A RESULT RELATED TO HERSTEIN THEOREM

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Abstract. The purpose of this paper is to prove the following result. Let R be a prime ring of characteristic different from two and let $D : R \to R$ be an additive mapping satisfying the relation $2D(x^3) = D(x^2)x + x^2D(x) + D(x)x^2 + xD(x^2)$ for all $x \in R$. In this case D is a derivation. This result is related to a classical result of Herstein, which states that any Jordan derivation on a prime ring of characteristic different from two is a derivation.

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This research is a continuation of the recent work of Vukman [15]. Throughout R will represent an associative ring with center $Z(R)$. As usual we write $[x, y]$ for $xy - yx$. Given an integer $n \geq 2$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. We denote by Q_{mr}, Q_r, Q_s and C the maximal right ring of quotients, the right ring of quotients, the symmetric Martindale ring of quotients and the extended centroid of a semiprime ring R, respectively. For the explanation of Q_{mr}, Q_r, Q_s and C we refer the reader to [1]. An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists $a \in R$ such that $D(x) = [x, a]$ for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [13] asserts that any Jordan derivation on a prime ring of characteristic different from two is a derivation. A brief proof of Herstein's result can be found in [6]. Cusack [7] generalized Herstein's result to 2-torsion free semiprime rings (see also [2] for an alternative proof). In last few decades a lot of results on certain identities with derivations on prime and semiprime rings has been obtained (see for example [2, 3, 8, 9, 10, 15]).

Brešar [3] has proved the following result.

Theorem 1. *Let R be a 2-torsion free semiprime ring and let $D : R \to R$ be an additive mapping satisfying the relation*

$$D(xyx) = D(x)yx + xD(y)x + xyD(x). \tag{1}$$

In this case D is a derivation.

An additive mapping $D : R \to R$, where R is an arbitrary ring, satisfying the relation (1) is called a Jordan triple derivation. One can easily prove that any Jordan derivation on an arbitrary 2-torsion free ring is a Jordan triple derivation, which means that Theorem 1 generalizes Cusack's generalization of Herstein theorem. Motivated by Theorem 1, Vukman [15] has recently proved the following result.

Theorem 2. *Let R be a 2-torsion free semiprime ring and let $D : R \to R$ be an additive mapping. Suppose that either*

$$D(xyx) = D(xy)x + xyD(x) \tag{2}$$

or

$$D(xyx) = D(x)yx + xD(yx) \tag{3}$$

holds for all pairs $x, y \in R$. In both cases D is a derivation.

Vukman [15] conjectured that in case there exists an additive mapping $D : R \to R$, where R is a 2-torsion free semiprime ring, satisfying the relation

$$2D(xyx) = D(xy)x + xyD(x) + D(x)yx + xD(yx)$$

for all pairs $x, y \in R$, then D is a derivation.

It is our aim in this paper to prove the following result, which is related to the conjecture we have just mentioned above.

Theorem 3. *Let R be a prime ring of characteristic different from two and let $D : R \to R$ be an additive mapping satisfying the relation*

$$2D(x^3) = D(x^2)x + x^2D(x) + D(x)x^2 + xD(x^2) \tag{4}$$

for all $x \in R$. In this case D is a derivation.

Any Jordan derivation $D : R \to R$ satisfies the relation

$$D(xy + yx) = D(x)y + xD(y) + D(y)x + yD(x)$$

for all pairs $x, y \in R$. The substitution $y = x^2$ in the relation above gives the relation (4), which means, that Theorem 3 generalizes Herstein theorem. In the proof of Theorem 3 we shall use as the main tool the theory of functional identities (Brešar-Beidar-Chebotar theory). The theory of functional identities considers set-theoretic mappings on rings that satisfy some identical relations. When treating such relations one usually concludes that the form of the maps involved can be described, unless the ring is very special. We refer the reader to [4] for an introductory account on functional identities and to [5] for full treatment of this theory.

Let R be an algebra over a commutative ring ϕ and let

$$p(x_1,x_2,x_3)=\sum_{\pi\in S_3}x_{\pi(1)}x_{\pi(2)}x_{\pi(3)} \tag{5}$$

be a fixed multilinear polynomial in noncommuting indeterminates x_i over ϕ. Here S_3 stands for the symmetric group of order 3. Let $\mathcal{L}$ be a subset of R closed under p, i.e. $p(\bar{x}_3)\in\mathcal{L}$ for all $x_1,x_2,x_3\in\mathcal{L}$, where $\bar{x}_3=(x_1,x_2,x_3)$. We shall consider a mapping $D:\mathcal{L}\to R$ satisfying

$$\begin{aligned} 2D(p(\bar{x}_3)) &= \sum_{\pi\in S_3}D(x_{\pi(1)}x_{\pi(2)})x_{\pi(3)}+\sum_{\pi\in S_3}x_{\pi(1)}x_{\pi(2)}D(x_{\pi(3)}) \\ &+\sum_{\pi\in S_3}D(x_{\pi(1)})x_{\pi(2)}x_{\pi(3)}+\sum_{\pi\in S_3}x_{\pi(1)}D(x_{\pi(2)}x_{\pi(3)}) \end{aligned} \tag{6}$$

for all $x_1,x_2,x_3\in\mathcal{L}$.

For the proof of Theorem 3 we need Theorem 4 which might be of independent interest.

