Relationships between the factors of the upper and the lower central series of a group*

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Abstract

In the paper we study groups in which the factor-group by k-th hypercenter is locally finite and has finite exponent. We proved that in these groups the (k+1)-th term of lower central series is locally finite and has finite exponent. Moreover we are able to find bounds for the exponent of $\gamma_{k+1}(G)$ and for the exponent of the locally nilpotent residual of G.

Key Words: upper central series of a group; lower central series of a group; locally nilpotent residual of a group; Baer class of groups; Schur class of groups; locally finite group of finite exponent; Z–decomposition; RG–nilpotent module.
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Introduction

The aim of this paper is to establish a relationship between the factors of the upper and the lower central series of a group. Given a group G, we recall that *the upper central series of* G is the series

$$\langle 1\rangle = \zeta_0(G) \leq \zeta_1(G) \leq \zeta_2(G) \leq \cdots \leq \zeta_\alpha(G) \leq \zeta_{\alpha+1}(G) \leq \cdots \zeta_\gamma(G),$$

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where $\zeta_1(G) = \zeta(G)$ is the center of G, $\zeta_{\alpha+1}(G)/\zeta_\alpha(G) = \zeta(G/\zeta_\alpha(G))$ for every ordinal α, $\zeta_\lambda(G) = \bigcup_{\mu<\lambda} \zeta_\mu(G)$ for every limit ordinal λ, and $\zeta(G/\zeta_\gamma(G)) = \langle 1\rangle$. The term $\zeta_\alpha(G)$ is said to be *the* α^{th}*–hypercenter of* G, and the last term $\zeta_\gamma(G)$ of this series is said to be *the upper hypercenter of* G. The ordinal γ is said to be *the central length of* G and is denoted by $zl(G)$. On the other hand, *the lower central series of* G is the series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \gamma_\alpha(G) \geq \gamma_{\alpha+1}(G) \geq \cdots \gamma_\delta(G),$$

where $\gamma_2(G) = [G, G]$ is the derived group of G, $\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G]$ for every ordinal α, $\gamma_\lambda(G) = \bigcap_{\mu<\lambda} \gamma_\lambda(G)$ for every limit ordinal, and $\gamma_\delta(G) = [\gamma_\delta(G), G]$. The term $\gamma_\alpha(G)$ is said to be *the* α^{th}*–hypocenter of* G, and the last term $\gamma_\delta(G)$ of this series is said to be *the lower hypocenter of* G.

Let G be a nilpotent group. Then there exits a positive integer k such that $G = \zeta_k(G)$. Equivalently $\gamma_{k+1}(G) = \langle 1\rangle$. Extending this well-know fact, R. Baer [1] has been able to show the following result:

Theorem. *Given a group G, suppose that the factor-group $G/\zeta_k(G)$ is finite for some positive integer k. Then $\gamma_{k+1}(G)$ is likewise finite.*

To express properly these results in a general and unified way, we introduce the following concept. A class of groups $\mathfrak{X}$ is said to be *a Baer class* if whenever G is a group and we have $G/\zeta_k(G) \in \mathfrak{X}$ for some positive integer k, then $\gamma_{k+1}(G) \in \mathfrak{X}$. A natural question here is *Finding Baer classes of groups.* Obviously the trivial class $\mathfrak{I} = \{\langle 1\rangle\}$ is a Baer class, and Baer's theorem shows that the class $\mathfrak{F}$ of all finite groups is also a Baer class. Another important precedent appeared if one considers the case $k = 1$. I. Schur has studied the relationship between the central factor-group $G/\zeta(G)$ of a group G and the derived subgroup $[G, G]$ of G [6]. In particular, from Schur's results it follows that if $G/\zeta(G)$ is finite then $[G, G]$ is also finite. Inspired by this and related facts, in [2] a class of groups $\mathfrak{X}$ of groups is called *a Schur class* if for every group G such that $G/\zeta_1(G) \in \mathfrak{X}$ it follows that derived subgroup $\gamma_2(G)$ always belong to $\mathfrak{X}$; examples of Schur classes are related in the mentioned paper [2]. Therefore $\mathfrak{I}$ and $\mathfrak{F}$ are Schur classes.

