WEIGHTED ERGODIC THEOREM FOR CONTRACTIONS OF ORLICZ-KANTOROVICH LATTICE $L_M(\widehat{\nabla},\widehat{\mu})$

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Abstract. In the present paper we prove a Besicovich weighted ergodic theorem for positive contractions acting on Orlich-Kantorovich space. Our main tool is the use of methods of measurable bundles of Banach-Kantorovich lattices.

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1. Introduction

It is known that a pioneering work of von Neumman [27] stimulated the development of the theory of Banach bundles (see [12]). It was proved such a theory has vast applications in analysis. Moreover, it was proved that such a theory is well connected with vector-valued Banach spaces, which has several application (see for example, [18]). We recall that in the theory of Banach bundles L_0-valued Banach spaces are considered, and such spaces are called *Banach–Kantorovich spaces.* In [11, 12, 17]) the theory of Banach–Kantorovich spaces were developed. In this theory, analogues of many well known functional spaces have been defined and studied. For example, in [7] Banach-Kantorovich lattice $L_p(\widehat{\nabla},\widehat{\mu})$ is represented as a measurable bundle of classical L_p –lattices. In [25, 26] an analogue of the Orlicz spaces has been considered. Naturally, these functional Kantorovich spaces should have many properties similar to the classical ones, constructed by the real valued measures.

We note that in [2] (see also [14]) weighted ergodic theorems an analogue for Dunford-Schwarz operators acting on L_p-spaces were proved. In [22] some properties of the convergence of Banach-valued martingales were described and their connections with the geometrical properties of Banach spaces were established too. Therefore, with the development of the theory Banach–Kantorovich spaces there naturally arises the necessity to study some ergodic type theorems for positive contractions and martingales defined on such spaces.

To investigate the properties of Banach–Kantorovich spaces is naturally to use measurable bundles of such spaces. Since, one has a sufficiently well explored theory of measurable bundles of Banach lattices [11], it is an effective tool which gives well opportunity to obtain various properties of Banach–Kantorovich spaces [5],[6]. It is worth mentioning that using this way, in [3, 10] weighted ergodic theorems for positive contractions of Banach-Kantorovich lattices $L_p(\widehat{\nabla},\widehat{\mu})$, have been established. In [6] the convergence of martingales on such lattices is proved. Further, in [9] the "zero-two" law for positive contractions of Banach-Kantorovich lattice $L_p(\widehat{\nabla},\widehat{\mu})$ has been proved. In [23] an individual ergodic theorem has been proved for positive contractions of Orlicz-Kantorovich lattices.

In the present paper we are going to prove a Besicovich weighted ergodic theorem for positive contractions acting on Orlich-Kantorovich space. Our results extend and improve the results of [10, 23]. To prove the main result of this paper we are going to use measurable bundles of Banach–Kantorovich lattices. We note that there are many papers devoted to the study of limit theorems in classical Orlich spaces (see, for example [1, 19, 20]). We point out that another effective tool to study of Banach-Kantorovich spaces are the methods of Boolean-valued analysis and measurable bundles (see [17]).

2. Preliminaries

In this section we recall necessary definitions and results concerning Banach-Kantorovich lattices.

Let $(\Omega, \Sigma, \lambda)$ be a measurable space with a finite measure λ, and $L_0(\Omega)$ be the algebra of all measurable functions on Ω (here the functions equal a.e. are identified) and let $\nabla(\Omega)$ be the Boolean algebra of all idempotents in $L_0(\Omega)$. By ∇ we denote an arbitrary complete Boolean subalgebra of $\nabla(\Omega)$. By $\mathcal{L}^\infty(\Omega)$ we denote the set of all measurable essentially bounded functions on Ω, and $L^\infty(\Omega)$ denotes an algebra of equivalence classes of essentially bounded measurable functions.

Let E be a linear space over the real field $\mathbb{R}$. By $\|\cdot\|$ we denote an $L_0(\Omega)$-valued norm on E. Then the pair $(E, \|\cdot\|)$ is called a *lattice-normed space (LNS) over* $L_0(\Omega)$. An LNS E is said to be *d-decomposable* if for every $x \in E$ and the decomposition $\|x\| = f + g$ with f and g disjoint positive elements in $L_0(\Omega)$ there exist $y, z \in E$ such that $x = y + z$ with $\|y\| = f$, $\|z\| = g$.

Suppose that $(E, \|\cdot\|)$ is an LNS over $L_0(\Omega)$. A net $\{x_\alpha\}$ of elements of E is said to be (bo)*-converging* to $x \in E$ (in this case we write $x = (bo)\text{-}\lim x_\alpha$), if the net $\{\|x_\alpha - x\|\}$ (o)-converges to zero (here (o)-convergence means the order convergence) in $L_0(\Omega)$ (written as $(o)\text{-}\lim \|x_\alpha - x\| = 0$). A net $\{x_\alpha\}_{\alpha\in A}$ is called (bo)*-fundamental* if $(x_\alpha - x_\beta)_{(\alpha,\beta)\in A\times A}$ (bo)-converges to zero.