Theorem 4. *Let $\mathcal{L}$ be a 6-free Lie subring of R closed under p. If $D:\mathcal{L}\to R$ is an additive mapping satisfying (6), then D is a derivation.*

Proof. For any $a\in R$ and $\bar{x}_3\in\mathcal{L}^3$ we have

$$[p(\bar{x}_3),a]=p([x_1,a],x_2,x_3)+p(x_1,[x_2,a],x_3)+p(x_1,x_2,[x_3,a])$$

and therefore

$$\begin{aligned} 2D\left[p(\bar{x}_3),a\right] &= \sum_{\pi\in S_3}D\left[x_{\pi(1)}x_{\pi(2)},a\right]x_{\pi(3)}+\sum_{\pi\in S_3}D(x_{\pi(1)}x_{\pi(2)})\left[x_{\pi(3)},a\right] \\ &+ \sum_{\pi\in S_3}\left[x_{\pi(1)}x_{\pi(2)},a\right]D(x_{\pi(3)})+\sum_{\pi\in S_3}x_{\pi(1)}x_{\pi(2)}D\left[x_{\pi(3)},a\right] \\ &+ \sum_{\pi\in S_3}D\left[x_{\pi(1)},a\right]x_{\pi(2)}x_{\pi(3)}+\sum_{\pi\in S_3}D(x_{\pi(1)})\left[x_{\pi(2)}x_{\pi(3)},a\right] \\ &+ \sum_{\pi\in S_3}\left[x_{\pi(1)},a\right]D(x_{\pi(2)}x_{\pi(3)})+\sum_{\pi\in S_3}x_{\pi(1)}D\left[x_{\pi(2)}x_{\pi(3)},a\right]. \end{aligned}$$

In particular, we have

$$\begin{aligned} & 2D\left[p(\bar{x}_3),p(\bar{y}_3)\right] \\ =& \sum_{\pi\in S_3}D\left[x_{\pi(1)}x_{\pi(2)},p(\bar{y}_3)\right]x_{\pi(3)}+\sum_{\pi\in S_3}D(x_{\pi(1)}x_{\pi(2)})\left[x_{\pi(3)},p(\bar{y}_3)\right] \\ +& \sum_{\pi\in S_3}\left[x_{\pi(1)}x_{\pi(2)},p(\bar{y}_3)\right]D(x_{\pi(3)})+\sum_{\pi\in S_3}x_{\pi(1)}x_{\pi(2)}D\left[x_{\pi(3)},p(\bar{y}_3)\right] \\ +& \sum_{\pi\in S_3}D\left[x_{\pi(1)},p(\bar{y}_3)\right]x_{\pi(2)}x_{\pi(3)}+\sum_{\pi\in S_3}D(x_{\pi(1)})\left[x_{\pi(2)}x_{\pi(3)},p(\bar{y}_3)\right] \\ +& \sum_{\pi\in S_3}\left[x_{\pi(1)},p(\bar{y}_3)\right]D(x_{\pi(2)}x_{\pi(3)})+\sum_{\pi\in S_3}x_{\pi(1)}D\left[x_{\pi(2)}x_{\pi(3)},p(\bar{y}_3)\right]. \end{aligned} \tag{7}$$

It is easy to verify that

$$
\begin{aligned}
& f(x_{\pi(1)}x_{\pi(2)}, p(\bar{y}_3)) = 2D\left[x_{\pi(1)}x_{\pi(2)}, p(\bar{y}_3)\right] = -2D\left[p(\bar{y}_3), x_{\pi(1)}x_{\pi(2)}\right] \\
= & \sum_{\sigma\in S_3} D\left[x_{\pi(1)}x_{\pi(2)}, y_{\sigma(1)}y_{\sigma(2)}\right] y_{\sigma(3)} + \sum_{\sigma\in S_3} D(y_{\sigma(1)}y_{\sigma(2)})\left[x_{\pi(1)}x_{\pi(2)}, y_{\sigma(3)}\right] \\
+ & \sum_{\sigma\in S_3} \left[x_{\pi(1)}x_{\pi(2)}, y_{\sigma(1)}y_{\sigma(2)}\right] D(y_{\sigma(3)}) + \sum_{\sigma\in S_3} y_{\sigma(1)}y_{\sigma(2)}D\left[x_{\pi(1)}x_{\pi(2)}, y_{\sigma(3)}\right] \\
+ & \sum_{\sigma\in S_3} D\left[x_{\pi(1)}x_{\pi(2)}, y_{\sigma(1)}\right] y_{\sigma(2)}y_{\sigma(3)} + \sum_{\sigma\in S_3} D(y_{\sigma(1)})\left[x_{\pi(1)}x_{\pi(2)}, y_{\sigma(2)}y_{\sigma(3)}\right] \\
+ & \sum_{\sigma\in S_3} \left[x_{\pi(1)}x_{\pi(2)}, y_{\sigma(1)}\right] D(y_{\sigma(2)}y_{\sigma(3)}) + \sum_{\sigma\in S_3} y_{\sigma(1)}D\left[x_{\pi(1)}x_{\pi(2)}, y_{\sigma(2)}y_{\sigma(3)}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
& f(x_{\pi(3)}, p(\bar{y}_3)) = 2D\left[x_{\pi(3)}, p(\bar{y}_3)\right] = -2D\left[p(\bar{y}_3), x_{\pi(3)}\right] \\
= & \sum_{\sigma\in S_3} D\left[x_{\pi(3)}, y_{\sigma(1)}y_{\sigma(2)}\right] y_{\sigma(3)} + \sum_{\sigma\in S_3} D(y_{\sigma(1)}y_{\sigma(2)})\left[x_{\pi(3)}, y_{\sigma(3)}\right] \\
+ & \sum_{\sigma\in S_3} \left[x_{\pi(3)}, y_{\sigma(1)}y_{\sigma(2)}\right] D(y_{\sigma(3)}) + \sum_{\sigma\in S_3} y_{\sigma(1)}y_{\sigma(2)}D\left[x_{\pi(3)}, y_{\sigma(3)}\right] \\
+ & \sum_{\sigma\in S_3} D\left[x_{\pi(3)}, y_{\sigma(1)}\right] y_{\sigma(2)}y_{\sigma(3)} + \sum_{\sigma\in S_3} D(y_{\sigma(1)})\left[x_{\pi(3)}, y_{\sigma(2)}y_{\sigma(3)}\right] \\
+ & \sum_{\sigma\in S_3} \left[x_{\pi(3)}, y_{\sigma(1)}\right] D(y_{\sigma(2)}y_{\sigma(3)}) + \sum_{\sigma\in S_3} y_{\sigma(1)}D\left[x_{\pi(3)}, y_{\sigma(2)}y_{\sigma(3)}\right].
\end{aligned}
$$