Obviously, every Baer class is a Schur class. This raises in a natural way the study of the converse: *Which Schur classes are Baer classes?*. Now we known many examples of Schur classes, most of them since a long time ago (see [2]). For example, the class $\mathfrak{F}$ of all finite groups, the class $L\mathfrak{F}_\pi$ of locally finite π–groups, for an arbitrary set π of prime numbers, the class $\mathfrak{P}$ of polycyclic-by-finite groups, the class $\mathfrak{C}$ of Chernikov groups, the class $\mathfrak{S}_1$ of soluble-by-finite minimax groups and many others. Many of these classes have been proved to be Baer classes. A few years ago, A. Mann [5] proved that the class $\mathfrak{L}$ of all locally finite groups having finite exponent is a Schur class. Moreover there exists a function m such that the exponent of the derived subgroup of a locally finite of exponent e is bounded by $m(e)$. Therefore the question of deciding whether this is a Baer class or not naturally appears. The first main result of this paper gives a positive answer on this question.

Theorem A. *Let G be a group and suppose that $G/\zeta_k(G)$ is a locally finite group, having finite exponent e. Then the subgroup $\gamma_{k+1}(G)$ is locally finite and has finite* *exponent. Moreover, there exists a function β_1 such that the exponent of $\gamma_{k+1}(G)$ is at most $\beta_1(e,k)$.*

For the groups described in Theorem A, we may ask another related question. Given a group G, we recall that *the locally nilpotent residual* L of G is the intersection of all normal subgroups H of G such that G/H is locally nilpotent. It is well-known that G/L need not to be locally nilpotent and therefore the case in which this factor-group is locally nilpotent is very interesting. In particular, such situation is obtained in our second main result.

Theorem B. *Let G be a group and suppose that $G/\zeta_k(G)$ is a locally finite group having finite exponent e. Then the locally nilpotent residual L of G is locally finite having finite exponent and G/L is locally nilpotent. Moreover, there exists a function β_2 such that the exponent of L is at most $\beta_2(e)$.*

It is worth mentioning that in fact the exponent of the locally nilpotent residual depends only of the exponent of $G/\zeta_k(G)$.

1 Proof of Theorem A

The proof rely on the following auxiliary results.

Lemma 1.1 *Suppose that A is an abelian normal subgroup of a group G such that $G/C_G(A) = \langle x_1 C_G(A), x_2 C_G(A)\rangle$ for some elements $x_1, x_2 \in G$. Then $[A,G] = [A,x_1][A,x_2]$.*

`Proof`. Put $U = [A,x_1][A,x_2]$. If $a \in A$, then

$$[a, x_j^2] = [a, x_j][y, x_j]^{x_j} = [y, x_j][y^{x_j}, x_j] \in [A, x_j] \leq U, \ j \in \{1, 2\}.$$

It follows that $[a, x_j^n] \in U$ for each $n \in \mathbb{Z}$. Let $n, k \in \mathbb{Z}$ and put $u = x_1^n$ and $v = x_2^k$. Given $a \in A$, we have

$$[a, uv] = [a, v][a, u]^v \text{ and } [a, u]^v = [vcv^{-1}, u]^v,$$

where $c = v^{-1}av \in A$. Put $d = [vcv^{-1}, u]$ so that

$$[vcv^{-1}, u]^v = d^v = dd^{-1}v^{-1}dv = d[d, v].$$

Clearly $d \in [A, u] = [A, x_1^n] \leq U$ and $[d, v] \in [A, v] = [A, x_2^k] \leq U$ and then

$$[a, u]^v = [vcv^{-1}, u]^v = d^v = d[d, v] \in U.$$

It follows that $[a, uv] \in U$. Proceeding in this way and applying induction, we see that

$$[a, x_1^{k_1} x_2^{t_1} \cdots x_1^{k_n} x_2^{t_n}] \in U, \text{ for } k_1, t_1, \cdots, k_n, t_n \in \mathbb{Z}.$$

Let g be an arbitrary element of G. Then

$$g = x_1^{r_1} x_2^{s_1} \cdots x_1^{r_m} x_2^{s_m} c,$$

for some element $c \in C_G(A)$ and integer numbers $r_1, s_1, \cdots, r_m, s_m \in \mathbb{Z}$. Then

$$[a, x_1^{r_1} x_2^{s_1} \cdots x_1^{r_m} x_2^{s_m} c] = [a, x_1^{r_1} x_2^{s_1} \cdots x_1^{r_m} x_2^{s_m}] \in U,$$

and hence we obtain that U is a G–invariant subgroup of A. By the choice of G, we have $A/U \leq \zeta(G/U)$ which gives that $[A, G] \leq U$, as required. □

