

SOLVABLE LEIBNIZ ALGEBRAS WITH FILIFORM NILRADICAL

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ABSTRACT. In this paper we continue the description of solvable Leibniz algebras whose nilradical is a filiform algebra. In fact, solvable Leibniz algebras whose nilradical is a naturally graded filiform Leibniz algebra are known. Here we extend the description to solvable Leibniz algebras whose nilradical is a filiform algebra. Moreover, we establish that solvable Leibniz algebras with filiform Lie nilradical are Lie algebras.

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1. INTRODUCTION

Leibniz algebras were introduced by Loday [12] as a non-skew symmetric version of Lie algebras. These algebras generalize Lie algebras in natural way. The theory of Leibniz algebras has been actively investigated in the last two decades. Many results of the theory of Lie algebras have been extended to Leibniz algebras. For instance, the classical results on Cartan subalgebras [17], Levi's decomposition [5], the properties of solvable algebras with given nilradical [7] and other from the theory of Lie algebras are also true for Leibniz algebras [3, 4, 19, 20, 21].

As far as physical applications are concerned, we note that solvable Lie algebras often occur as Lie algebras of symmetry groups of differential equations [16]. Group invariant solutions can be obtained by symmetry reduction, using the subalgebras of the symmetry algebra [24]. In this procedure an important step is to identify the symmetry algebra and its subalgebras as abstract Lie algebras. A detailed identification presupposes the existence of a classification of Lie algebras into isomorphism classes.

In this paper we continue the description of solvable algebras with a given nilradical. The first work which was devoted to the description of such Lie algebras is the paper [13]. In fact, it was proved that the complemented space to nilradical forms an abelian subalgebra, consisting of semisimple elements of an algebra. However, the structure of the nilradical depends on this subalgebra. Later, Mubarakzjanov G.M. proposed the description of solvable Lie algebras with a given structure of nilradical [14] by means of outer derivations of the nilradical. Papers [1, 2, 6, 15, 22, 23] were devoted to the application of Mubarakzjanov's method for solvable Lie algebras with different kinds of nilradicals. Some results of the Lie algebra theory generalized to Leibniz algebras in [3] allow us to apply the Mubarakzjanov's method to the case of Leibniz algebras. In this direction papers [7] and [8] deal with the description of solvable Leibniz algebras with null-filiform and naturally graded filiform nilradicals, respectively.

The aim of the present paper is to classify solvable Leibniz algebras with filiform nilradical. Thanks to papers [10] and [11], we already have the classification of filiform Leibniz algebras. It should be noted that the description of Lie algebras with filiform nilradicals obtained in this work is a new result for Lie algebras, as well.

In section 2 the necessary definitions and results for understanding the main parts of the paper are given. In section 3 the classification of solvable Leibniz algebras with filiform non-Lie Leibniz nilradicals is obtained. The last section deals with the description of solvable Leibniz algebras with filiform Lie nilradical.

Throughout the paper we consider finite-dimensional vector spaces and algebras over the field of the complex numbers. Moreover, in the multiplication table of an algebra omitted products are assumed to be zero and if it is not noted, we shall consider non-nilpotent solvable algebras.

2. PRELIMINARIES

In this section we give the necessary definitions and preliminary results on outer derivations of nilradicals of Leibniz algebras and the descriptions of filiform Leibniz algebras.

Definition 2.1. An algebra $(L, [-, -])$ over a field F is called a Leibniz algebra if for any $x, y, z \in L$, the so-called Leibniz identity

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]$$

holds.

From the Leibniz identity we conclude that for any $x, y \in L$ the elements $[x, x]$ and $[x, y] + [y, x]$ lie in the right annihilator of the algebra L (denoted by $\text{Ann}_r(L) = \{x \in L : [y, x] = 0, \text{ for all } y \in L\}$).

Definition 2.2. A linear map $d : L \rightarrow L$ of a Leibniz algebra L is said to be a derivation if for any $x, y \in L$, the following condition holds:

$$d([x, y]) = [d(x), y] + [x, d(y)].$$

For a given element x of a Leibniz algebra L the operator of right multiplication $R_x : L \rightarrow L$, defined as $R_x(y) = [y, x]$ for $y \in L$, is a derivation. This kind of derivations are called *inner derivations*.

Any Leibniz algebra L is associated with the algebra of right multiplications $R(L) = \{R_x \mid x \in L\}$, which is endowed with a structure of Lie algebra by means of the bracket $[R_x, R_y] = R_x R_y - R_y R_x$. Thank to the Leibniz identity the equality $[R_x, R_y] = R_{[y, x]}$ holds true. In addition, the algebra $R(L)$ is antisymmetric isomorphic to the quotient algebra $L/\text{Ann}_r(L)$.

Definition 2.3. [14] Let d_1, d_2, \dots, d_n be derivations of a Leibniz algebra L . The derivations d_1, d_2, \dots, d_n are said to be *nil-independent* if

$$\alpha_1 d_1 + \alpha_2 d_2 + \dots + \alpha_n d_n$$

is not nilpotent for any scalars $\alpha_1, \alpha_2, \dots, \alpha_n \in F$.

In other words, if for any $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ there exists a natural number k such that $(\alpha_1 d_1 + \alpha_2 d_2 + \dots + \alpha_n d_n)^k = 0$, then $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

For a Leibniz algebra L , the sequences of two-sided ideals defined recursively as follows:

$$L^1 = L, \quad L^{k+1} = [L^k, L^1], \quad k \geq 1, \quad L^{[1]} = L, \quad L^{[s+1]} = [L^{[s]}, L^{[s]}], \quad s \geq 1$$

are said to be the *lower central* and the *derived series* of L , respectively.

Definition 2.4. A Leibniz algebra L is called *nilpotent* (respectively, *solvable*) if there exists $m \in \mathbb{N}$ ($t \in \mathbb{N}$) such that $L^m = 0$ (respectively, $L^{[t]} = 0$). The minimal number m (respectively, t) with such property is said to be the *index of nilpotency* (respectively, *solvability*) of the algebra L .

Obviously, the index of nilpotency of an n -dimensional Leibniz algebra is not greater than $n + 1$.

Since the sum of two nilpotent ideals is a nilpotent ideal, we can consider the maximal nilpotent ideal of a Leibniz algebra. The maximal nilpotent ideal is called the *nilradical*. Notice that the nilradical is not the radical in the sense of Kurosh, because the quotient Leibniz algebra by its nilradical may contain a nilpotent ideal (see [9]).

Definition 2.5. An n -dimensional Leibniz algebra L is said to be *filiform* if $\dim L^i = n - i$ for $2 \leq i \leq n$.

Let R be a solvable Leibniz algebra. Then it can be decomposed in the form $R = N \oplus Q$, where N is the nilradical and Q is the complementary vector space. Since the square of a solvable Leibniz algebra is contained into the nilradical [3], we get the nilpotency of the ideal R^2 and consequently, $Q^2 \subseteq N$.

Theorem 2.6. [7] Let R be a solvable Leibniz algebra and N be its nilradical. Then the dimension of the complementary vector space to N is not greater than the maximal number of nil-independent derivations of N .

A nilpotent Leibniz algebra is called *characteristically nilpotent* if all its derivations are nilpotent. If the nilradical N of a Leibniz algebra is characteristically nilpotent then, according to Theorem 2.6, a solvable Leibniz algebra is nilpotent. Therefore, we shall consider solvable Leibniz algebras with non-characteristically nilpotent filiform nilradical.