In exactly the same way we obtain results for $f(x_{\pi(1)}, p(\bar{y}_3))$ and $f(x_{\pi(2)}x_{\pi(3)}, p(\bar{y}_3))$. Using the last four relations in (7) we arrive at

$$
\begin{aligned}
& 4D\left[p(\bar{x}_3), p(\bar{y}_3)\right] \\
= & \sum_{\pi\in S_3} f(x_{\pi(1)}x_{\pi(2)}, p(\bar{y}_3))x_{\pi(3)} + \sum_{\pi\in S_3} 2D(x_{\pi(1)}x_{\pi(2)})\left[x_{\pi(3)}, p(\bar{y}_3)\right] \\
+ & \sum_{\pi\in S_3} 2\left[x_{\pi(1)}x_{\pi(2)}, p(\bar{y}_3)\right] D(x_{\pi(3)}) + \sum_{\pi\in S_3} x_{\pi(1)}x_{\pi(2)}f(x_{\pi(3)}, p(\bar{y}_3)) \\
+ & \sum_{\pi\in S_3} f(x_{\pi(1)}, p(\bar{y}_3))x_{\pi(2)}x_{\pi(3)} + \sum_{\pi\in S_3} 2D(x_{\pi(1)})\left[x_{\pi(2)}x_{\pi(3)}, p(\bar{y}_3)\right] \\
+ & \sum_{\pi\in S_3} 2\left[x_{\pi(1)}, p(\bar{y}_3)\right] D(x_{\pi(2)}x_{\pi(3)}) + \sum_{\pi\in S_3} x_{\pi(1)}f(x_{\pi(2)}x_{\pi(3)}, p(\bar{y}_3)).
\end{aligned} \tag{8}
$$

Note that also

$$
\begin{aligned}
(9)\qquad & 4D\left[p(\bar{x}_3), p(\bar{y}_3)\right] = -4D\left[p(\bar{y}_3), p(\bar{x}_3)\right] \\
= & \sum_{\sigma\in S_3} -f(y_{\sigma(1)}y_{\sigma(2)}, p(\bar{x}_3))y_{\sigma(3)} + \sum_{\sigma\in S_3} 2D(y_{\sigma(1)}y_{\sigma(2)})\left[p(\bar{x}_3), y_{\sigma(3)}\right] \\
+ & \sum_{\sigma\in S_3} 2\left[p(\bar{x}_3), y_{\sigma(1)}y_{\sigma(2)}\right] D(y_{\sigma(3)}) - \sum_{\pi\in S_3} y_{\sigma(1)}y_{\sigma(2)}f(y_{\sigma(3)}, p(\bar{x}_3)) \\
- & \sum_{\sigma\in S_3} f(y_{\sigma(1)}, p(\bar{x}_3))y_{\sigma(2)}y_{\sigma(3)} + \sum_{\sigma\in S_3} 2D(y_{\sigma(1)})\left[p(\bar{x}_3), y_{\sigma(2)}y_{\sigma(3)}\right] \\
+ & \sum_{\sigma\in S_3} 2\left[p(\bar{x}_3), y_{\sigma(1)}\right] D(y_{\sigma(2)}y_{\sigma(3)}) - \sum_{\sigma\in S_3} y_{\sigma(1)}f(y_{\sigma(2)}y_{\sigma(3)}, p(\bar{x}_3)).
\end{aligned}
$$

Comparing relations (8) and (9) we arrive at

$$
\begin{aligned}
(10)\qquad & 0 = \sum_{\pi\in S_3}\sum_{\sigma\in S_3}\Big(f(x_{\pi(1)}x_{\pi(2)}, y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}) \\
- & 2D(x_{\pi(1)}x_{\pi(2)})y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)} + f(x_{\pi(1)}, y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)})x_{\pi(2)} \\
- & 2D(x_{\pi(1)})y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}x_{\pi(2)} + 2D(y_{\sigma(1)}y_{\sigma(2)})y_{\sigma(3)}x_{\pi(1)}x_{\pi(2)} \\
+ & 2D(y_{\sigma(1)})y_{\sigma(2)}y_{\sigma(3)}x_{\pi(1)}x_{\pi(2)}\Big)x_{\pi(3)} \\
+ & \sum_{\pi\in S_3}\sum_{\sigma\in S_3}\Big(f(y_{\sigma(1)}y_{\sigma(2)}, x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}) \\
- & 2D(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(1)}x_{\pi(2)}x_{\pi(3)} + f(y_{\sigma(1)}, x_{\pi(1)}x_{\pi(2)}x_{\pi(3)})y_{\sigma(2)} \\
- & 2D(y_{\sigma(1)})x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}y_{\sigma(2)} + 2D(x_{\pi(1)}x_{\pi(2)})x_{\pi(3)}y_{\sigma(1)}y_{\sigma(2)} \\
+ & 2D(x_{\pi(1)})x_{\pi(2)}x_{\pi(3)}y_{\sigma(1)}y_{\sigma(2)}\Big)y_{\sigma(3)} \\
+ & \sum_{\pi\in S_3}\sum_{\sigma\in S_3} x_{\pi(1)}\Big(f(x_{\pi(2)}x_{\pi(3)}, y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}) \\
+ & x_{\pi(2)}f(x_{\pi(3)}, y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}) + 2y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}D(x_{\pi(2)}x_{\pi(3)}) \\
+ & 2x_{\pi(2)}y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}D(x_{\pi(3)}) - 2x_{\pi(2)}x_{\pi(3)}y_{\sigma(1)}y_{\sigma(2)}D(y_{\sigma(3)}) \\
- & 2x_{\pi(2)}x_{\pi(3)}y_{\sigma(1)}D(y_{\sigma(2)}y_{\sigma(3)})\Big) \\
+ & \sum_{\pi\in S_3}\sum_{\sigma\in S_3} y_{\sigma(1)}\Big(f(y_{\sigma(2)}y_{\sigma(3)}, x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}) \\
+ & y_{\sigma(2)}f(y_{\sigma(3)}, x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}) + 2x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}D(y_{\sigma(2)}y_{\sigma(3)}) \\
+ & 2y_{\sigma(2)}x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}D(y_{\sigma(3)}) - 2y_{\sigma(2)}y_{\sigma(3)}x_{\pi(1)}x_{\pi(2)}D(x_{\pi(3)}) \\
- & 2y_{\sigma(2)}y_{\sigma(3)}x_{\pi(1)}D(x_{\pi(2)}x_{\pi(3)})\Big).
\end{aligned}
$$