Corollary 1.2 *Let A be an abelian normal subgroup of a group G and suppose we have that $G = \langle C_G(A), M \rangle$ for a certain subset M of G. Then $[A, G]$ is the product of all $[A, x]$, when x runs M.*

`Proof`. Put $V = \langle [A, x] \mid x \in M \rangle$. Clearly $V \leq [A, G]$. Let $w \in [A, G]$ so that

$$w = [a_1, y_1] \cdots [a_n, y_n]$$

for suitable elements $a_1, \cdots, a_n \in A$ and $y_1, \cdots, y_n \in G$. Then there exist elements $x_1, \cdots, x_m \in M$ such that

$$y_1, \cdots, y_n \in \langle x_1, \cdots, x_m, C_G(A) \rangle = H$$

and therefore $w \in [A, H]$. Since the product $[A, x_j][A, x_k]$ is $\langle x_j, x_k \rangle$–invariant for any choice of $j, k \in \{1, \cdots, m\}$ by Lemma 1.1, the subgroup $[A, x_1] \cdots [A, x_m] = U$ is H–invariant. Then the center of the section H/U includes A/U, that is $[A, H] \leq U$. Since the converse inclusion is also true, we deduce that $[A, H] = U$. Therefore

$$w \in [A, x_1] \cdots [A, x_m] \leq V$$

and hence $[A, G] = V$, as required. □

Lemma 1.3 *Let A be an abelian normal subgroup of a group G and suppose that $A/(\zeta(G) \cap A)$ is locally finite and has finite exponent e. Then $[A, G]$ is a locally finite subgroup having finite exponent at most e.*

`Proof`. We pick a subset M of G such that $G = \langle C_G(A), M \rangle$. Given $g \in G$, we consider the mapping $\xi_g : a \to [a, g]$, $a \in A$ so that ξ_g is an endomorphism of A. Since $\zeta(G) \cap A \leq C_A(g) = \mathrm{Ker}(\xi_g)$, $A/\mathrm{Ker}(\xi_g)$ is locally finite and has finite exponent at most e. Since

$$A/\mathrm{Ker}(\xi_g) \cong_G \mathrm{Im}(\xi_g) = [A, g]$$

$[A, g]$ is locally finite and has finite exponent at most e. By Corollary 1.2, $[A, G]$ is the product of the subgroups $[A, g]$, when g runs M. Since every subgroup $[A, g]$ is locally finite and has finite exponent at most e, the same is true for $[A, G]$. □

We are now in a position to prove our first main result.

`Proof of Theorem A`. Let

$$\langle 1 \rangle = Z_0 \leq Z_1 \leq \cdots \leq Z_{k-1} \leq Z_k = Z$$

be the upper central series of G. We proceed by induction on k.

If $k = 1$, then G/Z_1 is a locally finite group having finite exponent e. Application of Mann's theorem [5] shows that $\gamma_2(G) = [G, G]$ is locally finite and there exists a function m such that the exponent of $\gamma_2(G)$ is bounded by $m(e)$.

We now suppose that $k > 1$ and we have already proved that $\gamma_k(G/Z_1)$ is locally finite of finite exponent and there exists a function β_1 such that the exponent of $\gamma_k(G/Z_1)$ is at most $\beta_1(e, k-1)$. Put $K/Z_1 = \gamma_k(G/Z_1)$ and $L = \gamma_k(G)$ so that $L \leq K$. Applying Mann's theorem [5] to K, we obtain that $D = [K, K]$ is locally finite and has finite exponent at most $m(\beta_1(e, k-1))$. Since the factor-group K/D is abelian, LD/D is also abelian. We have

$$(LD/D)(LD/D \cap Z_1D/D) = (LD/D)((LD \cap Z_1D)/D) \cong LD/(LD \cap Z_1D) \cong$$