Theorem 2.7. [10] Let \mathfrak{g} be an $(n+1)$ -dimensional non-characteristically nilpotent filiform Lie algebra. Then, it is isomorphic to one of the following non-isomorphic algebras:

$L_n :$

$$\{ [e_0, e_i] = e_{i+1}, \quad 1 \leq i \leq n - 1;$$

$Q_n, n - \text{odd} :$

$$\left\{ \begin{array}{ll} [e_0, e_i] = e_{i+1}, & 1 \leq i \leq n - 2, \\ [e_i, e_{n-i}] = (-1)^i e_n, & 1 \leq i \leq n - 1; \end{array} \right.$$

$A_{n+1}^r(\alpha_1, \dots, \alpha_t)$, $1 \leq r \leq n-3$, $t = \lfloor \frac{n-r-1}{2} \rfloor$:

$$\begin{aligned} [e_0, e_i] &= e_{i+1}, & 1 \leq i \leq n-1, \\ [e_i, e_j] &= \left(\sum_{k=i}^t (-1)^{k-i} \alpha_k \binom{j-k-1}{k-i} \right) e_{i+j+r}, & 1 \leq i < j \leq n-2, \quad i+j+r \leq n; \end{aligned}$$

$B_{n+1}^r(\alpha_1, \dots, \alpha_t)$, $1 \leq r \leq n-4$, $t = \lfloor \frac{n-r-2}{2} \rfloor$, n - odd, $n = 2m+1$:

$$\begin{aligned} [e_0, e_i] &= e_{i+1}, & 1 \leq i \leq n-2, \\ [e_i, e_{n-i}] &= (-1)^i e_n, & 1 \leq i \leq m, \\ [e_i, e_j] &= \left(\sum_{k=i}^t (-1)^{k-i} \alpha_k \binom{j-k-1}{k-i} \right) e_{i+j+r}, & 1 \leq i, j \leq n-1, \quad i+j+r \leq n-1, \end{aligned}$$

where the parameters $(\alpha_1, \dots, \alpha_t)$ satisfy the polynomial relations emanating from the Jacobi identity and at least one parameter $\alpha_i \neq 0$.

In the following theorem, all $(n+1)$ -dimensional filiform Leibniz algebras decomposes into three families of algebras.

Theorem 2.8. [18] *Any complex $(n+1)$ -dimensional filiform Leibniz algebra admits a basis $\{e_0, e_1, \dots, e_n\}$ such that the table of multiplication of the algebra has one of the following forms:*

$$\begin{aligned} F_1(\alpha_3, \alpha_4, \dots, \alpha_n, \theta) : & \begin{cases} [e_0, e_0] = e_2, \\ [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1, \\ [e_0, e_1] = \sum_{k=3}^{n-1} \alpha_k e_k + \theta e_n, \\ [e_i, e_1] = \sum_{k=i+2}^n \alpha_{k+1-i} e_k, & 1 \leq i \leq n-2; \end{cases} \\ F_2(\beta_3, \beta_4, \dots, \beta_n, \gamma) : & \begin{cases} [e_0, e_0] = e_2, \\ [e_i, e_0] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_0, e_1] = \sum_{k=3}^n \beta_k e_k, \\ [e_1, e_1] = \gamma e_n, \\ [e_i, e_1] = \sum_{k=i+2}^n \beta_{k+1-i} e_k, & 2 \leq i \leq n-2; \end{cases} \\ F_3(\theta_1, \theta_2, \theta_3) : & \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1, \\ [e_0, e_i] = -e_{i+1}, & 2 \leq i \leq n-1, \\ [e_0, e_0] = \theta_1 e_n, \\ [e_0, e_1] = -e_2 + \theta_2 e_n, \\ [e_1, e_1] = \theta_3 e_n, \\ [e_i, e_j] = -[e_j, e_i] \in \text{lin} \langle e_{i+j+1}, e_{i+j+2}, \dots, e_n \rangle, & \begin{cases} 1 \leq i \leq n-2, \\ 2 \leq j \leq n-i, \quad i < j \end{cases} \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = \alpha(-1)^i e_n, & 1 \leq i \leq n-1, \end{cases} \end{cases} \end{aligned}$$

where $\alpha \in \{0, 1\}$ for odd n and $\alpha = 0$ for even n . Moreover, the structure constants of an algebra from $F_3(\theta_1, \theta_2, \theta_3)$ should satisfy the Leibniz identity.

It is easy to see that algebras of the first and the second families are non-Lie algebras. Moreover, an algebra of the third family is a Lie algebra if and only if $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)$.

From the list of Theorem 2.8 we only indicate the non-characteristically nilpotent filiform non-Lie Leibniz algebras.

Theorem 2.9. [11] *An arbitrary non-characteristically nilpotent filiform non-Lie Leibniz algebra is isomorphic to one of the following non-isomorphic algebras:*

- $F_1(0, 0, \dots, 0, 1)$ and $F_1^s(\alpha_3, \alpha_4, \dots, \alpha_{n-1}, \alpha_n, \alpha_n)$, $3 \leq s \leq n$, where

$$\alpha_k = \begin{cases} 0, & k \not\equiv s \pmod{(s-2)} \\ (-1)^t C_{t+1}^{s-1}, & k \equiv s \pmod{(s-2)} \end{cases} \quad \text{and } t = \frac{k-s}{s-2}, \quad 3 \leq k \leq n \text{ and } C_n^p \text{ is the } p\text{-th Catalan number,}$$

- ✓ for odd n :
 $F_2(0, 0, \dots, 0, 0, 1)$ and $F_2^j(0, 0, \dots, 0, \underbrace{1}_j, 0, \dots, 0, 0, 0)$ with $n \geq 4$ and $3 \leq j \leq n$,
- ✓ for even n :
 $F_2^1(0, 0, \dots, 0, \beta_{\frac{n+2}{2}}, 0, \dots, 0, 0, 1)$ and $F_2^j(0, 0, \dots, 0, \underbrace{1}_j, 0, \dots, 0, 0, 0)$ with $n \geq 4$ and $3 \leq j \leq n$,
- $F_3(1, 0, 0), \quad F_3(0, 1, 0), \quad F_3(0, 0, 1).$

3. SOLVABLE LEIBNIZ ALGEBRAS WITH FILIFORM NON-LIE LEIBNIZ NILRADICAL

In this section we investigate solvable Leibniz algebras whose nilradical is one from the list of Theorem 2.9. In order to demonstrate the considering cases for three families of Theorem 2.8, we divide this section into three subsections.

3.1. Solvable Leibniz algebras with nilradical a non-characteristically nilpotent algebra of the family $F_1(\alpha_3, \alpha_4, \dots, \alpha_n, \theta)$.

It follows the matrix form of any derivation of an algebra of the family $F_1(\alpha_3, \alpha_4, \dots, \alpha_n, \theta)$.