An LNS in which every (bo)-fundamental net (bo)-converges is called (bo)*-complete*. A *Banach-Kantorovich space (BKS) over* $L_0(\Omega)$ is a (bo)-complete d-decomposable LNS over $L_0(\Omega)$. It is well known [16, 17] that every BKS E over $L_0(\Omega)$ admits an $L_0(\Omega)$-module structure such that $\|fx\| = |f| \cdot \|x\|$ for every $x \in E$, $f \in L_0(\Omega)$, where $|f|$ is the modulus of a function $f \in L_0(\Omega)$. A BKS $(\mathcal{U}, \|\cdot\|)$ is called a *Banach-Kantorovich lattice* if $\mathcal{U}$ is a vector lattice and the norm $\|\cdot\|$ is monotone, i.e. $|u_1| \le |u_2|$ implies $\|u_1\| \le \|u_2\|$. It is known [16] that the cone $\mathcal{U}_+$ of positive elements is (bo)-closed.

Let $(\Omega, \Sigma, \lambda)$ be the same as above and X be a mapping assigning a real Banach space $(X(\omega), \|\cdot\|_{X(\omega)})$ to each point $\omega \in \Omega$, where $X(\omega) \ne \{0\}$ for all $\omega \in \Omega$. A *section* of X is a function u defined λ-almost everywhere in Ω that takes values $u(\omega) \in X(\omega)$ for all ω in the domain $dom(u)$ of u. Let L be a set of sections. The pair (X, L) is called a *measurable Banach bundle over* Ω if

(1) $\alpha_1 u_1 + \alpha_2 u_2 \in L$ for every $\alpha_1, \alpha_2 \in \mathbb{R}$ and $u_1, u_2 \in L$, where $\alpha_1 u_1 + \alpha_2 u_2 : \omega \in dom(u_1) \cap dom(u_2) \to \alpha_1 u_1(\omega) + \alpha_2 u_2(\omega)$;
(2) the function $\|u\| : \omega \in dom(u) \to \|u(\omega)\|_{X(\omega)}$ is measurable for every $u \in L$;
(3) the set $\{u(\omega) : u \in L, \omega \in dom(u)\}$ is dense in $X(\omega)$ for every $\omega \in \Omega$.

A measurable Banach bundle (X, L) is called *measurable bundle of Banach lattices (MBBL)* if $(X(\omega), \|\cdot\|_{X(\omega)})$ is a Banach lattice for all $\omega \in \Omega$ and for every $u_1, u_2 \in L$ one has $u_1 \vee u_2 \in L$, where $u_1 \vee u_2$: $\omega \in \text{dom } (u_1) \cap \text{dom } (u_2) \to u_1(\omega) \vee u_2(\omega)$.

A section s is called *step-section* if it has a form

$$s(\omega) = \sum_{i=1}^{n} \chi_{A_i}(\omega) u_i(\omega),$$

for some $u_i \in L$, $A_i \in \Sigma$, $A_i \cap A_j = \emptyset$, $i \neq j$, $i, j = 1, \cdots, n$, $n \in \mathbb{N}$, where χ_A is the indicator of a set A. A section u is called *measurable* there exists a sequence of step-functions $\{s_n\}$ such that $s_n(\omega) \to u(\omega)$ λ-a.e.

By $M(\Omega, X)$ we denote the set all measurable sections, and by $L_0(\Omega, X)$ the factor space of $M(\Omega, X)$ with respect to the equivalence relation of the equality almost everywhere. Clearly, $L_0(\Omega, X)$ is an $L_0(\Omega)$-module. The equivalence class of an element $u \in M(\Omega, X)$ is denoted by $\hat{u}$. The norm of $\hat{u} \in L_0(\Omega, X)$ is defined as a class of equivalence in $L_0(\Omega)$ containing the function $\|u(\omega)\|_{X(\omega)}$, namely $\|\hat{u}\| = (\widehat{\|u(\omega)\|_{X(\omega)}})$. In [11] it was proved that $L_0(\Omega, X)$ is a BKS over $L_0(\Omega)$. Furthermore, for every BKS E over $L_0(\Omega)$ there exists a measurable Banach bundle (X, L) over Ω such that E is isomorphic to $L_0(\Omega, X)$.

Let X be a MBBL. We put $\hat{u} \leq \hat{v}$ if $u(\omega) \leq v(\omega)$ a.e. One can see that the relation $\hat{u} \leq \hat{v}$ is a partial order in $L_0(\Omega, X)$. If X is a MBBL, then $L_0(\Omega, X)$ is a Banach-Kantorovich lattice [5, 7].

A mapping $\mu : \nabla \to L_0(\Omega)$ is called an $L_0(\Omega)$*-valued measure* if the following conditions are satisfied:

1) $\mu(e) \geq 0$ for all $e \in \nabla$;
2) if $e \wedge g = 0, e, g \in \nabla$, then $\mu(e \vee g) = \mu(e) + \mu(g)$;
3) if $e_n \downarrow 0$, $e_n \in \nabla$, $n \in \mathbb{N}$, then $\mu(e_n) \downarrow 0$.

An $L_0(\Omega)$-valued measure μ is called *strictly positive* if $\mu(e) = 0$, $e \in \nabla$ implies $e = 0$.

Let a Boolean algebra $\nabla(\Omega)$ of all idempotents of $L_0(\Omega)$ be a regular subalgebra of ∇.

In the sequel, we will consider a strictly positive $L_0(\Omega)$-valued measure μ with the following property $\mu(ge) = g\mu(e)$ for all $e \in \nabla$ and $g \in \nabla(\Omega)$.