Let us define mappings $E, F : \mathcal{L}^5 \to R$ by the rule

$$\begin{aligned} E(u_1,u_2,u_3,u_4,u_5) &= f(u_1u_2,u_3u_4u_5) - 2D(u_1u_2)u_3u_4u_5 + \\ &+ f(u_1,u_3u_4u_5)u_2 - 2D(u_1)u_3u_4u_5u_2 + \\ &+ 2D(u_3u_4)u_5u_1u_2 + 2D(u_3)u_4u_5u_1u_2 \end{aligned}$$

and

$$\begin{aligned} F(u_1,u_2,u_3,u_4,u_5) &= f(u_1u_2,u_3u_4u_5) + u_1f(u_2,u_3u_4u_5) + \\ &+ 2u_3u_4u_5D(u_1u_2) + 2u_1u_3u_4u_5D(u_2) - \\ &- 2u_1u_2u_3u_4D(u_5) - 2u_1u_2u_3D(u_4u_5) \end{aligned}$$

for all $\overline{u}_5 \in \mathcal{L}^5$. Accordingly, (10) can be rewritten as

$$\begin{aligned} 0 &= \sum_{\pi\in S_3}\sum_{\sigma\in S_3} E(x_{\pi(1)},x_{\pi(2)},y_{\sigma(1)},y_{\sigma(2)},y_{\sigma(3)})x_{\pi(3)} \\ &+ \sum_{\pi\in S_3}\sum_{\sigma\in S_3} E(y_{\sigma(1)},y_{\sigma(2)},x_{\pi(1)},x_{\pi(2)},x_{\pi(3)})y_{\sigma(3)} \\ &+ \sum_{\pi\in S_3}\sum_{\sigma\in S_3} x_{\pi(1)}F(x_{\pi(2)},x_{\pi(3)},y_{\sigma(1)},y_{\sigma(2)},y_{\sigma(3)}) \\ &+ \sum_{\pi\in S_3}\sum_{\sigma\in S_3} y_{\sigma(1)}F(y_{\sigma(2)},y_{\sigma(3)},x_{\pi(1)},x_{\pi(2)},x_{\pi(3)}). \end{aligned}$$

and hence

$$\begin{aligned} 0 &= \sum_{i=1}^{3}\Big(\sum_{\substack{\pi\in S_3\\ \pi(3)=i}}\sum_{\sigma\in S_3} E(x_{\pi(1)},x_{\pi(2)},y_{\sigma(1)},y_{\sigma(2)},y_{\sigma(3)})\Big)x_i \\ &+ \sum_{i=4}^{6}\Big(\sum_{\pi\in S_3}\sum_{\substack{\sigma\in S_3\\ \sigma(3)=i}} E(y_{\sigma(1)},y_{\sigma(2)},x_{\pi(1)},x_{\pi(2)},x_{\pi(3)})\Big)y_i \\ &+ \sum_{j=1}^{3}x_j\Big(\sum_{\substack{\pi\in S_3\\ \pi(1)=j}}\sum_{\sigma\in S_3} F(x_{\pi(2)},x_{\pi(3)},y_{\sigma(1)},y_{\sigma(2)},y_{\sigma(3)})\Big) \\ &+ \sum_{j=4}^{6}y_j\Big(\sum_{\pi\in S_3}\sum_{\substack{\sigma\in S_3\\ \sigma(1)=j}} F(y_{\sigma(2)},y_{\sigma(3)},x_{\pi(1)},x_{\pi(2)},x_{\pi(3)})\Big). \end{aligned}$$

Let $s : \mathbb{Z} \to \mathbb{Z}$ be a mapping defined by $s(i) = i - 3$. For each $\sigma \in S_3$ the mapping $s^{-1}\sigma s : \{4,5,6\} \to \{4,5,6\}$ will be denoted by $\overline{\sigma}$. Writing x_{3+i} instead of y_i, $i = 1,2,3$, we can express so obtained relation as

$$\sum_{i=1}^{6} E_i^i(\overline{x}_6)x_i + \sum_{j=1}^{6} x_jF_j^j(\overline{x}_6) = 0$$

for all $\overline{x}_6 = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathcal{L}^6$, where $E_i, F_j : \mathcal{L}^5 \to R$ and $E^i, F^j : \mathcal{L}^6 \to R$ are mappings

$$E^i(\bar{x}^6) = E(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_6)$$

and

$$F^j(\bar{x}^6) = E(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_6).$$

Now we simply apply the definition of 6−freeness $\mathcal{L}$. There exist maps $p_{6,j} : \mathcal{L}^4 \to R$, $j = 1, \ldots, 5$ and $\lambda_6 : \mathcal{L}^5 \to C(\mathcal{L})$ such that