Proposition 3.1. [11] *Any derivation of a filiform Leibniz algebras from the family $F_1(\alpha_3, \alpha_4, \dots, \alpha_n, \theta)$ has the following matrix form:*

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots & a_{n-2} & a_{n-1} & a_n \\ 0 & a_0 + a_1 & a_2 & a_3 & \dots & a_{n-2} & b_{n-1} & b_n \\ 0 & 0 & 2a_0 + a_1 & a_2 + a_1\alpha_3 & \dots & a_{n-3} + a_1\alpha_{n-2} & a_{n-2} + a_1\alpha_{n-1} & a_{n-1} + a_1\alpha_n \\ 0 & 0 & 0 & 3a_0 + a_1 & \dots & a_{n-4} + 2a_1\alpha_{n-2} & a_{n-3} + 2a_1\alpha_{n-1} & a_{n-2} + 2a_1\alpha_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & (n-2)a_0 + a_1 & a_2 + (n-3)a_1\alpha_3 & a_3 + (n-3)a_1\alpha_4 \\ 0 & 0 & 0 & 0 & \dots & 0 & (n-1)a_0 + a_1 & a_2 + (n-2)a_1\alpha_3 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & na_0 + a_1 \end{pmatrix},$$

where

$$a_0(\theta - \alpha_n) = 0, \quad a_1(\alpha_n - \theta) = a_{n-1} - b_{n-1}, \quad \alpha_3(a_1 - a_0) = 0,$$

$$\alpha_k(a_1 - (k-2)a_0) = \frac{k}{2}a_1 \sum_{j=4}^k \alpha_{j-1}\alpha_{k-j+3}, \quad 4 \leq k \leq n.$$

Case $F_1(0, 0, \dots, 0, 1)$.

From Proposition 3.1 we conclude that the number of nil-independent outer derivations of algebra $F_1(0, 0, \dots, 0, 1)$ is equal to one. Thus, if there is any solvable Leibniz algebra with $F_1(0, 0, \dots, 0, 1)$ nilradical, then should be dimension $n + 2$.

Proposition 3.2. *There are not any $(n + 2)$ -dimensional solvable Leibniz algebras with nilradical $F_1(0, 0, \dots, 0, 1)$.*

Proof. Let L be a solvable Leibniz algebra with $F_1(0, 0, \dots, 0, 1)$ nilradical. We complement the basis $\{e_0, e_1, \dots, e_n\}$ of nilradical by a basis element x of Q .

From the table of multiplication of $F_1(0, 0, \dots, 0, 1)$ we conclude that $\langle e_2, e_3, \dots, e_n \rangle \subseteq \text{Ann}_r(L)$. Using Proposition 3.1, we derive the following products in the algebra L :

$$\begin{aligned} [e_0, e_0] &= e_2, \\ [e_i, e_0] &= e_{i+1}, & 1 \leq i \leq n-1, \\ [e_0, e_1] &= e_n, \\ [e_0, x] &= \sum_{i=1}^n a_i e_i, & [x, e_0] &= \sum_{i=0}^n \beta_i e_i, \\ [e_1, x] &= \sum_{i=1}^n a_i e_i + (a_{n-1} + a_1)e_{n-1} + b_n e_n, & [x, e_1] &= \sum_{i=0}^n \gamma_i e_i, \\ [e_i, x] &= \sum_{k=i}^n a_{k-i+1} e_k, & 2 \leq i \leq n, & [x, x] &= \sum_{i=0}^n \delta_i e_i. \end{aligned}$$

Taking the change as follows:

$$x' = x - \sum_{i=2}^{n-1} \beta_{i+1} e_i,$$

we can assume $[x, e_0] = \beta_0 e_0 + \beta_1 e_1 + \beta_2 e_2$.

The equalities

$$0 = [e_0, [e_0, x] + [x, e_0]] = [e_0, [x, x]] = [e_0, [e_1, x] + [x, e_1]]$$

imply

$$\beta_0 = 0, \beta_1 = -a_1, \delta_0 = \delta_1 = 0, \gamma_0 = 0, \gamma_1 = -a_1.$$

Considering

$$0 = [x, e_2] = -a_1 e_2 + \sum_{i=3}^n \gamma_i e_i,$$

we obtain $a_1 = 0$. Consequently, the restriction of the operator R_x to nilradical $F_1(0, 0, \dots, 0, 1)$ is a nilpotent derivation. Therefore, we get a contradiction with the existence of $(n+2)$ -dimensional solvable Leibniz algebra. \square

Case $F_1^s(\alpha_3, \alpha_4, \dots, \alpha_{n-1}, \alpha_n, \alpha_n)$.

Let us fix the first non zero parameter $\alpha_s \neq 0$ of the algebra $F_1^s(\alpha_3, \alpha_4, \dots, \alpha_{n-1}, \alpha_n, \alpha_n)$. Then, from the relations of Proposition 3.1, we deduce $a_1 = (s-2)a_0$ and $b_{n-1} = a_{n-1}$. Therefore, the number of nil-independent outer derivations of nilradical $F_1^s(\alpha_3, \alpha_4, \dots, \alpha_{n-1}, \alpha_n, \alpha_n)$ is equal to one.

Proposition 3.3. *There are not any $(n+2)$ -dimensional solvable Leibniz algebras with nilradical $F_1^s(\alpha_3, \alpha_4, \dots, \alpha_{n-1}, \alpha_n, \alpha_n)$.*

Proof. Let L be a solvable Leibniz algebra with nilradical $F_1^s(\alpha_3, \alpha_4, \dots, \alpha_{n-1}, \alpha_n, \alpha_n)$. Since in the general form of a non-nilpotent derivation of nilradical $F_1^s(\alpha_3, \alpha_4, \dots, \alpha_{n-1}, \alpha_n, \alpha_n)$ the parameter $a_0 \neq 0$ (otherwise due to equality $a_1 = (s-2)a_0$ a derivation is nilpotent), without loss of generality, one can assume $a_0 = 1$.

Since for a basis element of the space Q the general form of the derivation R_x is presented in Proposition 3.1, we have the following multiplications:

$$\begin{aligned} [e_0, e_0] &= e_2, & [e_i, e_0] &= e_{i+1}, & 1 \leq i \leq n-1, \\ [e_0, e_1] &= \sum_{k=3}^n \alpha_k e_k, & [e_i, e_1] &= \sum_{k=i+2}^n \alpha_{k+1-i} e_k, & 1 \leq i \leq n-2, \\ [e_0, x] &= e_0 + (s-2)e_1 + \sum_{i=2}^n a_i e_i, & [x, e_0] &= \sum_{i=0}^n \beta_i e_i, \\ [e_1, x] &= (s-1)e_1 + \sum_{i=2}^{n-1} a_i e_i + b_n e_n, & [x, e_1] &= \sum_{i=0}^n \gamma_i e_i, \\ [e_i, x] &= (s-2+i)e_i + \sum_{j=i+1}^n (a_{j+1-i} + (i-1)(s-2)\alpha_{j-i+2})e_j, & 2 \leq i \leq n, \\ [x, x] &= \sum_{i=0}^n \delta_i e_i. \end{aligned}$$

Evidently, $\langle e_2, e_3, \dots, e_n \rangle \subseteq \text{Ann}_r(L)$, consequently, $[x, e_i] = 0$ with $2 \leq i \leq n$.
 The equality $[e_0, [x, x]] = 0$ implies $\delta_0 = \delta_1 = 0$.
 Making the following change of basis:

$$x' = x - \sum_{i=2}^{n-1} \beta_{i+1} e_i$$

we obtain

$$\begin{aligned} [x', e_0] &= \beta_0 e_0 + \beta_1 e_1 + \beta_2 e_2, & [e_0, x'] &= [e_0, x] = e_0 + (s-2)e_1 + \sum_{i=2}^n a_i e_i, \\ [e_1, x'] &= [e_1, x] = (s-1)e_1 + \sum_{i=2}^{n-1} a_i e_i + b_n e_n, & [x', e_1] &= \sum_{i=0}^n \gamma'_i e_i, \\ [x', x'] &= \sum_{i=0}^n \delta'_i e_i. \end{aligned}$$

From the equalities

$$[e_0, [e_0, x] + [x, e_0]] = [e_0, [e_1, x] + [x, e_1]] = 0$$

we conclude

$$\beta_0 = -1, \quad \beta_1 = -s + 2, \quad \gamma_0 = 0, \quad \gamma_1 = -(s-1).$$

A contradiction obtained from $0 = [x, e_2] = [x, [e_1, e_0]] = -(s-1)e_2 + \sum_{k=3}^n \Delta_k e_k$ with $s \geq 3$ completes the proof of the proposition. \square

3.2. Solvable Leibniz algebras with nilradical a non-characteristically nilpotent algebra of the family $F_2(\beta_3, \beta_4, \dots, \beta_n, \gamma)$.