Let ∇_ω, $\omega \in \Omega$ be complete Boolean algebras with strictly positive real-valued measures μ_ω. Put $\rho_\omega(e, g) = \mu_\omega(e \bigtriangleup g)$, $e, g \in \nabla_\omega$. Then $(\nabla_\omega, \mu_\omega)$ is a complete metric space. Let us consider a mapping ∇, which assigns to each $\omega \in \Omega$ a Boolean algebra ∇_ω. Such a mapping is called a section.

Assume that L is a nonempty set of sections ∇. A pair (∇, L) is called *a measurable bundle of Boolean algebras over* Ω if one has

1) (∇, L) is a measurable bundle of metric spaces (see [7]);
2) if $e \in L$, then $e^\perp \in L$, where $e^\perp : \omega \in \text{dom}\,(e) \to e^\perp(\omega)$;
3) if $e_1, e_2 \in L$, then $e_1 \vee e_2 \in L$, where $e_1 \vee e_2 : \omega \in \text{dom}\,(e_1) \cap \text{dom}\,(e_2) \to e_1(\omega) \vee e_2(\omega)$.

Let $M(\Omega, \nabla)$ be the set of all measurable sections, and $\hat{\nabla}$ be the factorization of $M(\Omega, \nabla)$ with respect to equivalence relation the equality a.e. Let us define a mapping $\hat{\mu} : \hat{\nabla} \to L_0(\Omega)$ by $\hat{\mu}(\hat{e}) = \hat{f}$, where $\hat{f}$ is the class containing the function $f(\omega) = \mu_\omega(e(\omega))$. It is clear that the mapping $\hat{\mu}$ is well-defined. It is known that $(\hat{\nabla}, \hat{\mu})$ is a complete Boolean algebra with a strictly positive $L_0(\Omega)$-valued measure $\hat{\mu}$. Note that a Boolean algebra $\nabla(\Omega)$ of all idempotents of $L_0(\Omega)$ is identified with a regular subalgebra of $\hat{\nabla}$, and one has $\widehat{\mu}(g\widehat{e}) = g\widehat{\mu}(\widehat{e})$ for all $g \in \nabla(\Omega)$ and $\widehat{e} \in \widehat{\nabla}$.

The reverse is also tree, namely one has the following

Theorem 2.1. [7] *Let $\tilde{\nabla}$ be a complete Boolean algebra, $\tilde{\mu}$ be a strictly positive $L_0(\Omega)$-valued measure on $\tilde{\nabla}$, and let $\nabla(\Omega)$ be a regular subalgebra of $\tilde{\nabla}$ and $\tilde{\mu}(g\tilde{e}) = g\tilde{\mu}(\tilde{e})$ for all $g \in \nabla(\Omega)$, $\tilde{e} \in \tilde{\nabla}$. Then there exists a measurable bundle of Boolean algebras (∇, L) such that $\hat{\nabla}$ is isometrically isomorphic to $\tilde{\nabla}$.*

By $L_0(\hat{\nabla},\hat{\mu})$ we denote an order complete vector lattice $C_\infty(Q(\hat{\nabla}))$, where $Q(\hat{\nabla})$ is the Stonian compact associated with complete Boolean algebra $\hat{\nabla}$. For $\widehat{f},\widehat{g}\in L_0(\hat{\nabla},\hat{\mu})$ we let $\widehat{\rho}(\widehat{f},\widehat{g})=\int\frac{|\hat{f}-\hat{g}|}{1+|\hat{f}-\hat{g}|}d\hat{\mu}$. Then it is know [7] that $\widehat{\rho}$ is an $L_0(\Omega)$-valued metric on $L_0(\hat{\nabla},\hat{\mu})$ and $(L_0(\hat{\nabla},\hat{\mu}),\widehat{\rho})$ is isometrically isomorphic to the measurable bundle of metric spaces $L_0(\nabla_\omega,\mu_\omega)$, where $\rho_\omega(a,b)=\int\frac{|a-b|}{1_\omega+|a-b|}d\mu_\omega$. In particularly, each element $\widehat{f}\in L_0(\hat{\nabla},\hat{\mu})$ can be identified with the measurable section $\{f(\omega)\}_{\omega\in\Omega}$, here $f(\omega)\in L_0(\nabla_\omega,\mu_\omega)$.

Following the well known scheme of the construction of L_p-spaces, a space $L_p(\hat{\nabla},\hat{\mu})$ can be defined by

$$L_p(\hat{\nabla},\hat{\mu})=\left\{\hat{f}\in L_0(\hat{\nabla}):\int|\hat{f}|^p d\hat{\mu}-\text{exist}\right\},\quad p\geq 1$$

where $\hat{\mu}$ is an $L_0(\Omega)$-valued measure on $\hat{\nabla}$.

It is known [16] that $L_p(\hat{\nabla},\hat{\mu})$ is a BKS over $L_0(\Omega)$ with respect to the $L_0(\Omega)$-valued norm $\|\hat{f}\|_{L_p(\hat{\nabla},\hat{\mu})}=\left(\int|\hat{f}|^p d\hat{\mu}\right)^{1/p}$. Moreover, $L_p(\hat{\nabla},\hat{\mu})$ is a Banach-Kantorovich lattice (see [17],[7]).