$$\sum_{\substack{\pi\in S_3\\ \pi(3)=3}} \sum_{\sigma\in S_3} E(x_{\pi(1)}, x_{\pi(2)}, x_{\bar{\sigma}(4)}, x_{\bar{\sigma}(5)}, x_{\bar{\sigma}(6)}) = \sum_{i=1}^{5} x_i p_{6,i}(\bar{x}_5^i) + \lambda_6(\bar{x}_5)$$

for all $\bar{x}_5 \in \mathcal{L}^5$. In view of definition of a mapping E, we arrive at

$$\begin{aligned}
&\sum_{\substack{\pi\in S_3\\ \pi(3)=3}} \sum_{\sigma\in S_3} && x_{\pi(1)}\Big(x_{\pi(2)}x_{\bar{\sigma}(4)}x_{\bar{\sigma}(5)}D(x_{\bar{\sigma}(6)}) + x_{\pi(2)}x_{\bar{\sigma}(4)}D(x_{\bar{\sigma}(5)}x_{\bar{\sigma}(6)}) \\
& &+\ & x_{\bar{\sigma}(4)}D(x_{\bar{\sigma}(5)}x_{\bar{\sigma}(6)})x_{\pi(2)}\Big) \\
&+\sum_{\substack{\pi\in S_3\\ \pi(3)=3}} \sum_{\sigma\in S_3} && x_{\bar{\sigma}(4)}\Big(D\left[x_{\pi(1)}x_{\pi(2)}, x_{\bar{\sigma}(5)}x_{\bar{\sigma}(6)}\right] + x_{\bar{\sigma}(5)}D\left[x_{\pi(1)}x_{\pi(2)}, x_{\bar{\sigma}(6)}\right] \\
& &-\ & x_{\pi(1)}x_{\pi(2)}D(x_{\bar{\sigma}(5)}x_{\bar{\sigma}(6)}) - x_{\bar{\sigma}(5)}x_{\pi(1)}x_{\pi(2)}D(x_{\bar{\sigma}(6)}) \\
& &-\ & x_{\bar{\sigma}(5)}x_{\pi(1)}D(x_{\bar{\sigma}(6)})x_{\pi(2)} + x_{\bar{\sigma}(5)}D\left[x_{\pi(1)}, x_{\bar{\sigma}(6)}\right]x_{\pi(2)} \\
& &-\ & x_{\pi(1)}D(x_{\bar{\sigma}(5)}x_{\bar{\sigma}(6)})x_{\pi(2)} + D\left[x_{\pi(1)}, x_{\bar{\sigma}(5)}x_{\bar{\sigma}(6)}\right]x_{\pi(2)}\Big) \\
&+\sum_{\substack{\pi\in S_3\\ \pi(3)=3}} \sum_{\sigma\in S_3} && \Big(D\left[x_{\pi(1)}x_{\pi(2)}, x_{\bar{\sigma}(4)}x_{\bar{\sigma}(5)}\right] + D(x_{\bar{\sigma}(4)}x_{\bar{\sigma}(5)})x_{\pi(1)}x_{\pi(2)} \\
& &+\ & D\left[x_{\pi(1)}x_{\pi(2)}, x_{\bar{\sigma}(4)}\right]x_{\bar{\sigma}(5)} - 2D(x_{\pi(1)}x_{\pi(2)})x_{\bar{\sigma}(4)}x_{\bar{\sigma}(5)} \\
& &+\ & D(x_{\bar{\sigma}(4)})x_{\pi(1)}x_{\pi(2)}x_{\bar{\sigma}(5)}\Big)x_{\bar{\sigma}(6)} \\
&+\sum_{\substack{\pi\in S_3\\ \pi(3)=3}} \sum_{\sigma\in S_3} && \Big(D\left[x_{\pi(1)}, x_{\bar{\sigma}(4)}x_{\bar{\sigma}(5)}\right]x_{\bar{\sigma}(6)} + D(x_{\bar{\sigma}(4)}x_{\bar{\sigma}(5)})x_{\pi(1)}x_{\bar{\sigma}(6)} \\
& &+\ & D\left[x_{\pi(1)}, x_{\bar{\sigma}(4)}\right]x_{\bar{\sigma}(5)}x_{\bar{\sigma}(6)} + D(x_{\bar{\sigma}(4)})x_{\pi(1)}x_{\bar{\sigma}(5)}x_{\bar{\sigma}(6)} \\
& &-\ & 2D(x_{\pi(1)})x_{\bar{\sigma}(4)}x_{\bar{\sigma}(5)}x_{\bar{\sigma}(6)} + x_{\pi(1)}x_{\bar{\sigma}(4)}x_{\bar{\sigma}(5)}D(x_{\bar{\sigma}(6)})\Big)x_{\pi(2)} \\
&-\sum_{i=1}^{5} x_i p_{6,i}(\bar{x}_5^i) &\in\ & C(\mathcal{L})
\end{aligned}$$

for all $\bar{x}_5 \in \mathcal{L}^5$. Applying the theory of functional identities gives

$$\sum_{\sigma\in S_3}\Big(D\left[x_1, x_{\bar\sigma(4)}x_{\bar\sigma(5)}\right] + D(x_{\bar\sigma(4)}x_{\bar\sigma(5)})x_1 + D\left[x_1, x_{\bar\sigma(4)}\right]x_{\bar\sigma(5)}$$
$$+D(x_{\bar\sigma(4)})x_1x_{\bar\sigma(5)} - 2D(x_1)x_{\bar\sigma(4)}x_{\bar\sigma(5)}\Big)x_{\bar\sigma(6)}$$
$$+ \quad \sum_{\sigma\in S_3} x_1x_{\bar\sigma(4)}x_{\bar\sigma(5)}D(x_{\bar\sigma(6)}) - \sum_{i=1}^{4} x_i p_{5,i}(\bar{x}_4^i) \in C(\mathcal{L}),$$