In this subsection we consider the family of algebras $F_2(\beta_3, \beta_4, \dots, \beta_n, \gamma)$. Similar to the above subsection, firstly we describe the derivations of such algebras.

Proposition 3.4. [11] *Any derivation of a filiform Leibniz algebra of the family $F_2(\beta_3, \beta_4, \dots, \beta_n, \gamma)$ has the following matrix form:*

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots & a_{n-2} & a_{n-1} & a_n \\ 0 & b_1 & 0 & 0 & \dots & 0 & -a_1 \gamma & b_n \\ 0 & 0 & 2a_0 & a_2 + a_1 \beta_3 & \dots & a_{n-3} + a_1 \beta_{n-2} & a_{n-2} + a_1 \beta_{n-1} & a_{n-1} + a_1 \beta_n \\ 0 & 0 & 0 & 3a_0 & \dots & a_{n-4} + 2a_1 \beta_{n-3} & a_{n-3} + 2a_1 \beta_{n-2} & a_{n-2} + 2a_1 \beta_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & (n-2)a_0 & a_2 + (n-3)a_1 \beta_3 & a_3 + (n-3)a_1 \beta_4 \\ 0 & 0 & 0 & 0 & \dots & 0 & (n-1)a_0 & a_2 + (n-2)a_1 \beta_3 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & na_0 \end{pmatrix},$$

where

$$\begin{aligned} \gamma(2b_1 - na_0) &= 0, & \beta_3(b_1 - 2a_0) &= 0, \\ \beta_k(b_1 - (k-1)a_0) &= \frac{k}{2} a_1 \sum_{j=4}^k \beta_{j-1} \beta_{k-j+3}, & 4 \leq k \leq n-1, \\ \beta_n(b_1 - (n-1)a_0) &= -a_1 \gamma + \frac{n}{2} a_1 \sum_{j=4}^n \beta_{j-1} \beta_{n-j+3}. \end{aligned}$$

Case $F_2(0, 0, \dots, 0, 0, 1)$ and n -odd.

From the relations of Proposition 3.4, it follows $b_1 = \frac{n}{2} a_0$ and $a_1 = 0$. Therefore, the number of nil-independent outer derivations of the algebra $F_2(0, 0, \dots, 0, 0, 1)$ is equal to one. According to Theorem 2.6, we conclude that any solvable Leibniz algebra whose nilradical is $F_2(0, 0, \dots, 0, 0, 1)$ has dimension $n + 2$.

Theorem 3.5. *Any solvable $(n+2)$ -dimensional (the case of odd n) Leibniz algebra with nilradical $F_2(0,0,\dots,0,0,1)$ is isomorphic to the following algebra:*

$$L_1 : \begin{cases} [e_0, e_0] = e_2, & [e_1, e_1] = e_n, & [e_0, x] = e_0, & [x, e_0] = -e_0, \\ [e_i, e_0] = e_{i+1}, & 2 \leq i \leq n-1, & [x, e_1] = -\frac{n}{2}e_1, & [e_1, x] = \frac{n}{2}e_1, & [e_i, x] = ie_i, & 2 \leq i \leq n. \end{cases}$$

Proof. Using Proposition 3.4, we have the products:

$$\begin{aligned} [e_0, e_0] &= e_2, & [e_1, e_1] &= e_n, & [e_0, x] &= a_0e_0 + \sum_{i=2}^n a_i e_i, \\ [e_i, e_0] &= e_{i+1}, & 2 \leq i \leq n-1, & [x, e_1] &= \sum_{i=0}^n \gamma_i e_i, & [e_1, x] &= \frac{n}{2}a_0e_1 + b_n e_n, \\ [x, e_0] &= \sum_{i=0}^n \mu_i e_i, & [e_i, x] &= ie_i + \sum_{j=i+1}^n a_{j+1-i} e_j, & 2 \leq i \leq n, \\ [x, x] &= \sum_{i=0}^n \delta_i e_i. \end{aligned}$$

Without loss of generality we can suppose $a_0 = 1$. It is easy to see that $\langle e_2, e_3, \dots, e_n \rangle \subseteq \text{Ann}_r(L)$. Hence, $[x, e_i] = 0$ for $2 \leq i \leq n$.

Let us take the following transformation of basis:

$$e'_0 = e_0 + \sum_{i=2}^n A_i e_i, \quad e'_1 = e_1, \quad e'_i = e_i + \sum_{k=i+1}^n A_{k-i+1} e_k, \quad 2 \leq i \leq n, \quad x' = x$$

with $A_2 = -a_2$, $A_i = \frac{1}{1-i} \left(a_i + \sum_{j=2}^{i-1} A_j a_{i-j+1} \right)$, $3 \leq i \leq n$. Then we obtain

$$\begin{aligned} [x', e'_0] &= [x, e_0] = \sum_{i=0}^n \mu_i e_i, \\ [e'_0, x'] &= e_0 + \sum_{i=2}^n a_i e_i + \sum_{i=2}^n A_i \left(ie_i + \sum_{j=i+1}^n a_{j+1-i} e_j \right) = e_0 + \sum_{i=2}^n a_i e_i + A_2(2e_2 + \sum_{j=3}^n a_{j-1} e_j) + \\ &A_3(3e_3 + \sum_{j=4}^n a_{j-2} e_j) + \dots + A_{n-2}((n-2)e_{n-2} + \sum_{j=n-1}^n a_{j-n+3} e_j) + A_{n-1}((n-1)e_{n-1} + a_2 e_n) + \\ &+ A_n(n e_n) = e_0 + (a_2 + 2A_2)e_2 + \sum_{i=3}^n (iA_i + a_i + \sum_{j=2}^{i-1} A_j a_{i-j+1}) e_i = e_0 + \sum_{i=2}^n A_i e_i = e'_0. \end{aligned}$$

Thus, we can assume $a_i = 0$ for $2 \leq i \leq n$.

Now, making the change $x' = x - \sum_{i=2}^{n-1} \mu_{i+1} e_i$ we obtain the family:

$$\begin{aligned} [e_0, e_0] &= e_2, & [e_1, e_1] &= e_n, & [e_0, x] &= e_0, \\ [e_i, e_0] &= e_{i+1}, & 2 \leq i \leq n-1, & [x, e_1] &= \sum_{i=0}^n \gamma_i e_i, & [e_1, x] &= \frac{n}{2}e_1 + b_n e_n, \\ [x, e_0] &= \mu_0 e_0 + \mu_1 e_1 + \mu_2 e_2, & [e_i, x] &= ie_i, & 2 \leq i \leq n, \\ [x, x] &= \sum_{i=0}^n \delta_i e_i. \end{aligned}$$

By setting $e'_1 = e_1 - \frac{2}{n} b_n e_n$ we get $b'_n = 0$.