Let X be a mapping assigning an L_p-space constructed by a real-valued measure μ_ω, i.e. $L_p(\nabla_\omega,\mu_\omega)$ to each point $\omega\in\Omega$ and let

$$L=\left\{\sum_{i=1}^n\alpha_i e_i:\alpha_i\in\mathbb{R},\ \ e_i\in M(\Omega,\nabla),\ i=\overline{1,n},\ n\in\mathbb{N}\right\}$$

be the set of sections. In [7, 8] it has been established that the pair (X,L) is a measurable bundle of Banach lattices and $L_0(\Omega,X)$ is modulo ordered isomorphic to $L_p(\hat{\nabla},\hat{\mu})$.

Let $\mathcal{L}^\infty(\Omega,X)$=$\{u\in M(\Omega,X):\|u(\omega)\|_{X(\omega)}\in\mathcal{L}^\infty(\Omega)\}$.

Let us consider a linear subspace of $L^p(\hat{\nabla},\hat{\mu})$ defined by $L^\infty(\hat{\nabla},\hat{\mu})=\{\hat{f}\in L^p(\hat{\nabla},\hat{\mu}):\exists\lambda\in R,\lambda\geq 0,|\hat{f}|\leq\lambda\mathbf{1}\}$ with the numerical norm $\|\hat{f}\|_\infty=\text{vrai}\ \sup|\hat{f}|$.

Let ρ be a lifting on $L^\infty(\Omega)$ with values in $\mathcal{L}^\infty(\Omega)$ (see [11]).

In [7] it has been proven the existence of a linear mapping $\ell:L^\infty(\hat{\nabla},\hat{\mu})\to\mathcal{L}^\infty(\Omega,\mathcal{X})$ with the following properties: for any $\hat{f},\hat{g}\in L^\infty(\hat{\nabla},\hat{\mu})$ one has

1) $\ell(\hat{f})\in\hat{f}$ with dom $\ell(\hat{f})=\Omega$;
2) $\|\ell(\hat{f})(\omega)\|_{L^p(\nabla_\omega,\mu_\omega)}=\rho(\,|\hat{f}|_p)(\omega)$;
3) $\ell(\hat{f})(\omega)\geq 0$ whenever $\hat{f}\geq 0$;
4) $\ell(h\hat{f})=\rho(h)\ell(\hat{f})$ for every $h\in L^\infty(\Omega)$;
5) the set $\{\ell(\hat{f})(\omega):\hat{f}\in L^\infty(\hat{\nabla},\hat{\mu})\}$ is dense in $X(\omega)$ for all $\omega\in\Omega$;
6) $\ell(\hat{f}\vee\hat{g})=\ell(\hat{f})\vee\ell(\hat{g})$ for any $f,g\in L^\infty(\hat{\nabla},\hat{\mu})$.

The map $\ell:L^\infty(\hat{\nabla},\hat{\mu})\to\mathcal{L}^\infty(\Omega,\mathcal{X})$ is called *vector-valued lifting* on $L^\infty(\hat{\nabla},\hat{\mu})$, associated with the lifting ρ.

An even continuous convex function $M:R\to[0,\infty)$ is called an N-function, if $\lim\limits_{t\to 0}\frac{M(t)}{t}=0$ and $\lim\limits_{t\to\infty}\frac{M(t)}{t}=\infty$. Every N-function M has the form $M(t)=$

$\int_0^{|t|} p(s)ds$, where $p(t)$ is a nondecreasing function that is positive for $t > 0$, right-continuous for $t \geq 0$, and such that $p(0) = 0$ and $\lim_{t\to\infty} p(t) = \infty$. Put $q(s) := \sup\{t : p(t) \leq s\}$, $s \geq 0$. The function $N(t) := \int_0^{|t|} q(s)ds$ is an N-function which is called *the complementary N-function* to M. An N-function M is said to satisfy $\triangle_2$*-condition* on $[s_0, \infty), s_0 \geq 0$, if there exists a constant k such that $M(2s) \leq kM(s)$ for every $s \geq s_0$ (see [13, 21]).

The set

$$L_M^0 := L_M^0(\hat{\nabla}, \hat{\mu}) := \{x \in L_0(\hat{\nabla}) : M(x) \in L_1(\hat{\nabla}, \hat{\mu})\}$$

is called the Orlicz L_0-class, and the vector space

$$L_M := L_M(\hat{\nabla}, \hat{\mu}) := \{x \in L_0(\hat{\nabla}, \hat{\mu}) : xy \in L_1(\hat{\nabla}, \hat{\mu}) for \quad all \quad y \in L_N^0\}$$

is called the Orlicz L_0-space, here as before N is the complementary N-function to M (see for classical Orlich spaces [21, 20]).

The following are valid: $L_M(\hat{\nabla}, \hat{\mu}) \subset L_1(\hat{\nabla}, \hat{\mu})$.

Define the L_0-valued Orlicz norm on $L_M(\hat{\nabla}, \hat{\mu})$ as follows

$$\|x\|_M := \sup\{|\int xyd\hat{\mu}| : y \in A(N)\}, x \in L_M(\hat{\nabla}, \hat{\mu}),$$

where $A(N) = \{y \in L_N^0 : \int N(y)d\hat{\mu} \leq \mathbf{1}\}$. The pair $(L_M(\hat{\nabla}, \hat{\mu}), \|\cdot\|_M)$ is a Banach–Kantorovich lattice which is called the Orlicz–Kantorovich lattice associated with the L_0-valued measure [23].