which implies the existence of functions $t, u, v : \mathcal{L}^2 \to R$ and $\kappa : \mathcal{L}^3 \to C(\mathcal{L})$ such that

$$\begin{aligned}\sum_{\substack{\sigma\in S_3\\ \bar\sigma(6)=6}} & D\left[x_1, x_{\bar\sigma(4)}x_{\bar\sigma(5)}\right] + D(x_{\bar\sigma(4)}x_{\bar\sigma(5)})x_1 + D\left[x_1, x_{\bar\sigma(4)}\right]x_{\bar\sigma(5)}\\ + \; & D(x_{\bar\sigma(4)})x_1x_{\bar\sigma(5)} - 2D(x_1)x_{\bar\sigma(4)}x_{\bar\sigma(5)} =\\ = \; & x_1t(x_4, x_5) + x_4u(x_1, x_5) + x_5v(x_1, x_4) + \kappa(x_1, x_4, x_5).\end{aligned}$$

Putting $x_1 = x_4 = x_5 = x$ in the above relation gives

$$2D(x^2)x - 2D(x)x^2 = xt(x, x) + xu(x, x) + xv(x, x) + \kappa(x, x, x). \tag{11}$$

After the complete linearization of the above identity and considering that $\mathcal{L}$ is a 6-free subset of R, we get

$$2D(xy) + 2D(yx) - 2D(x)y - 2D(y)x = xf(y) + yg(x) + \lambda(x, y), \tag{12}$$

where $f, g : \mathcal{L} \to R$ and $\lambda : \mathcal{L}^2 \to C(\mathcal{L})$. The symmetric analogue in which maps F are involved, is clearly proved in the same way. Therefore

$$2D(xy) + 2D(yx) - 2xD(y) - 2yD(x) = f'(x)y + g'(y)x + \lambda'(x, y), \tag{13}$$

where $f', g' : \mathcal{L} \to R$ and $\lambda' : \mathcal{L}^2 \to C(\mathcal{L})$. Replacing the roles of denotations x and y in (12) and comparing so obtained identities leads to $0 = xf(y) + yg(x) - yf(x) - xg(y) + \lambda(x, y) - \lambda(y, x)$, which yields $f(x) = g(x)$ and $\lambda(x, y) = \lambda(y, x)$ for all $x, y \in \mathcal{L}$. Putting x for y in (12) leads to

$$4D(x^2) = 4D(x)x + 2xf(x) + \lambda(x, x). \tag{14}$$

Using the same arguments, it follows from (13) that $f'(x) = g'(x)$ and $\lambda'(x, y) = \lambda'(y, x)$ for all $x, y \in \mathcal{L}$. Therefore

$$4D(x^2) = 4xD(x) + 2f'(x)x + \lambda'(x, x). \tag{15}$$

Comparing the above relations gives

$$x(4D(x) - 2f(x)) - (4D(x) - 2f'(x))x \in C(\mathcal{L}). \tag{16}$$

Hence, there exist $r \in R$ and $\mu : \mathcal{L} \to C(\mathcal{L})$ such that

$$4D(x) - 2f(x) = rx + \mu(x). \tag{17}$$

Considering $2f(x) = 4D(x) - rx - \mu(x)$ in (14) gives

$$4D(x^2) = 4D(x)x + 4xD(x) - xrx - x\mu(x) + \lambda(x, x). \tag{18}$$

Replacing y for x and x for x^2 in (12) gives

$$4D(x^3) = 2D(x^2)x + 2D(x)x^2 + x^2 f(x) + xf(x^2) + \lambda(x^2, x).$$

Using (4) and (14) in the above relation leads to

$$2xf(x^2) = 4xD(x)x + 4x^2 D(x) + x\lambda(x, x) - 2\lambda(x^2, x).$$

Considering $2f(x) = 4D(x) - rx - \mu(x)$ in the above relation gives

$$\begin{aligned} &4xD(x^2) - xrx^2 - x\mu(x^2) = \\ &= 4xD(x)x + 4x^2 D(x) + x\lambda(x, x) - 2\lambda(x^2, x). \end{aligned} \tag{19}$$

Using (18) in the above relation we obtain $-x^2rx - xrx^2 = x^2\mu(x) + x\mu(x^2) - 2\lambda(x^2, x)$. The complete linearization of this relation and using the theory of functional identities leads to $-xrx - rx^2 = x\mu(x) + \mu(x^2)$. This identity implies that $-xr - rx \in C(\mathcal{L})$. Therefore

$$-xr - rx = \nu(x), \tag{20}$$

where $\nu(x) : \mathcal{L} \to C(\mathcal{L})$. Left multiplication of the above relation by y gives $-yxr - yrx = y\nu(x)$. Putting yx for x in (20) leads to $-yxr - ryx = \nu(yx)$. On comparing the last two identities, we obtain $[y, r]x = \nu(yx) - y\nu(x)$, whence it follows that $[y, r] = 0$ and $\nu(x) = 0$ for all $x \in R$. From (20) we obtain $-xr - rx = 0$, which together with $[x, r] = 0$ gives $xr = 0$. Since R is prime, the last relation implies $r = 0$. We now have $x\mu(x) + \mu(x^2) = 0$ and therefore also $\mu(x) = 0$. Considering these ascertainments in (17) we obtain

$$f(x) = 2D(x) \tag{21}$$

for all $x \in R$. Using this in (14) gives

$$4D(x^2) = 4D(x)x + 4xD(x) + \lambda(x, x). \tag{22}$$

Putting x^2 for x in (22) and using (22) leads to

$$\begin{aligned} 4D(x^4) &= 4D(x)x^3 + 4xD(x)x^2 + 4x^2D(x)x + 4x^3D(x) + \\ &+ \lambda(x, x)x^2 + x^2\lambda(x, x) + \lambda(x^2, x^2). \end{aligned} \tag{23}$$