The equalities

$$0 = [e_0, [x, x]] = [e_1, [x, x]] = [e_0, [e_0, x] + [x, e_0]] = [e_1, [e_0, x] + [x, e_0]] = [e_1, [e_1, x] + [x, e_1]]$$

derive

$$\delta_0 = \delta_1 = \mu_1 = \gamma_0 = 0, \quad \mu_0 = -1, \quad \gamma_1 = -\frac{n}{2}.$$

Applying Leibniz identity for the triples $\{x, e_0, e_1\}$, $\{x, x, e_0\}$ and $\{x, x, e_1\}$, we conclude

$$\gamma_i = 0, \quad 2 \leq i \leq n, \quad \delta_i = 0, \quad 2 \leq i \leq n-1, \quad \mu_2 = 0.$$

Finally, putting $x' = x - \frac{\delta_n}{n}e_n$, we obtain the family of algebras L_1 . \square

Case $F_2(0, 0, \dots, 0, \beta_{\frac{n+2}{2}}, 0, \dots, 0, 0, 1)$ and n -even.

Similarly, we get $a_0 = 1$, $b_1 = \frac{n}{2}$, $a_1 = 0$ and that any solvable Leibniz algebra whose nilradical is $F_2^1(0, 0, \dots, 0, \beta_{\frac{n+2}{2}}, 0, \dots, 0, 0, 1)$ has dimension $n + 2$.

Theorem 3.6. *Any solvable Leibniz algebra with nilradical $F_2^1(0, 0, \dots, 0, \beta_{\frac{n+2}{2}}, 0, \dots, 0, 0, 1)$ (the case of even n) is isomorphic to an algebra of the following family of algebras:*

$$L_2^{\beta_{\frac{n+2}{2}}} : \begin{cases} [e_0, e_0] = e_2, & [e_0, e_1] = \beta_{\frac{n+2}{2}} e_{\frac{n+2}{2}}, & [e_0, x] = e_0, \\ [e_i, e_0] = e_{i+1}, \quad 2 \leq i \leq n-1, & [e_1, e_1] = e_n, & [e_1, x] = \frac{n}{2} e_1, \\ [x, e_0] = -e_0, & [e_i, e_1] = \beta_{\frac{n+2}{2}} e_{\frac{n+2}{2}i}, \quad 2 \leq i \leq \frac{n}{2}, & [e_i, x] = i e_i, \quad 2 \leq i \leq n. \\ & [x, e_1] = -\frac{n}{2} e_1 - \beta_{\frac{n+2}{2}} e_{\frac{n}{2}}, \end{cases}$$

Proof. Using the previous arguments and Proposition 3.4, we obtain the multiplications:

$$\begin{aligned} [e_0, e_0] &= e_2, & [e_0, e_1] &= \beta_{\frac{n+2}{2}} e_{\frac{n+2}{2}}, & [e_0, x] &= e_0 + \sum_{i=2}^n a_i e_i, \\ [e_i, e_0] &= e_{i+1}, \quad 2 \leq i \leq n-1, & [e_1, e_1] &= e_n, & [e_1, x] &= \frac{n}{2} e_1 + b_n e_n, \\ [x, e_0] &= \sum_{i=0}^n \mu_i e_i, & [e_i, e_1] &= \beta_{\frac{n+2}{2}} e_{\frac{n+2}{2}i}, \quad 2 \leq i \leq \frac{n}{2}, & [e_i, x] &= i e_i + \sum_{j=i+1}^n a_{j+1-i} e_j, \quad 2 \leq i \leq n, \\ & & [x, e_1] &= \sum_{i=0}^n \gamma_i e_i, & [x, x] &= \sum_{i=0}^n \delta_i e_i. \end{aligned}$$

Taking the following transformation of basis:

$$e'_0 = e_0 + \sum_{i=2}^n A_i e_i, \quad e'_1 = e_1, \quad e'_i = e_i + \sum_{k=i+1}^n A_{k-i+1} e_k, \quad 2 \leq i \leq n, \quad x' = x$$

with

$$A_2 = -a_2, \quad A_i = \frac{1}{1-i} \left(a_i + \sum_{j=2}^{i-1} A_j a_{i-j+1} \right), \quad 3 \leq i \leq n,$$

we can assume that $a_i = 0$ for $2 \leq i \leq n$.

By setting

$$x' = x - \sum_{i=2}^{n-1} \mu_{i+1} e_i, \quad e'_1 = e_1 - \frac{2}{n} b_n e_n$$

we reduce the above multiplication to the following one:

$$\begin{aligned} [e_0, e_0] &= e_2, & [e_0, e_1] &= \beta_{\frac{n+2}{2}} e_{\frac{n+2}{2}}, & [e_0, x] &= e_0, \\ [e_i, e_0] &= e_{i+1}, \quad 2 \leq i \leq n-1, & [e_1, e_1] &= e_n, & [e_1, x] &= \frac{n}{2} e_1, \\ [x, e_0] &= \mu_0 e_0 + \mu_1 e_1 + \mu_2 e_2, & [e_i, e_1] &= \beta_{\frac{n+2}{2}} e_{\frac{n+2}{2}i}, \quad 2 \leq i \leq \frac{n}{2}, & [e_i, x] &= i e_i, \quad 2 \leq i \leq n, \\ & & [x, e_1] &= \sum_{i=0}^n \gamma_i e_i, & [x, x] &= \sum_{i=0}^n \delta_i e_i. \end{aligned}$$

From the equalities

$$\begin{aligned} [e_1, [x, x]] &= [e_0, [x, x]] = [e_1, [e_0, x] + [x, e_0]] = [e_0, [e_0, x] + [x, e_0]] = \\ &= [e_1, [e_1, x] + [x, e_1]] = [e_0, [e_1, x] + [x, e_1]] = 0 \end{aligned}$$

we derive

$$\delta_1 = \delta_0 = \mu_1 = \gamma_0 = 0, \quad \mu_0 = -1, \quad \gamma_1 = -\frac{n}{2}$$

Applying the Leibniz identity for the triples $\{x, e_0, x\}$, $\{x, e_0, e_1\}$ and $\{x, x, e_1\}$, we obtain $\mu_2 = 0$, $\delta_i = 0$ with $2 \leq i \leq n-1$, $\gamma_i = 0$ with $2 \leq i \leq n-1$, $i \neq \frac{n}{2}$ and $\gamma_{\frac{n}{2}} = -\beta_{\frac{n+2}{2}}$.

The following change: $x' = x - \frac{\delta_n}{n} e_n$ deduce $\delta'_n = 0$. \square

Case $F_2^j(0, 0, \dots, 0, \underbrace{1}_j, 0, \dots, 0, 0, 0)$, $n \geq 4$ and $3 \leq j \leq n$.

Similar to the above cases and using Proposition 3.4, we derive

$$b_1 = (j-1)a_0, \quad \begin{cases} a_1 = 0, & \text{if } 3 \leq j \leq \lfloor \frac{n+2}{2} \rfloor, \\ a_1, & \text{in other case.} \end{cases}$$

Similar to the previous cases, we can suppose $a_0 = 1$ and $b_1 = (j-1)$. Therefore, the number of nil-independent outer derivations of the algebras $F_2^j(0, 0, \dots, 0, \underbrace{1}_j, 0, \dots, 0, 0, 0)$ is equal to one. We

use F_2^j to denote $F_2^j(0, 0, \dots, 0, \underbrace{1}_j, 0, \dots, 0, 0, 0)$.