As in the case of classical Orlicz spaces, along with the Orlicz norm $\|\cdot\|_{(M)}$ on $L_M(\hat{\nabla}, \hat{\mu})$, we may consider the L_0-valued Luxemburg norm $\|x\|_{(M)} := \inf\{\lambda \in L_0 : \int M(\lambda^{-1}x)d\hat{\mu} \leq \mathbf{1}, \lambda \quad is \quad an \quad invertible \quad positive \quad element\}$; moreover, the pair $(L_M(\hat{\nabla}, \hat{\mu}), \|\cdot\|_{(M)})$ is also a Banach-Kantorovich lattice [24] (see also [25, 26]).

Let $(L_M(\hat{\nabla}, \hat{\mu}), \|\cdot\|_{(M)})$ be an Orlicz–Kantorovich lattice and $T : L_M(\hat{\nabla}, \hat{\mu}), \hat{\mu}) \to L_M(\hat{\nabla}, \hat{\mu})$ be a linear mapping. As usual we will say that T is *positive* if $T\hat{f} \geq 0$ whenever $\hat{f} \geq 0$. We say that T is an $L_0(\Omega)$*-bounded mapping* if there exists a function $k \in L_0(\Omega)$ such that $\|T\hat{f}\|_{L_M(\hat{\nabla},\hat{\mu})} \leq k\|\hat{f}\|_{L_M(\hat{\nabla},\hat{\mu})}$ for all $\hat{f} \in L_M(\hat{\nabla}, \hat{\mu})$. For a such mapping one can define an element of $L_0(\Omega)$ as follows

$$\|T\| = \sup_{\|\hat{f}\|_{L_M(\hat{\nabla},\hat{\mu})} \leq \mathbb{1}} \|T\hat{f}\|_{L_M(\hat{\nabla},\hat{\mu})},$$

which is called an $L_0(\Omega)$*-valued norm* of T. If $\|T\hat{f}\|_{L_M(\hat{\nabla},\hat{\mu})} \leq \|\hat{f}\|_{L_M(\hat{\nabla},\hat{\mu})}$ then the mapping T is said to be an $L_M(\hat{\nabla}, \hat{\mu})$*-contraction*.

The set of all essentially bounded functions w.r.t. $\hat{f}$ taken from $L_0(\hat{\nabla}, \hat{\mu})$ is denoted by $L^\infty(\hat{\nabla}, \hat{\mu})$.

3. (o)-convergence

In this section we give some auxiliary facts related to (o)-convergence of sequence $\hat{f}_n$ from $L_0(\hat{\nabla}, \hat{\mu})$ and (o)-convergence of the sequence $\{f_n(\omega)\}$ from $L_0(\nabla_\omega, \mu_\omega)$ (see [10]). For the sake of completeness we provide the proofs.

Theorem 3.1. *Let $\widehat{f_n} \in L_0(\hat{\nabla}, \hat{\mu})$. Then $\sup\limits_n \hat{f}_n$ exists in $L_0(\hat{\nabla}, \hat{\mu})$ if and only if $\sup\limits_n f_n(\omega)$ exists in $L_0(\nabla_\omega, \mu_\omega)$ for a.e. $\omega \in \Omega$. In this case one has $(\sup\limits_n \hat{f}_n)(\omega) = \sup\limits_n f_n(\omega)$ for a.e. $\omega \in \Omega$.*

Proof. Assume that $g(\omega) = \sup\limits_n f_n(\omega)$ exists in $(L_0(\nabla_\omega, \mu_\omega)$ for a.e. $\omega \in \Omega$. Denote $\hat{g}_n = \sup\limits_{1 \leq k \leq n} \hat{f}_k$ in $L_0(\hat{\nabla}, \hat{\mu})$. Then $g_n(\omega) = \sup\limits_{1 \leq k \leq n} f_k(\omega)$ for a.e. $\omega \in \Omega$.

Obviously, that $g_n(\omega) \uparrow g(\omega)$ as $n \to \infty$ for a.e. $\omega \in \Omega$. The relation $g_n(\omega) \uparrow g(\omega)$ implies that $g_n(\omega) \stackrel{\rho_\omega}{\to} g(\omega)$ for a.e. $\omega \in \Omega$, this means $g \in M(\Omega, X)$ and $\hat{g} \in L_0(\hat{\nabla}, \hat{\mu})$.

Let us prove that $\hat{g} = \sup\limits_n \hat{f}_n$ in $L_0(\hat{\nabla}, \hat{\mu})$. It is clear that $g(\omega) \geq f_n(\omega)$ for a.e. $\omega \in \Omega$. Therefore, $\widehat{g} \geq \widehat{f_n}$ for all $n \in \mathbb{N}$.

Let $\widehat{\varphi} \in L^0(\hat{\nabla}, \hat{\mu})$ and $\widehat{\varphi} \geq \widehat{f_n}$ for all $n \in \mathbb{N}$. Then $\varphi(\omega) \geq f_n(\omega)$ for any $n \in \mathbb{N}$. Hence, $\varphi(\omega) \geq g(\omega)$, for a.e. $\omega \in \Omega$, i.e. $\hat{\varphi} \geq \hat{g}$. This yields that $\hat{g} = \sup\limits_{n \in \mathbb{N}} \hat{f}_n$.

Conversely, let us assume that there exists $\hat{\psi} \in L_0(\hat{\nabla}, \hat{\mu})$ such that $\hat{\psi} = \sup\limits_{n \in \mathbb{N}} \hat{f}_n = \sup\limits_{n \in \mathbb{N}} \hat{g}_n$.