$$\cong (LD)(Z_1D)/(Z_1D) = (LZ_1D)/(Z_1D) \cong L/(L \cap Z_1D),$$

which shows that $(LD/D)(LD/D \cap Z_1D/D)$ is an epimorphic image of $L/(L \cap Z_1)$. Since $L/(L \cap Z_1) \cong LZ_1/Z_1 \leq K/Z_1$, $L/(L \cap Z_1)$ is a locally finite group of finite exponent at most $\beta_1(e, k-1)$. Therefore, the same is true also for $(LD/D)(LD/D \cap Z_1D/D)$. Applying Lemma 1.3 to the factor-group G/D, we see that its subgroup $V/D = [LD/D, G/D]$ is locally finite and has finite exponent at most $\beta_1(e, k-1)$. Since the center of G/V includes LV/V and $(G/V)/(LV/V)$ is nilpotent of class at most k, $\gamma_{k+1}(G) \leq V$. It follows that $\gamma_{k+1}(G)$ is a locally finite subgroup having exponent at most $m(\beta_1(e, k-1))\beta_1(e, k-1) = \beta_1(e, k)$, and we are done. □

It is worth mentioning that the function $\beta_1(t, k)$ constructed in this theorem is defined recursively by $\beta_1(e, 1) = m(e)$, $\beta_1(e, 2) = m(m(e))m(e)$ and

$$\beta_1(t, k) = m(\beta_1(e, k-1))\beta_1(e, k-1).$$

2 Proof of Theorem B

To show the auxiliary results that lead to the proof of this theorem, we need the following module-theoretical concepts.

Let G be a group, R a ring and A an RG–module. Then the set

$$\zeta_{RG}(A) = \{a \in A \mid a(g-1) = 0 \text{ for each element } g \in G\}$$

is a submodule called *the RG–center of A. The upper RG–central series of A* is,

$$\{0\} = A_0 \leq A_1 \leq \cdots \leq A_\alpha \leq A_{\alpha+1} \leq \cdots A_\gamma,$$

where $A_1 = \zeta_{RG}(A)$, $A_{\alpha+1}/A_\alpha = \zeta_{RG}(A/A_\alpha)$, $\alpha < \gamma$, and $\zeta_{RG}(A/A_\gamma) = \{0\}$. The last term A_γ of this series is called *the upper RG–hypercenter of A* and will be denoted by $\zeta^{\infty}_{RG}(A)$, while the ordinal γ is said to be *the RG–central length of A* and will be denoted by $zl_{RG}(A)$. The RG–module A is said to be *RG–hypercentral* if $A = A_\gamma$ happens and *RG–nilpotent* if γ is finite.

If B and C are RG–submodules of A and $B \leq C$, then the factor C/B is said to be *G–central* if $G = C_G(C/B)$ and *G-eccentric* otherwise. An RG–submodule C of A is said to be *RG–hypereccentric* if C has an ascending series of RG–submodules

$$\{0\} = C_0 \leq C_1 \leq \cdots \leq C_\alpha \leq C_{\alpha+1} \leq \cdots C_\gamma = C$$

whose factors $C_{\alpha+1}/C_\alpha$ are G–eccentric simple RG–modules.

Following D.I. Zaitsev [7], an RG–module A is said to have *the Z–decomposition* if one has

$$A = \zeta^{\infty}_{RG}(A) \oplus E^{\infty}_{RG}(A),$$

where $E^{\infty}_{RG}(A)$ is the unique maximal RG–hypereccentric RG–submodule of A. We actually note that a given maximal E includes every RG–hypereccentric RG–submodule B and, in particular, it is unique. For, if $(B+E)/E$ is non-zero, it hve to include a non-zero simple RG–submodule U/E. Since $(B+E)/E \cong B/(B\cap E)$, U/E is RG–isomorphic to some simple RG–factor of B and it follows that $G/C_G(U/E) \neq G$. On the other hand, $(B+E)/E \leq A/E \leq \zeta^{\infty}_{RG}(A)$, that is $G/C_G(U/E) = G$. This contradiction shows that $B \leq E$, as claimed.

Lemma 2.1 *Let G be a finite nilpotent group and A be a $\mathbb{Z}G$–module. Suppose that A includes a $\mathbb{Z}G$–nilpotent $\mathbb{Z}G$–submodule C such that A/C is a finite group of order t and exponent e. Then A includes a finite $\mathbb{Z}G$–submodule K such that $|K| \mid t$, the exponent of K is at most e and A/K is $\mathbb{Z}G$–nilpotent.*