Theorem 3.7. *Any solvable $(n+2)$ -dimensional Leibniz algebra with nilradical F_2^j with $3 \leq j \leq n$ is isomorphic to the following algebra:*

$$L_3^j : \begin{cases} [e_0, e_0] = e_2, & [e_0, e_1] = e_j, & [e_0, x] = e_0, \\ [e_i, e_0] = e_{i+1}, \quad 2 \leq i \leq n-1, & [e_i, e_1] = e_{j+i-1}, \quad 2 \leq i \leq n-1-j, & [e_1, x] = (j-1)e_1, \\ [x, e_0] = -e_0, & [x, e_1] = -(j-1)e_1 - e_{j-1}, & [e_i, x] = ie_i, \quad 2 \leq i \leq n. \end{cases}$$

Proof. The proof is carried out by applying arguments used in Theorems 3.5 and 3.6. \square

3.3. Solvable Leibniz algebras with nilradical a non-characteristically nilpotent algebra of the family $F_3(\theta_1, \theta_2, \theta_3)$.

Let L be a filiform Leibniz algebra from the family $F_3(\theta_1, \theta_2, \theta_3)$ and let $\{e_0, e_1, \dots, e_n\}$ be a basis. The following proposition describes the derivations of such algebras.

Proposition 3.8. [11] *A derivation d of a filiform Leibniz algebra of the family $F_3(\theta_1, \theta_2, \theta_3)$ has the following form:*

$$\begin{aligned} d(e_0) &= \sum_{i=0}^n a_i e_i, & d(e_1) &= \sum_{i=1}^n b_i e_i, \\ d(e_i) &= ((i-1)a_0 + b_1)e_i + \sum_{j=i+1}^{n-1} b_{j-i+1} e_j + (b_{n-i+1} + (-1)^{i-1} \alpha a_{n-i+1}) e_n, \\ d(e_n) &= ((n-1)a_0 + b_1 + \alpha a_1) e_n \end{aligned}$$

with the following restrictions:

$$\begin{aligned} \theta_1((n-3)a_0 + b_1) &= a_1 \theta_2, \\ 2a_1 \theta_3 &= (n-2)a_0 \theta_2, \\ \theta_3((n-1)a_0 - b_1) &= 0. \end{aligned}$$

We consider the case of solvable Leibniz algebras with non-Lie filiform nilradical of the family $F_3(\theta_1, \theta_2, \theta_3)$ bellow.

Theorem 3.9. *There is not any solvable Leibniz algebra whose nilradical is a non-characteristically filiform non-Lie algebra of the family $F_3(\theta_1, \theta_2, \theta_3)$.*

Proof. Let us consider firstly the **Case** $F_3^1(1, 0, 0)$. Proposition 3.8 leads to $b_1 = (3-n)a_0$ and we can suppose $a_0 = 1$ making the change $x' = \frac{1}{a_0}x$. Thus, we have the following products:

$$\begin{aligned}
[e_i, e_0] &= -[e_0, e_i] = e_{i+1}, & 1 \leq i \leq n-1, \\
[e_0, e_0] &= e_n, \\
[e_i, e_{n-i}] &= -[e_{n-i}, e_i] = (-1)^i \alpha e_n, & 1 \leq i \leq n-1, \\
[e_0, x] &= e_0 + \sum_{i=1}^n a_i e_i, \\
[e_1, x] &= (3-n)e_1 + \sum_{i=2}^n b_i e_i, \\
[e_i, x] &= ((i-1) + b_1)e_i + \sum_{j=i+1}^{n-1} b_{j-i+1} e_j + (b_{n-i+1} + (-1)^{i-1} \alpha a_{n-i+1})e_n, & 2 \leq i \leq n-1 \\
[e_n, x] &= ((n-1) + b_1 + \alpha a_1)e_n, \\
[x, e_0] &= \sum_{i=0}^n \beta_i e_i, \\
[x, e_1] &= \sum_{i=0}^n \gamma_i e_i, \\
[x, x] &= \sum_{i=0}^n \delta_i e_i.
\end{aligned}$$

It is easy to check that by the change $x' = x + \sum_{i=1}^{n-1} a_{i+1} e_i$ we can suppose $a_i = 0$ with $2 \leq i \leq n$.

Applying the Leibniz identity for the elements $\{e_2, e_{n-2}, x\}$ and $\{e_0, e_0, x\}$, we get a contradiction with $\alpha = 1$. Thus, $\alpha = 0$.

Taking the following basis transformation:

$$e'_0 = e_0 + \frac{a_1}{n-2} e_1, \quad e'_i = e_i, \quad 1 \leq i \leq n, \quad x' = x + \frac{a_1}{n-2} \sum_{i=1}^{n-1} b_{i+1} e_i,$$

we can suppose $a_1 = 0$.

Putting

$$e'_0 = e_0, \quad e'_i = e_i + \sum_{j=i+1}^n A_{j-i+1} e_j, \quad 1 \leq i \leq n,$$

with

$$A_2 = -b_2, \quad A_i = \frac{1}{1-i} (b_i + \sum_{j=2}^{i-1} A_j b_{i-j+1}), \quad 3 \leq i \leq n,$$

one can assume $b_i = 0$ for $2 \leq i \leq n$.

Let us resume the products of the family

$$\begin{aligned}
[e_i, e_0] &= -[e_0, e_i] = e_{i+1}, & 1 \leq i \leq n-1, \\
[e_0, e_0] &= e_n, \\
[e_0, x] &= e_0, & [x, e_0] &= \sum_{i=0}^n \beta_i e_i, \\
[e_i, x] &= (i-n+2)e_i, & 1 \leq i \leq n, & [x, e_1] &= \sum_{i=0}^n \gamma_i e_i, \\
[x, x] &= \sum_{i=0}^n \delta_i e_i.
\end{aligned}$$

Considering Leibniz identity for the elements of the form

$$\{e_0, x, e_0\}, \quad \{e_0, x, e_1\}, \quad \{e_1, x, e_1\}, \quad \{e_1, x, e_0\},$$

we obtain that

$$\beta_0 = -1, \quad \beta_i = 0, \quad 1 \leq i \leq n-2, \quad \gamma_0 = 0, \quad \gamma_1 = n-3, \quad \gamma_i = 0, \quad 2 \leq i \leq n-1.$$

Using the induction method, we get $[x, e_i] = -(i-n+2)e_i$, with $2 \leq i \leq n$.

From the equality $0 = [x, [e_0, e_0]]$, we have $[x, e_n] = 0$, but this is a contradiction, because $[x, e_n] = -2e_n$. Therefore, there are not any solvable Leibniz algebras with $F_3^1(1, 0, 0)$ -nilradical.

Similar study of **Case** $F_3^2(0, 1, 0)$ and **Case** $F_3^2(0, 0, 1)$ leads to non-existence of solvable Leibniz algebras with nilradicals $F_3^2(0, 1, 0)$ and $F_3^2(0, 0, 1)$. \square

4. SOLVABLE LEIBNIZ ALGEBRAS WITH FILIFORM LIE NILRADICAL

In this section we study solvable Leibniz algebras whose nilradical is a filiform Lie algebra. Since we consider non-nilpotent solvable Leibniz algebras, it is sufficient to consider non-characteristically nilpotent filiform Lie nilradicals. In this section we restrict ourselves to the study of the families $A_{n+1}^r(\alpha_1, \dots, \alpha_t)$, $B_{n+1}^r(\alpha_1, \dots, \alpha_t)$ of Theorem 2.7, because the other two algebras of Theorem 2.7 have been already studied in [8].