From $g_n(\omega) = \sup\limits_{1 \leq k \leq n} f_k(\omega)$ for a.e. $\omega \in \Omega$, we find $\psi(\omega) \geq f_n(\omega)$ for all $n \in \mathbb{N}$ for a.e. $\omega \in \Omega$. Hence, one gets $\psi(\omega) \geq \sup\limits_{n \in \mathbb{N}} f_n(\omega) = \sup\limits_{n \in \mathbb{N}} g_n(\omega)$ for a.e. $\omega \in \Omega$. As $\hat{g}_n \to \hat{\psi}$ in metric $\hat{\rho}$, then $g_n(\omega) \to \psi(\omega)$ in metric ρ_ω for a.e. $\omega \in \Omega$.

Since $\{g_n(\omega)\}$ is incrasing then $\psi(\omega) = \sup\limits_{n \in \mathbb{N}} g_n(\omega)$ for a.e. $\omega \in \Omega$. □

From this theorem immediately follow two corollaries.

Corollary 3.2. *Let $\{\hat{f}_n\} \subset L_0(\hat{\nabla}, \hat{\mu})$. Then $\inf\limits_{n \in \mathbb{N}} \hat{f}_n$ exists in $L_0(\hat{\nabla}, \hat{\mu})$ if and only if $\inf\limits_{n \in \mathbb{N}} f(\omega)$ exists in $L_0(\nabla_\omega, \mu_\omega)$ for a.e. $\omega \in \Omega$. In this case one has $(\inf\limits_{n \in \mathbb{N}} \hat{f}_n)(\omega) = \inf\limits_{n \in \mathbb{N}} f_n(\omega)$ for a.e. $\omega \in \Omega$.*

Corollary 3.3. *Let $\hat{f}_n \in L_0(\hat{\nabla}, \hat{\mu})$. If $\hat{f}_n \stackrel{(o)}{\to} \hat{f}$ for some $\hat{f} \in L_0(\hat{\nabla}, \hat{\mu})$, then $f_n(\omega) \stackrel{(o)}{\to} f(\omega)$ in $L_0(\nabla_\omega, \mu_\omega)$ for a.e. $\omega \in \Omega$. Conversely, if $f_n(\omega) \stackrel{(o)}{\to} g(\omega)$ for some $g(\omega) \in L_0(\nabla_\omega, \mu_\omega)$ for a.e. $\omega \in \Omega$, then $\hat{g} \in L_0(\hat{\nabla}, \hat{\mu})$ and $\hat{f}_n \stackrel{(o)}{\to} \hat{g}$ in $L_0(\hat{\nabla}, \hat{\mu})$.*

4. Weighted ergodic theorem

In this section we are going to prove the main result of the paper. We will prove (o)-convergence of Besicovich weighted ergodic averages for positive contractions of the Orlich-Kantorovich space.

Proposition 4.1. *Let M be an N-function, strictly convex in some interval and let $T : L_M(\hat{\nabla}, \hat{\mu}) \to L_M(\hat{\nabla}, \hat{\mu})$ be a positive L_0-linear operator such that*

(i) $\int M(|T\hat{f}|d\hat{\mu} \leq \int M(|\hat{f}|)d\hat{\mu}$;
(ii) $\|T\|_{L_1(\hat{\nabla},\hat{\mu}) \to L_1(\hat{\nabla},\hat{\mu})} \leq \mathbf{1}$;
(iii) *there exists $h \in L_M(\hat{\nabla}, \hat{\mu}), h(\omega) \neq 0$ almost everywhere, such that $Th = h$.*

Then there exists a family $T_\omega : L_M(\nabla_\omega, \mu_\omega) \to L_M(\nabla_\omega, \mu_\omega)$ *of positive linear operators such that for each* $\hat{f} \in L_M(\hat{\nabla}, \hat{\mu})$ *the equality* $T_\omega f(\omega) = (T\hat{f})(\omega)$ *holds for almost all* $\omega \in \Omega$*. Moreover, one has*

(a) $\int M(|T_\omega f(\omega)|d\mu_\omega \leq \int M(|f(\omega)|)d\mu_\omega$ *for almost all* $\omega \in \Omega$*;*
(b) $\|T_\omega\|_{L_M(\nabla_\omega,\mu_\omega)\to L_M(\nabla_\omega,\mu_\omega)} \leq 1$*;*
(c) *There exists* $h(\omega) \in L_M(\nabla_\omega, \mu_\omega), h(\omega) \neq 0$ *a.e. such that* $T_\omega h(\omega) = h(\omega)$*.*

Proof. Since $\|T\|_{L_1(\hat{\nabla},\hat{\mu})\to L_1(\hat{\nabla},\hat{\mu})} \leq \mathbf{1}$ by Theorem 2.1 [9] there exists a family $T_\omega : L_1(\nabla_\omega, \mu_\omega) \to L_1(\nabla_\omega, \mu_\omega)$ of positive linear operators such that for each $\hat{f} \in L_M(\hat{\nabla}, \hat{\mu})$ the equality $T_\omega f(\omega) = (T\hat{f})(\omega)$ holds for almost all $\omega \in \Omega$. Moreover, one has $\|T_\omega\|_{L_1(\nabla_\omega,\mu_\omega)\to L_1(\nabla_\omega,\mu_\omega)} \leq 1$.