`Proof`. We first remark that $A\zeta^{\infty}_{\mathbb{Z}G}(A)$ is a finite of order divisor of t and exponent e. Pick a finite subset M of elements of A such that

$$M\mathbb{Z}G + C = A/C.$$

Put $V = M\mathbb{Z}G$ and $U = C \cap V$ so that U is clearly $\mathbb{Z}G$–nilpotent. Since

$$V/U = V/(V \cap C) \cong (V + C)/C = A/C,$$

$|V/U| = t$ and V/U has exponent at most e. Since G is finite, the natural semidirect product $V \rtimes G$ is a nilpotent-by-finite group. Being finitely generated, it satisfies the maximal condition on all subgroups, and it follows that U is a finitely generated subgroup. Therefore the periodic part T of U is finite and hence $U = T \oplus W$, for some torsion-free subgroup W. Put $Y = U^{|T|}$ so that Y is a characteristic subgroup of U. In particular, Y is a $\mathbb{Z}G$–submodule and U/Y is finite whence V/Y is finite too. Since G is nilpotent, the finite factor-module V/Y has the Z–decomposition [7], that is

$$V/Y = Z/Y \bigoplus E/Y,$$

where $Z/Y = \zeta^{\infty}_{ZG}(V/Y)$ and $E/Y = E^{\infty}_{ZG}(V/Y)$. Since U/Y is $\mathbb{Z}G$–nilpotent, $U/Y \leq Z/Y$. Applying the latter, the isomorphisms

$$E/Y \cong (V/Y)/(Z/Y) \cong V/Z$$

and the inclusion $(\zeta^{\infty}_{ZG}(A)+Y)/Y \leq Z/Y$ at once, we obtain that E/Y is isomorphic to some factor-module of $A/\zeta^{\infty}_{\mathbb{Z}G}(A)$. In particular, E/Y is finite, $|E| \mid t$ and the exponent of E/Y is at most e.

The choice of E yields that E is a $\mathbb{Z}G$–submodule of V. Then the periodic part K of E is also a $\mathbb{Z}G$–submodule. Since Y is torsion-free, $K \cap Y = \{0\}$ and then K is isomorphic to some section of E/Y. Therefore K is finite, $|K| \mid t$ and the exponent of K is at most e. The choice of E yields $|K| \mid |E/Y| \mid t$. The factor-module E/K is $\mathbb{Z}$–torsion-free and includes a $\mathbb{Z}G$–nilpotent submodule $(Y+K)/K$ having finite index. It follows that E/K is also $\mathbb{Z}G$–nilpotent. The isomorphisms

$$V/E \cong (V/Y)/(E/Y) \cong \zeta^{\infty}_{\mathbb{Z}G}(V/Y)$$

give that V/K is $\mathbb{Z}G$–nilpotent. Since $A = V + C$ and $C \leq \zeta^{\infty}_{\mathbb{Z}G}(A)$, A/K is $\mathbb{Z}G$–nilpotent, as required. □

An RG–module is said to be *locally RG–nilpotent* if for every finitely generated subgroup F of G and every finite subset M of A the $\mathbb{Z}F$–submodule $M\mathbb{Z}F$ generated by M is $\mathbb{Z}F$–nilpotent.

Corollary 2.2 *Let G be a periodic locally nilpotent group and A be a $\mathbb{Z}G$–module. Suppose that A includes a $\mathbb{Z}G$–nilpotent $\mathbb{Z}G$–submodule C such that the additive group of A/C is periodic and has finite exponent e. Then A includes a $\mathbb{Z}G$–submodule K the additive group of K is periodic and has finite exponent at most e and A/K is locally $\mathbb{Z}G$–nilpotent.*

`Proof`. Let M be an arbitrary finite subset of A. If $\mathcal{L}$ is the local system of G consisting of all its finite subgroups and $F \in \mathcal{L}$, we consider the $\mathbb{Z}F$–submodule $M_F = C + M\mathbb{Z}F$. Since A/C is $\mathbb{Z}$–periodic and F is finite, M_F/C is finite (perhaps trivial if $M \subseteq C$). By Lemma 2.1, M_F includes a finite $\mathbb{Z}F$–submodule R such that M_F/R is $\mathbb{Z}F$–nilpotent and the exponent of R is at most e. Then R includes a unique minimal finite $\mathbb{Z}F$–submodule K_F such that M_F/K_F is $\mathbb{Z}F$–nilpotent. Let $H \in \mathbb{L}$ be such that $F \leq H$. Obviously $M_F \leq M_H$. Since the factor-module M_H/K_H is $\mathbb{Z}H$–nilpotent, it is clearly $\mathbb{Z}F$–nilpotent. It follows that $M_F/(K_H \cap M_F)$ is $\mathbb{Z}F$–nilpotent and then $K_F \leq K_H \cap M_F$ whence $K_F \leq K_H$ by the election of K_F. From the equation $G = \bigcup_{F\in\mathcal{L}} F$,