Case $A_{n+1}^r(\alpha_1, \dots, \alpha_t)$ **with** $1 \leq r \leq n-3$ **and** $t = \lfloor \frac{n-r-1}{2} \rfloor$.

Below we present some description of the derivations of the family of algebras $A_{n+1}^r(\alpha_1, \dots, \alpha_t)$.

Proposition 4.1. *Any derivation of a filiform Lie algebra of the family $A_{n+1}^r(\alpha_1, \dots, \alpha_t)$ with $1 \leq r \leq n-3$ and $t = \lfloor \frac{n-r-1}{2} \rfloor$ has the form:*

$$\begin{aligned} d(e_0) &= \sum_{i=0}^n a_i e_i, \\ d(e_1) &= (1+r)a_0 e_1 + \sum_{i=2}^n b_i e_i, \\ d(e_i) &= (i+r)a_0 e_i + \sum_{j=i+1}^n (*) e_j, \quad 2 \leq i \leq n, \end{aligned}$$

where $\{e_0, e_1, \dots, e_n\}$ is a basis of the family A_{n+1}^r .

Proof. Let us denote

$$d(e_0) = \sum_{i=0}^n a_i e_i, \quad d(e_1) = \sum_{i=0}^n b_i e_i.$$

Using the induction method and the properties of derivation, we establish

$$d(e_i) = ((i-1)a_0 + b_1)e_i + \sum_{j=i+1}^n (*) e_j, \quad 2 \leq i \leq n.$$

From the equalities

$$0 = d([e_1, e_{n-1}]) = [d(e_1), e_{n-1}] + [e_1, d(e_{n-1})] = b_0 e_n$$

we get $b_0 = 0$.

- If $\alpha_1 \neq 0$, then from the equality $d([e_1, e_2]) = d(\alpha_1 e_{r+3})$ we have $b_1 = (r+1)a_0$.
- If $\alpha_1 = 0$, then there exists i , $2 \leq i \leq t$ such that $\alpha_i \neq 0$. The equality $d([e_i, e_{i+1}]) = d(\alpha_i e_{2i+1+r})$ implies $b_1 = (r+1)a_0$.

Thus, we obtain the form of derivation which is asserted in the proposition. \square

From Proposition 4.1, we conclude that the number of nil-independent outer derivations of the algebra $A_{n+1}^r(\alpha_1, \dots, \alpha_t)$ is equal to one. Consequently, according to Theorem 2.6, any solvable Leibniz algebra whose nilradical is $A_{n+1}^r(\alpha_1, \dots, \alpha_t)$ has dimension $n+2$.

In this section we use similar arguments to those from the above section.

Theorem 4.2. *Any solvable Leibniz algebra with nilradical $A_{n+1}^r(\alpha_1, \dots, \alpha_t)$ is isomorphic to the following family of Lie algebras:*

$$\begin{aligned} [e_0, e_i] &= e_{i+1}, \quad 1 \leq i \leq n-1, \\ [e_i, e_j] &= \left(\sum_{k=i}^j (-1)^{k-i} \alpha_k \binom{j-k-1}{k-i} \right) e_{i+j+r}, \quad 1 \leq i < j \leq n-2, \quad i+j+r \leq n, \\ [e_0, x] &= e_0 + a_1 e_1, \\ [e_1, x] &= (1+r)e_1 + \sum_{i=2}^n b_i e_i, \\ [e_2, x] &= (2+r)e_2 + \sum_{i=3}^n b_{i-1} e_i, \\ [e_i, x] &= (i+r)e_i + \sum_{j=i+1}^{i-1+r} b_{j-i+1} e_j + \left(b_{1+r} + a_1 \left(\sum_{k=2}^{i-1} \left(\sum_{s=1}^t (-1)^{s-1} \alpha_s \binom{k-s-1}{s-1} \right) \right) \right) e_{i+r} \\ &\quad + \sum_{j=i+1+r}^n b_{j-i+1} e_j \quad \text{with } 3 \leq i \leq n-r, \\ [e_i, x] &= (i+r)e_i + \sum_{j=i+1}^n b_{j-i+1} e_j, \quad n-r+1 \leq i \leq n. \end{aligned}$$

Proof. From Proposition 4.1 we have the following products:

$$\begin{aligned} [e_0, e_i] &= e_{i+1}, & 1 \leq i \leq n-1, \\ [e_i, e_j] &= \left(\sum_{k=i}^t (-1)^{k-i} \alpha_k \binom{j-k-1}{k-i} \right) e_{i+j+r}, & 1 \leq i < j \leq n-2, \ i+j+r \leq n, \\ [e_0, x] &= \sum_{i=0}^n a_i e_i, \\ [e_1, x] &= (1+r)a_0 e_1 + \sum_{i=2}^n b_i e_i, \\ [e_i, x] &= (i+r)a_0 e_i + \sum_{j=i+1}^n (*) e_j, & 2 \leq i \leq n. \end{aligned}$$

Since $a_0 \neq 0$, by scaling of basis element x , we can suppose $a_0 = 1$. The change $x' = x - \sum_{i=1}^{n-1} a_{i+1} e_i$ allow us to suppose $a_i = 0$ for $2 \leq i \leq n$.

Thus, we have

$$\begin{aligned} [e_0, e_i] &= e_{i+1}, & 1 \leq i \leq n-1, \\ [e_i, e_j] &= \left(\sum_{k=i}^t (-1)^{k-i} \alpha_k \binom{j-k-1}{k-i} \right) e_{i+j+r}, & 1 \leq i < j \leq n-2, \ i+j+r \leq n, \\ [e_0, x] &= e_0 + a_1 e_1, \\ [e_1, x] &= (1+r)e_1 + \sum_{i=2}^n b_i e_i, \end{aligned}$$

Using Jacobi identity and the induction method, we obtain the expressions of the products $[e_i, x]$ with $2 \leq i \leq n$, which complete the proof of theorem. \square

Proposition 4.3. *There are not any solvable non-Lie Leibniz algebras with nilradical $A_{n+1}^r(\alpha_1, \dots, \alpha_t)$.*

Proof. From Proposition 4.1 and applying similar arguments to the ones used in the proof of Theorem 4.2, we have

$$\begin{aligned} [e_0, e_i] &= e_{i+1}, \quad 1 \leq i \leq n-1, \\ [e_i, e_j] &= \left(\sum_{k=i}^t (-1)^{k-i} \alpha_k \binom{j-k-1}{k-i} \right) e_{i+j+r}, \quad 1 \leq i < j \leq n-2, \ i+j+r \leq n, \\ [e_0, x] &= e_0 + a_1 e_1, \\ [e_1, x] &= (1+r)e_1 + \sum_{i=2}^n b_i e_i, \\ [e_2, x] &= (2+r)e_2 + \sum_{i=3}^n b_{i-1} e_i, \\ [e_i, x] &= (i+r)e_i + \sum_{j=i+1}^{i-1+r} b_{j-i+1} e_j + \left(b_{1+r} + a_1 \left(\sum_{k=2}^{i-1} \left(\sum_{s=1}^t (-1)^{s-1} \alpha_s \binom{k-s-1}{s-1} \right) \right) \right) e_{i+r} + \\ &\quad + \sum_{j=i+1+r}^n b_{j-i+1} e_j \quad \text{with } 3 \leq i \leq n-r, \\ [e_i, x] &= (i+r)e_i + \sum_{j=i+1}^n b_{j-i+1} e_j, \quad n-r+1 \leq i \leq n. \end{aligned}$$