(a) Let $\hat{f} \in L^\infty(\hat{\nabla}, \hat{\mu})$. Due to $T_\omega(\ell(\hat{f})(\omega)) = \ell(T\hat{f})(\omega)$ for all $\omega \in \Omega$ and for all $\hat{f} \in L^\infty(\hat{\nabla}, \hat{\mu})$ (see [9]) we then get

$$
\begin{aligned}
\int M(|T_\omega(\ell(\hat{f})(\omega)|)d\mu_\omega &= \int M(|\ell(T\hat{f})(\omega)|)d\mu_\omega = \int M(\ell(|T\hat{f}|)(\omega))d\mu_\omega \\
&= \int \ell(M(((|T\hat{f}|))(\omega))d\mu_\omega = \rho\Big(\int M(|T\hat{f}|d\hat{\mu}\Big)(\omega) \\
&\leq \rho\Big(\int M(|\hat{f}|)d\hat{\mu}\Big)(\omega) = \int \ell(M(|\hat{f}|)(\omega))d\mu_\omega \\
&= \int M(\ell(|\hat{f}|)(\omega))d\mu_\omega
\end{aligned}
$$

for all $\omega \in \Omega$.

Let $f(\omega) \in L_M(\nabla_\omega, \mu_\omega)$, then we choose a sequence $\hat{f}_n \in L^\infty(\hat{\nabla}, \hat{\mu})$ such that $f(\omega) = \lim_{n\to\infty} \ell(\hat{f}_n)(\omega)$. Now we extend the operator T_ω to $L_M(\nabla_\omega, \mu_\omega)$ as follows: $T_\omega f(\omega) = \lim_{n\to\infty} T_\omega(\ell(\hat{f}_n)(\omega))$ and

$$
\begin{aligned}
\int M(|T_\omega f(\omega)|)d\mu_\omega &= \lim_{n\to\infty} \int M(|T_\omega \ell(\hat{f}_n)(\omega)|)d\mu_\omega \\
&\leq \lim_{n\to\infty} \int M(|\ell(\hat{f}_n)(\omega)|)d\mu_\omega \\
&= \int M(|f(\omega)|)d\mu_\omega.
\end{aligned}
$$

(c) Since, there exists $h \in L_M(\hat{\nabla}, \hat{\mu}), h(\omega) \neq 0$ for almost all $\omega \in \Omega$ such that $Th = h$ we have $T_\omega h(\omega) = (Th)(\omega) = h(\omega)$.

(b) Using (a), (c), $\|T_\omega\|_{L_1(\nabla_\omega,\mu_\omega)\to L_1(\nabla_\omega,\mu_\omega)} \leq 1$ and Theorem 2.5 [4] one finds $\|T_\omega f(\omega)\|_\infty \leq \|f(\omega)\|_\infty$ for every $f(\omega) \in L_1(\nabla_\omega, \mu_\omega) \cap L^\infty(\nabla_\omega, \mu_\omega)$, consequently the last inequality holds for every $f(\omega) \in L_M(\nabla_\omega, \mu_\omega) \cap L^\infty(\nabla_\omega, \mu_\omega)$. The positivity of T_ω implies $T_\omega \mathbf{1}_\omega \leq \mathbf{1}_\omega$, i.e. $\|T_\omega\|_{L^\infty(\nabla_\omega,\mu_\omega)\to L^\infty(\nabla_\omega,\mu_\omega)} \leq 1$. Due to

$$
\|T_\omega\|_{L_1(\nabla_\omega,\mu_\omega)\to L_1(\nabla_\omega,\mu_\omega)} \leq 1, \quad \|T_\omega\|_{L^\infty(\nabla_\omega,\mu_\omega)\to L^\infty(\nabla_\omega,\mu_\omega)} \leq 1
$$

and the Orlicz space $L_M(\nabla_\omega, \mu_\omega)$ is an interpolation space (see [15, II, sec.4, item 6]), i.e.

$$
\|T_\omega\|_{L_M(\nabla_\omega,\mu_\omega)\to L_M(\nabla_\omega,\mu_\omega)} \leq \max\{\|T_\omega\|_{L_1(\nabla_\omega,\mu_\omega)\to L_1(\nabla_\omega,\mu_\omega)}, \|T_\omega\|_{L^\infty(\nabla_\omega,\mu_\omega)\to L^\infty(\nabla_\omega,\mu_\omega)}\}
$$

we infer that $T_\omega(L_M(\nabla_\omega, \mu_\omega)) \subset L_M(\nabla_\omega, \mu_\omega)$ and $\|T_\omega\|_{L_M(\nabla_\omega,\mu_\omega)\to L_M(\nabla_\omega,\mu_\omega)} \leq 1$. This completes the proof. □

A sequence $\{b(k)\}$ is called *Besicovich sequence,* if for every $\varepsilon > 0$ there is a sequence of trigonometric polynomials ψ_ε, such that

$$\lim_{N\to\infty} \sup \frac{1}{N} \sum_{k=1}^{N} |b(k) - \psi_\varepsilon(k)| < \varepsilon$$

We say that $\{b(k)\}$ is *bounded Besicovich sequence,* if $b(k) \in \ell^\infty$. In the present paper we consider only bounded, real Besicovich sequences.