Putting $x = x^3$, $y = x$ in (12), considering (4), (21) and (22) we obtain

$$\begin{aligned} 16D(x^4) &= 8D(x^3)x + 8D(x)x^3 + 8x^3D(x) + 8xD(x^3) + 4\lambda(x^3, x) = \\ &= 16D(x)x^3 + 16x^3D(x) + 16xD(x)x^2 + 16x^2D(x)x + \\ &+ \lambda(x, x)x^2 + x^2\lambda(x, x) + 2x\lambda(x, x)x + 4\lambda(x^3, x). \end{aligned} \tag{24}$$

Comparing the last two identities gives

$$3\lambda(x, x)x^2 + 3x^2\lambda(x, x) + 4\lambda(x^2, x^2) - 2x\lambda(x, x)x - 4\lambda(x^3, x) = 0.$$

Since $\lambda(x, x) \in C(\mathcal{L})$, the above identity simplifies to

$$\lambda(x, x)x^2 + \lambda(x^2, x^2) - \lambda(x^3, x) = 0.$$

Because $\mathcal{L}$ is a 6-free subset of R, the above identity implies $\lambda(x, x) = 0$ for all $x \in R$. Consequently, it follows from (22) that D is a Jordan derivation.

By Herstein theorem, D is a derivation, which completes the proof of the theorem. □

We are now in the position to prove Theorem 3.

Proof of Theorem 3. The complete linearization of (4) gives us (6). First suppose that R is not a PI ring (satisfying the standard polynomial identity of degree less than 6). According to Theorem 4 the mapping D is a derivation.

Assume now that R is a PI ring. It is well-known that in this case R has a nonzero center (see [14]). Let c be a nonzero central element. Picking any $x \in R$ and set $x_1 = x_2 = cx$ and $x_3 = x$ in (6) we obtain

$$\begin{aligned} 6D(c^2x^3) &= D(c^2x^2)x + 2D(cx^2)cx + c^2x^2D(x) + 2cx^2D(cx) \\ &+ D(x)c^2x^2 + 2D(cx)cx^2 + xD(c^2x^2) + 2cxD(cx^2). \end{aligned}$$

Next, setting $x_1 = x_2 = c$ and $x_3 = x^3$ in (6) we get

$$\begin{aligned} 6D(c^2x^3) &= D(c^2)x^3 + 2D(cx^3)c + c^2D(x^3) + 2cx^3D(c) \\ &+ D(x^3)c^2 + 2D(c)cx^3 + x^3D(c^2) + 2cD(cx^3) \\ &= D(c^2)x^3 + 4D(cx^3)c + 2c^2D(x^3) + 2cx^3D(c) \\ &+ 2D(c)cx^3 + x^3D(c^2) \end{aligned}$$

for all $x \in R$. Comparing both identities and using (4) we obtain

$$\begin{aligned} & D(c^2x^2)x + 2D(cx^2)cx + 2cx^2D(cx) \\ + \ & 2D(cx)cx^2 + xD(c^2x^2) + 2cxD(cx^2) \\ = \ & D(c^2)x^3 + 4D(cx^3)c + c^2D(x^2)x + c^2xD(x^2) \\ + \ & 2cx^3D(c) + 2D(c)cx^3 + x^3D(c^2) \end{aligned} \tag{25}$$

for all $x \in R$. In case $x = c$ we arrive at $D(c^4) = 2D(c^2)c^2$. Setting $x_1 = x$ and $x_2 = x_3 = c$ in the complete linearization of (25) we get

$$\begin{aligned} & c^2D(c)x + c^2xD(c) + 4D(c^2x)c + 2cxD(c^2) + 2cD(c^2)x \\ = \ & 4D(c^3x) + 3xc^2D(c) + 3c^2D(c)x \end{aligned} \tag{26}$$

for all $x \in R$. Substituting x for cx in relation (26) we obtain

$$\begin{aligned} & c^3D(c)x + c^3xD(c) + 4D(c^3x)c + 2c^2xD(c^2) + 2c^2D(c^2)x \\ = \ & 4D(c^4x) + 3xc^3D(c) + 3c^3D(c)x. \end{aligned}$$

Multiplying identity (26) by c we get

$$\begin{aligned} & c^3D(c)x + c^3xD(c) + 4D(c^2x)c^2 + 2c^2xD(c^2) + 2c^2D(c^2)x \\ = \ & 4cD(c^3x) + 3xc^3D(c) + 3c^3D(c)x. \end{aligned} \tag{27}$$

Comparing the last two identities we obtain

$$2D(c^3x)c = D(c^4x) + D(c^2x)c^2. \tag{28}$$

for all $x \in R$. Substituting x by cx in (4) we get

$$\begin{aligned} 6D(c^3x^3) &= 3D(c^2x^2)cx + 3c^2x^2D(cx) \\ &+ 3cxD(c^2x^2) + 3D(cx)c^2x^2 \end{aligned}$$

for all $x \in R$. Next, setting $x_1 = x_2 = c$ and $x_3 = cx^3$ in the complete linearization of (4) we arrive at

$$\begin{aligned} 6D(c^3x^3) &= D(c^2)cx^3 + 4cD(c^2x^3) + 2c^2D(cx^3) \\ &+ 2c^2x^3D(c) + cx^3D(c^2) + 2D(c)c^2x^3. \end{aligned}$$

