s)ds\right]^2 \\
\qquad\qquad + \dfrac{\cos t}{32}\left[\displaystyle\int_0^1 t^2s^4u(s)ds\right]^2, \quad t \in J=[0,1], t \neq \dfrac{1}{2}, \\
\Delta u\left(\dfrac{1}{2}\right) = \dfrac{1}{15}\displaystyle\int_{\frac{1}{6}}^{\frac{5}{14}} u(s)ds, \quad k=1, \\
\Delta u'\left(\dfrac{1}{2}\right) = -\dfrac{13}{5}\displaystyle\int_{\frac{1}{6}}^{\frac{1}{4}} u'(s)ds, \quad k=1, \\
u(0)=u(1), \qquad u'(0)=u'(1),
\end{cases}
\tag{3.8}
$$

where $k(t,s)=t^2s^2$, $h(t,s)=t^2s^4$, $m=1$, $t_1=1/2$, $\tau_1=1/3$, $\sigma_1=1/7$, $\delta_1=1/3$, $\varepsilon_1=1/4$, $T=1$. Obviously, $\alpha_0=0$, $\beta_0=4$ are lower and upper solutions for (3.8), respectively, and $\alpha_0 \leq \beta_0$.

Let

$$
f(t,w_1,x_1,y_1,z_1) = \frac{1}{12}\sin t^3(w_1-3t) + \frac{t}{21}x_1 + \frac{\sin t}{40}y_1^2 + \frac{\cos t}{32}z_1^2,
$$

we have

$$
f(t,w_2,x_2,y_2,z_2)-f(t,w_1,x_1,y_1,z_1) \leq \frac{1}{12}(w_2-w_1)+\frac{1}{21}(x_2-x_1)+\frac{1}{15}(y_2-y_1)+\frac{1}{20}(z_2-z_1),
$$

where $\alpha(t) \le w_1 \le w_2 \le \beta(t)$, $\alpha(\theta(t)) \le x_1 \le x_2 \le \beta(\theta(t))$, $(K\alpha)(t) \le y_1 \le y_2 \le (K\beta)(t)$, $(S\alpha)(t) \le z_1 \le z_2 \le (S\beta)(t)$, $t \in J$. It is easy to see that

$$I_1\left(\int_{\frac{1}{6}}^{\frac{5}{14}} x(s)ds\right) - I_1\left(\int_{\frac{1}{6}}^{\frac{5}{14}} y(s)ds\right) = \frac{1}{15}\int_{\frac{1}{6}}^{\frac{5}{14}} x(s) - y(s)ds,$$

and

$$I_1^*\left(\int_{\frac{1}{6}}^{\frac{1}{4}} x'(s)ds\right) - I_1^*\left(\int_{\frac{1}{6}}^{\frac{1}{4}} y'(s)ds\right) = -\frac{13}{5}\int_{\frac{1}{6}}^{\frac{1}{4}} x'(s) - y'(s)ds,$$

whenever $\alpha_0(t) \le y(t) \le x(t) \le \beta_0(t)$.
Taking $M = 1/12$, $W = 1/21$, $N = 1/15$, $L = 1/20$, $L_1 = 1/15$, $L_1^* = 13/5$, it follows that

$$C_1 = 1 + \frac{1}{15}\left(\frac{1}{3} - \frac{1}{7}\right) = \frac{319}{315},$$

$$D_1 = C_1\int_0^{\frac{1}{6}} U(s)ds + \int_{\frac{1}{6}}^{\frac{5}{14}}\left[1 + \frac{1}{15}\left(\frac{1}{2} - \frac{1}{7} - s\right)\right]U(s)ds + \int_{\frac{5}{14}}^{\frac{1}{2}} U(s)ds,$$

$$\hat{L} = \frac{1}{15}\left(\frac{1}{3} - \frac{1}{7}\right) = \frac{4}{315},$$

$$U(t) = \frac{\prod_{0<t_k<t} A_k^*}{1 - \prod_{k=1}^m A_k^*}\left[\sum_{k=1}^m \prod_{j=k+1}^m A_j^* B_k^* + \int_{\frac{1}{2}}^1 r(s)ds\right] + \left[\sum_{0<t_k<t}\prod_{t_k<t_j<t} A_j^* B_k^* + \int_{t_l}^t r(s)ds\right],$$

$$A_1^* = 1 - \frac{13}{5}\left(\frac{1}{3} - \frac{1}{4}\right),$$

$$B_1^* = A_1^*\int_0^{\frac{1}{6}} r(s)ds + \int_{\frac{1}{6}}^{\frac{1}{4}}\left[1 - \frac{3}{15}\left(\frac{1}{2} - \frac{1}{4} - s\right)\right]r(s)ds + \int_{\frac{1}{4}}^{\frac{1}{2}} r(s)ds,$$

$$r(t) = \frac{1}{12} + \frac{1}{21} + \frac{1}{15}\int_0^t t^2 s^2 ds + \frac{1}{20}\int_0^1 t^2 s^4 ds.$$

Through a simple calculation we can get

$$\prod_{k=1}^m C_k\left[\hat{L} + \sum_{k=1}^m D_k + \int_0^T U(s)ds\right] = 0.9717485093 \le 1.$$

Furthermore, we have

$$\begin{aligned}\psi =: &\frac{1 + e^{\sqrt{\frac{1}{12}}}}{2\sqrt{\frac{1}{12}}\left(e^{\sqrt{\frac{1}{12}}} - 1\right)}\left[\int_0^1\left(\frac{1}{21} + \frac{1}{15}\int_0^s s^2 r^2 dr + \frac{1}{20}\int_0^1 s^2 r^4 dr\right)ds\right.\\ &\left. + \frac{1}{15}\left(\frac{1}{3} - \frac{1}{7}\right)\right] + \frac{13}{10}\left(\frac{1}{3} - \frac{1}{4}\right),\\ =&0.922192396 < 1,\end{aligned}$$

$$\begin{aligned}\mu =&:\frac{1}{2}\left[\int_0^1\left(\frac{1}{21}+\frac{1}{15}\int_0^s s^2r^2dr+\frac{1}{20}\int_0^1 s^2r^4dr\right)ds+\frac{1}{15}\left(\frac{1}{3}-\frac{1}{7}\right)\right]\\&+\frac{\sqrt{\frac{1}{12}}\left(1+e^{\sqrt{\frac{1}{12}}}\right)}{2\left(e^{\sqrt{\frac{1}{12}}}-1\right)}\frac{13}{5}\left(\frac{1}{3}-\frac{1}{4}\right),\\=&0.2566612741<1.\end{aligned}$$

Therefore, PBVP (3.8) satisfies all conditions of Theorem 3.2. Thus, PBVP (3.8) has minimal and maximal solutions in the segment $[\alpha_0, \beta_0]$.

4 Conclusion

This paper is devoted to establish two new maximum principles, i.e., Theorem 2.1-2.2. Theorem 2.2 can be used as a tool for proving the existence of extreme solutions for the periodic boundary value problem of impulsive functional integro-differential equation with integral jump conditions (3.1) as mentioned in Theorem 3.2.

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