$$M_0 = \bigcup_{F\in\mathcal{L}} M_F \text{ and } K(M) = \bigcup_{F\in\mathcal{L}} K_F$$

are $\mathbb{Z}G$–submodules. Let S be an arbitrary finite subset of M_0 and X be an arbitrary finite subgroup of G. Since M_0 is generated by M as $\mathbb{Z}G$–submodule, there exists a finite subgroup $F \in \mathcal{L}$ such that $S \leq M_F$. Pick $H \in \mathcal{L}$ such that $X, F \leq H$. Then $M\mathbb{Z}X \leq M_H$. Since M_H/K_H is $\mathbb{Z}F$–nilpotent, in particular, it is $\mathbb{Z}X$–nilpotent. Then $(M\mathbb{Z}X + K_H)/K_H$ is $\mathbb{Z}X$–nilpotent and therefore $(M\mathbb{Z}X + K(M))/K(M)$ is $\mathbb{Z}X$–nilpotent. Hence $M_0/K(M)$ is locally $\mathbb{Z}G$–nilpotent. Since K_F has exponent at most e for each $F \in \mathcal{L}$, $K(M)$ also has exponent at most e.

We now consider the local family $\mathcal{M}$ consisting of the finite subset of A. Let $M, S \in \mathcal{M}$ such that $M \subseteq S$ and pick $F \in \mathcal{L}$. Since $S_0/K(S)$ is locally $\mathbb{Z}G$–nilpotent, $(S\mathbb{Z}F + K(S))/K(S)$ is $\mathbb{Z}F$–nilpotent. It follows that $M\mathbb{Z}F/(M\mathbb{Z}F \cap K(S))$ is $\mathbb{Z}F$–nilpotent. Therefore $K_F \leq M\mathbb{Z}F \cap K(S)$ and then $K_F \leq K(S)$. Thus $\bigcup_{F\in\mathcal{L}} K_F \leq K(S)$. Thus $K(M) \leq K(S)$. This means that the family $\{K(M) \mid M \in \mathcal{M}\}$ is local, hence $K = \bigcup_{M\in\mathcal{M}} K(M)$ is a $\mathbb{Z}G$–submodule. Since $A = \bigcup_{M\in\mathcal{M}} M$, A/K is locally $\mathbb{Z}G$–nilpotent. By construction, K has exponent at most e. □

Lemma 2.3 *Let K be a locally finite normal subgroup of a group G such that G/K is locally nilpotent. Then the locally nilpotent residual L of G is locally finite. Moreover, if G satisfies locally the maximal condition on subgroups, then G/L is locally nilpotent.*

`Proof`. Since G/K is locally nilpotent, $L \leq K$ and it follows that L is locally finite. Replacing G by the factor-group G/L, we may suppose that $L = \langle 1 \rangle$. Then the thesis is to prove that G is locally nilpotent. Pick a family $\{G_\lambda \mid \lambda \in \Lambda\}$ of normal subgroups of G such that $\bigcap_{\lambda \in \Lambda} G_\lambda = \langle 1 \rangle$ and G/G_λ is locally nilpotent for every $\lambda \in \Lambda$. Since the result is trivial if Λ is finite, we suppose that the family is infinite. Put $K_\lambda = K \cap G_\lambda$ so that $\bigcap_{\lambda \in \Lambda} K_\lambda = \langle 1 \rangle$, every subgroup K_λ is G–invariant and G/K_λ is locally nilpotent for every $\lambda \in \Lambda$. Let F be an arbitrary finitely generated subgroup of G. Then $F/(F \cap K)$ is a finitely generated nilpotent group and the subgroup $F \cap K$ is locally finite. Since F satisfies the maximal condition on subgroups, $T = F \cap K$ have to be finite. Then there exists a finite subset M of Λ such that $T \cap (\bigcap_{\lambda \in M} K_\lambda) = \langle 1 \rangle$. Put $V = \bigcap_{\lambda \in M} K_\lambda$ so that G/V is locally nilpotent. We have now

$$F \cap V = F \cap (\bigcap_{\lambda \in M} K_\lambda) = \bigcap_{\lambda \in M} (F \cap K_\lambda) = \bigcap_{\lambda \in M} (F \cap (K \cap K_\lambda)) =$$

$$= \bigcap_{\lambda \in M} ((F \cap K) \cap K_\lambda) = \bigcap_{\lambda \in M} (T \cap K_\lambda) = T \cap (\bigcap_{\lambda \in M} K_\lambda) = \langle 1 \rangle.$$