Theorem 4.2. *Let M be an N-function, strictly convex in some interval with Δ_2-condition and $T : L_M(\hat{\nabla}, \hat{\mu}) \to L_M(\hat{\nabla}, \hat{\mu})$ be a positive L_0-linear operator such that*

(i) $\int M(|T\hat{f}|d\hat{\mu} \leq \int M(|\hat{f}|)d\hat{\mu}$*;*

(ii) $\|T\|_{L_1(\hat{\nabla},\hat{\mu})\to L_1(\hat{\nabla},\hat{\mu})} \leq \mathbf{1}$*;*

(iii) *There exists $h \in L_M(\hat{\nabla}, \hat{\mu})$, $h(\omega) \neq 0$ for almost all ω, such that $Th = h$.*

Then

(a) *There exists*

$$\sup_{n\geq 1} \frac{1}{n} \sum_{k=1}^{n-1} b(k) T^k |\hat{f}|$$

in $L_M(\hat{\nabla}, \hat{\mu})$ for any bounded Besicovich sequence $b(k)$.

(b) *If $\{b(k)\}$ is a bounded Besicovich sequence, then for every $\hat{f} \in L_M(\hat{\nabla}, \hat{\mu})$ there exists $\hat{f}^* \in L_M(\hat{\nabla}, \hat{\mu})$ such that the sequence*

$$\widetilde{A_n}(\widehat{f}) = \frac{1}{n} \sum_{k=1}^{n-1} b(k) T^k \widehat{f}$$

(o)-converges to $\hat{f}^*$ in $L_M(\widehat{\nabla}, \widehat{\mu})$.

Proof. (a). Let $\hat{f} \in L_M(\hat{\nabla}, \hat{\mu})$ and $T : L_M(\hat{\nabla}, \hat{\mu}) \to L_M(\hat{\nabla}, \hat{\mu})$ satisfy the conditions (i),(ii),(iii). Then by Proposition 4.1 there exists a family $\{T_\omega : L_M(\nabla_\omega, \mu_\omega) \to L_M(\nabla_\omega, \mu_\omega)\}$ of positive linear operators such that for each $\hat{f} \in L_M(\hat{\nabla}, \hat{\mu})$ the equality $T_\omega f(\omega) = (T\hat{f})(\omega)$ holds for almost all $\omega \in \Omega$. Moreover, one has $\int M(|T_\omega f(\omega)|d\mu_\omega \leq \int M(|f(\omega)|)d\mu_\omega$ for almost all $\omega \in \Omega$, and $\|T_\omega\|_{L_1(\nabla_\omega,\mu_\omega)\to L_1(\nabla_\omega,\mu_\omega)} \leq 1$. Besides, there exists $h(\omega) \in L_M(\nabla_\omega, \mu_\omega), h(\omega) \neq 0$ a.e. such that $T_\omega h(\omega) = h(\omega)$. This implies that for each ω the operator T_ω satisfies conditions of Theorem 3.1 [4]. Consequently, we get

$$\begin{aligned} \left\| \sup_{n\geq 1} \frac{1}{n} \sum_{k=1}^{n-1} b(k) T_\omega^k(|f(\omega)|) \right\|_{(M)} &\leq b \left\| \sup_{n\geq 1} \frac{1}{n} \sum_{k=1}^{n-1} T_\omega^k(|f(\omega)|) \right\|_{(M)} \\ &\leq bC\|f(\omega)\|_{(M)}, \end{aligned}$$

where $b = \sup_k |b_k|$ and C is a constant.

Therefore, $\sup_{n\geq 1} \frac{1}{n} \sum_{k=1}^{n-1} b(k) T_\omega^k(|f(\omega)|)$ exists in $L_M(\nabla_\omega, \mu_\omega)$. Thus, from Theorem 4.1 [7] and Proposition 2.3 [23] one finds the existence of

$$\sup_{n\geq 1} \frac{1}{n} \sum_{k=1}^{n-1} b(k) T^k |\hat{f}|$$

in $L_M(\hat{\nabla}, \hat{\mu})$.

(b). Since almost all $\omega \in \Omega$ the operator T_ω satisfies conditions of Theorem 3.1 [4], then there exists $f^*(\omega) \in L_M(\nabla_\omega, \mu_\omega)$ such that

$$\lim_{n\to\infty} A_n(f(\omega)) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} b(k) T_\omega^k(f(\omega)) = f^*(\omega)$$

μ_ω- almost everywhere for each $f(\omega) \in L_M(\nabla_\omega, \mu_\omega)$. Taking into account that μ_ω- almost everywhere convergence for sequence in $L_0(\nabla_\omega, \mu_\omega)$ coincides with (o)-convergence in $L_0(\nabla_\omega, \mu_\omega)$, we infer that the sequence $A_n(f(\omega))$ is (o)-convergent in $L_0(\nabla_\omega, \mu_\omega)$. Hence, by Corollary 3.3 one finds that $\widetilde{A_n}(\widehat{f})$ is (o)-convergent in $L_0(\widehat{\nabla}, \widehat{\mu})$.

The existence of $\sup\limits_{n\geq 1} \frac{1}{n}\sum\limits_{k=1}^{n-1} b(k) T^k |\hat{f}|$ in $L_M(\hat{\nabla}, \hat{\mu})$ yields that

$$\widetilde{A_n}(\widehat{f}) = \widehat{A_n(f(\omega))} \xrightarrow{(o)} \widehat{f^*} = \widehat{f^*(\omega)}$$

in $L_M(\widehat{\nabla}, \widehat{\mu})$. This completes the proof. □

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