Comparing the last two identities we arrive at

$$\begin{aligned} (29) \quad & 3D(c^2x^2)x + 3cx^2D(cx) + 3xD(c^2x^2) + 3D(cx)cx^2 \\ = \quad & D(c^2)x^3 + 4D(c^2x^3) + 2cD(cx^3) + 2cx^3D(c) + x^3D(c^2) + 2D(c)cx^3. \end{aligned}$$

Setting $x_1 = x_2 = c$ and $x_3 = x$ in the complete linearization of (29) and using (28) we get

$$\begin{aligned} & 3D(c^2)cx + 3cxD(c^2) + 2D(cx)c^2 + 4D(c^2x)c \\ = \; & 6D(c^3x) + 2xD(c)c^2 + 2D(c)c^2x. \end{aligned}$$

Using the last identity and (27) we obtain

$$\begin{aligned} (30) \qquad & D(c^2)cx + cxD(c^2) + 2D(cx)c^2 + 2xD(c)c^2 + 2D(c)c^2x \\ = \; & 2D(c^3x) + 2D(c)xc^2 + 2xD(c)c^2 \end{aligned}$$

and so

$$2D(c^3x) = 2D(cx)c^2 + D(c^2)cx + cxD(c^2)$$

for all $x \in R$. Setting $x_1 = x_2 = c$ and $x_3 = cx$ in (6) we get

$$\begin{aligned} 12D(c^3x) &= 2D(c^2)cx + 8D(c^2x)c + 4c^2D(cx) \\ &+ 4c^2xD(c) + 2cxD(c^2) + 4c^2D(c)x. \end{aligned}$$

Comparing the last two identities we obtain

(31) $2D(cx)c + D(c^2)cx + cxD(c^2) = 2D(c^2x) + cxD(c) + cD(c)x.$

Substituting x for cx in (30) and using (28) we get

$$D(c^2)c^2x + c^2xD(c^2) = 2D(c)c^3x + 2c^3xD(c)$$

for all $x \in R$. If $x = c$ we get $D(c^2) = 2D(c)c$. Let us write cx instead of x in (31). On the other hand we can multiply (31) by c. After comparing so obtained identities we arrive at

$$(32) \qquad 2D(c^2x)c = D(cx)c^2 + D(c^3x).$$

Using the last identity in (27) we obtain

$$(33) \qquad D(c)xc + xcD(c) = 2D(c^2x) - 2D(cx)c$$

for all $x \in R$. If $x = c$ we get $D(c^3) = 3D(c)c^2$. Set $x_1 = x_2 = c$ and $x_3 = x$ in (6). It follows that

$$6D(c^2x) = D(c^2)x + xD(c^2) + 4D(cx)c + 2c^2D(x) + 2cxD(c) + 2cD(c)x. \tag{34}$$

Using (33) we obtain

$$2D(cx) - 2D(x)c = D(c)x + xD(c). \tag{35}$$

Therefore

$$2D(cx^2) - 2D(x^2)c = D(c)x^2 + x^2D(c) \tag{36}$$

for all $x \in R$. Multiplying identity (35) by c we get

$$2D(cx)c - 2D(x)c^2 = D(c)cx + cxD(c).$$

Next substituting x by cx in (35) we arrive at

$$2D(c^2x) - 2D(cx)c = D(c)cx + cxD(c).$$

Then comparing the last two identities we get

$$2D(cx)c = D(x)c^2 + D(c^2x). \tag{37}$$

Setting $x_1 = x_2 = x$ and $x_3 = c$ in (6) and using (36) we have

$$\begin{aligned} 6D(cx^2) &= 2D(x^2)c + 2D(cx)x + 2xD(cx) + x^2D(c) \\ &+ D(c)x^2 + 2D(x)cx + 2cxD(x) \\ &= (2D(cx^2) - D(c)x^2 - x^2D(c)) \\ &+ 2D(cx)x + 2xD(cx) \\ &+ x(xD(c) + 2cD(x)) + (D(c)x + 2cD(x))x \\ &= 2D(cx^2) - D(c)x^2 - x^2D(c) \\ &+ 4D(cx)x + 4xD(cx) - 2xD(c)x. \end{aligned} \tag{38}$$

Comparing this identity and (36) we get

$$4D(cx)x + 4xD(cx) - 2xD(c)x = 3D(c)x^2 + 3x^2D(c) + 4D(x^2)c.$$

Hence

$$\begin{aligned} 4D(x^2)c &= 4(D(cx)x + xD(cx)) \\ &\quad -3(D(c)x^2 + x^2D(c)) - 2xD(c)x. \end{aligned} \tag{39}$$

Using (35) we arrive at

$$\begin{aligned} 4D(x^2)c &= 2(2D(x)xc + D(c)x^2 + xD(c)x \\ &+ 2cxD(x) + xD(c)x + x^2D(c)) \\ &- 3(D(c)x^2 + x^2D(c)) - 2xD(c)x \\ &= 4D(x)xc + 4xD(x)c - [[D(c), x], x]. \end{aligned} \tag{40}$$

Therefore, we also have

$$4D(x^2)c^2 = 4D(x)xc^2 + 4xD(x)c^2 - [[D(c^2), x], x].$$

Multiplying (40) by c we get

$$4D(x^2)c^2 \quad = \quad 4D(x)xc^2 + 4xD(x)c^2 - [[D(c), x], x]c.$$

Comparing so obtained identities we arrive at

$$[[D(c^2), x], x] - [[D(c), x], x]c = 0$$

for all $x \in R$. Since $[[2cD(c), x], x] = 2c[[D(c), x], x]$ we get $[[D(c), x], x] = 0$ which in turn implies

$$4D(x^2)c^2 = 4D(x)xc^2 + 4xD(x)c^2.$$

According to our assumptions it follows that $D(x^2) = D(x)x + xD(x)$ for all $x \in R$. In other words, D is a Jordan derivation. By Herstein theorem it follows that D is a derivation. The proof of the theorem is complete. □

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