It follows that $F \cong F/(F \cap V) \cong FV/V$. Since G/V is locally nilpotent, FV/V is nilpotent. Therefore an arbitrary finitely generated subgroup F of G is nilpotent and hence G is locally nilpotent, as required. □

Lemma 2.4 *Let Z be the upper hypercenter of a group G. If G/Z is locally finite, then every finitely generated subgroup of G is nilpotent-by-finite.*

`Proof`. Let F be an arbitrary finitely generated subgroup of G. The factor-group FZ/Z is finite since it is finitely generated and locally finite. Since $FZ/Z \cong F/(F \cap Z)$, $F \cap Z$ has finite index in F. Then $F \cap Z$ is finitely generated too (see [3, Corollary 7.2.1]), and being hypercentral, is nilpotent. □

`Proof of Theorem B`. Let

$$\langle 1 \rangle = Z_0 \leq Z_1 \leq \cdots \leq Z_{k-1} \leq Z_k = Z$$

be the upper central series of G so that every term Z_j is G–invariant and every factors Z_j/Z_{j-1} is G–central. By L.A. Kaluzhnin's theorem [4], the factor-group $G/C_G(Z)$ is nilpotent of nilpotency class at most $k-1$. Put $C = C_G(Z)$ so that $Z \leq C_G(C)$. In particular, $G/C_G(C)$ is locally finite and has finite exponent at most e. Clearly $C \cap Z \leq \zeta(C)$ and then $C/(Z \cap C) \cong CZ/Z$ is locally finite and has finite exponent at most e. By Mann's theorem [5], the derived subgroup $D = [C, C]$ is locally finite and there exists a function m such that the exponent of D is bounded by $m(e)$. The subgroup D is G–invariant and C/D is abelian. We think of C/D as a $\mathbb{Z}H$–module where $H = (G/D)/C_{G/D}(C/D)$. Since C/G is abelian, $C/D \leq$

$C_{G/D}(C/D)$ and then $(G/D)/C_{G/D}(C/D)$ is nilpotent. Since $G/C_G(C)$ is locally finite, $(G/D)/C_{G/D}(C/D)$ is also locally finite.

We have $(C \cap Z)D/D \leq \zeta^{\infty}_{\mathbb{Z}H}(C/D)$ and $(C/D)/((C \leq Z)D/D) \cong C/(C \cap Z)D$ is a locally finite group of finite exponent at most e. By Lemma 2.2, C/D includes a $\mathbb{Z}G$–submodule V/D such that the additive group of V/D is periodic and has finite exponent at most e. Moreover, the factor-module $(C/D)/(V/D)$ is locally $\mathbb{Z}G$–nilpotent. Put $B = C/V$ and pick an arbitrary subset M of $E = G/V$ and put $F = \langle M \rangle$. By Lemma 2.4, F is nilpotent-by-finite, in particular, it is noetherian, that is it satisfies the maximal condition on subgroups. Then its subgroup $K = F \cap B$ is finitely generated. In this case K is finitely generated as a $\mathbb{Z}F$–module. Since B is a $\mathbb{Z}G$–module locally $\mathbb{Z}G$–nilpotent, its finitely generated $\mathbb{Z}F$–submodule K is $\mathbb{Z}F$–nilpotent. In other words, the upper hypercenter of F includes K. Since $F/K = F/(F \cap B) \cong FB/B$ is nilpotent, F is likewise nilpotent. Thus G/V is locally nilpotent and hence V includes the locally nilpotent residual L. Since D (respectively V/D) is locally finite and has finite exponent at most $m(e)$ (respectively e), V is locally finite and has finite exponent at most $em(e)$. In particular, L is locally finite and has finite exponent at most $em(e)$.

Finally, Lemma 2.4 shows that G is locally noetherian, and it suffices to apply Lemma 2.3 to see that G/L is locally nilpotent, as required. □

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