Let us denote

$$[x, e_0] = \sum_{i=0}^n \beta_i e_i, \quad [x, e_1] = \sum_{i=0}^n \gamma_i e_i, \quad [x, x] = \sum_{i=0}^n \delta_i e_i.$$

Applying the Leibniz identity for the elements

$$\{e_0, x, x\}, \quad \{e_1, x, x\}, \quad \{e_0, x, e_0\}, \quad \{e_1, x, e_0\}$$

we derive

$$\delta_i = 0, \quad 0 \leq i \leq n-1, \quad \beta_i = 0, \quad 2 \leq i \leq n-1, \quad \beta_1 = -a_1, \quad \beta_0 = -1.$$

By means of the Leibniz identity for the products and the induction method, we compute the following products:

$$\begin{aligned} [x, e_2] &= -(1 - \gamma_1)e_2 + \sum_{i=3}^n \gamma_{i-1}e_i, \\ [x, e_i] &= -(i-1 - \gamma_1)e_i + \sum_{j=i+1}^{i-1+r} \gamma_{j-i+1}e_j + \left(\gamma_{1+r} + a_1 \left(\sum_{k=2}^{i-1} \left(\sum_{s=1}^t (-1)^{s-1} \alpha_s \binom{k-s-1}{s-1} \right) \right) \right) e_{i+r} + \\ &\quad + \sum_{j=i+1+r}^n \gamma_{j-i+1}e_j, \quad 3 \leq i \leq n-r, \\ [x, e_i] &= -(i-1 - \gamma_1)e_i + \sum_{j=i+1}^n \gamma_{j-i+1}e_j, \quad n-r+1 \leq i \leq n. \end{aligned}$$

Since $[e_1, x] + [x, e_1] \in \text{Ann}_r(L)$, we get

$$[e_{n-1}, [e_1, x] + [x, e_1]] = [e_0, [e_1, x] + [x, e_1]] = [x, [e_1, x] + [x, e_1]] = 0,$$

from which we obtain

$$\gamma_0 = 0, \quad \gamma_1 = -1 - r, \quad \gamma_i = -b_i, \quad 2 \leq i \leq n.$$

Similarly, from $[x, [e_0, x] + [x, e_0]] = 0$ we conclude $\beta_n = 0$.

Considering the equality $0 = [x, [x, x]]$ we get $\delta_n = 0$. Thus, we obtain a Lie algebra. \square

Case $B_{n+1}^r(\alpha_1, \dots, \alpha_t)$ with $1 \leq r \leq n-4$, $t = \lfloor \frac{n-r-2}{2} \rfloor$ and $n = 2m+1$.

The study of this case is similar to previous case.

Proposition 4.4. *Any derivation of a filiform Lie algebra of the family $B_{n+1}^r(\alpha_1, \dots, \alpha_t)$ with $1 \leq r \leq n-4$, $t = \lfloor \frac{n-r-2}{2} \rfloor$ and $n = 2m+1$ has the form:*

$$\begin{aligned} d(e_0) &= a_0e_0 + \sum_{i=2}^n a_i e_i, \\ d(e_1) &= (1+r)a_0e_1 + \sum_{i=2}^n b_i e_i, \\ d(e_i) &= (i+r)a_0e_i + \sum_{j=i+1}^n (*)e_j, \quad 2 \leq i \leq n-1, \\ d(e_n) &= (n+2r)a_0e_n, \end{aligned}$$

where $\{e_0, e_1, \dots, e_n\}$ is a basis of the family B_{n+1}^r .

Proof. In an analogous way to the proof of Proposition 4.1. \square

From Proposition 4.4 and Theorem 2.6, we conclude that any solvable Leibniz algebra whose nilradical is $B_{n+1}^r(\alpha_1, \dots, \alpha_t)$ has dimension $n+2$.

Theorem 4.5. *Any solvable Leibniz algebra with nilradical $B_{n+1}^r(\alpha_1, \dots, \alpha_t)$ is isomorphic to an algebra of the following family of Lie algebras:*

$$\begin{aligned} [e_0, e_i] &= e_{i+1}, & 1 \leq i \leq n-2, \\ [e_i, e_{n-i}] &= (-1)^i e_n, & 1 \leq i \leq n-1, \\ [e_i, e_j] &= \left(\sum_{k=i}^t (-1)^{k-i} \alpha_k \binom{j-k-1}{k-i} \right) e_{i+j+r}, & 1 \leq i, j \leq n-1, \quad i+j+r \leq n-1, \\ [e_0, x] &= e_0, \\ [e_1, x] &= (1+r)e_1 + \sum_{i=2}^{n-1} b_i e_i, \\ [e_i, x] &= (i+r)e_i + \sum_{j=i+1}^{n-1} b_{j-i+1} e_j, & 2 \leq i \leq n-1, \\ [e_n, x] &= (n+2r)e_n. \end{aligned}$$

Proof. The proof is similar to that of Theorem 4.2. \square

Proposition 4.6. *There are not any solvable non-Lie Leibniz algebras with nilradical $B_{n+1}^r(\alpha_1, \dots, \alpha_t)$.*

Proof. The proof is similar to that of Proposition 4.3. \square

From Propositions 4.3 and 4.6, we resume that any solvable Leibniz algebra whose nilradical is either $A_{n+1}^r(\alpha_1, \dots, \alpha_t)$ or $B_{n+1}^r(\alpha_1, \dots, \alpha_t)$ is a Lie algebra.

Conjecture.

(i) By the following transformation basis of an algebra of the family from Theorem 4.2:

$$e'_0 = e_0, \quad e'_1 = e_1 + \sum_{i=2}^{n-1} A_i e_i, \quad e'_i = e_i + \sum_{j=i+1}^n A_{j-i+1} e_j, \quad 2 \leq i \leq n, \quad (*)$$

with

$$A_2 = -b_2, \quad A_i = \frac{1}{1-i} (b_i + \sum_{j=2}^{i-1} A_j b_{i-j+1}), \quad 3 \leq i \leq n,$$

we can eliminate the parameters $b_i = 0$ with $2 \leq i \leq n$

(ii) By the transformation of basis (*), an algebra of the family from Theorem 4.5 with

$$A_2 = -b_2, \quad A_3 = \frac{b_2^2}{2}, \quad A_{2k} = \frac{1}{1-2k} (b_{2k} + \sum_{j=2}^k A_{2j-1} b_{2k-2j+2}), \quad 2 \leq k \leq \frac{n-1}{2},$$

$$A_{2k+1} = -\frac{1}{2k} (\sum_{j=1}^k A_{2j} b_{2k-2j+2}), \quad 2 \leq k \leq \frac{n-3}{2},$$

we can eliminate the parameters $b_i = 0$ with $2 \leq i \leq n$.

The correctness of Conjecture was checked for fixed low dimensions by program Mathematica. However, we could not generalize the calculation from low dimensions for any finite dimension because of the great number of complex calculations needed.

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