The Commutators of Intrinsic Square Functions on Weighted Herz Spaces

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Abstract In this paper, we consider the boundedness properties of commutator operators generated by $BMO(\mathbb{R}^n)$ function and intrinsic square function including the Lusin area integral, Littlewood-Paley g-function and g^*_λ-function on weighted Herz spaces $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)$ with general Muckenhoupt weights.

Keywords BMO; commutator; intrinsic square function; Muckenhoupt weight; weighted Herz space.

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1 Introduction and results

The intrinsic square functions were introduced by Wilson in [1] and [2]. For $0<\beta\le 1$, let $\mathcal{C}_\beta$ be the family of functions φ defined on $\mathbb{R}^n$ such that φ has support containing in $\{x\in\mathbb{R}^n : |x|\le 1\}$ and $\int_{\mathbb{R}^n}\varphi(x)dx=0$, and for all $x,x'\in\mathbb{R}^n$,

$$|\varphi(x)-\varphi(x')|\le |x-x'|^\beta.$$

For $(y,t)\in\mathbb{R}^{n+1}_+$ and $f\in L_{loc}(\mathbb{R}^n)$, set

$$A_\beta(f)(y,t)=\sup_{\varphi\in\mathcal{C}_\beta}|f*\varphi_t(y)|=\sup_{\varphi\in\mathcal{C}_\beta}\left|\int_{\mathbb{R}^n}\varphi_t(y-z)f(z)dz\right|.$$

Define the intrinsic square function of f (of order β) by the formula

$$\mathcal{S}_\beta(f)(x)=\left(\iint_{\Gamma(x)}\Big(A_\beta(f)(y,t)\Big)^2\frac{dydt}{t^{n+1}}\right)^{1/2},$$

where $\Gamma(x)=\{(y,t)\in\mathbb{R}^{n+1}_+ :\ |x-y|<t\}$. Also, define a cone of aperture γ for any $\gamma>0$ as follows.

$$\Gamma_\gamma(x)=\{(y,t)\in\mathbb{R}^{n+1} :\ |x-y|<\gamma t\}$$

and corresponding square function

$$\mathcal{S}_{\beta,\gamma}(f)(x)=\left(\iint_{\Gamma_\gamma(x)}\Big(A_\beta(f)(y,t)\Big)^2\frac{dydt}{t^{n+1}}\right)^{1/2}.$$

The intrinsic Littlewood-Paley $\mathcal{G}$-function and the intrinsic $\mathcal{G}^*_\lambda$-function are given by

$$\mathcal{G}_\beta(f)(x)=\left(\int_0^\infty\Big(A_\beta(f)(x,t)\Big)^2\frac{dt}{t}\right)^{1/2}$$

and

$$\mathcal{G}^*_{\lambda,\beta}(f)(x)=\left(\iint_{\mathbb{R}^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\Big(A_\beta(f)(y,t)\Big)^2\frac{dydt}{t^{n+1}}\right)^{1/2},\ \lambda>1,$$

respectively.

For any given weight function ω on $\mathbb{R}^n$ and $0 < p < \infty$, denote by $L^p_\omega(\mathbb{R}^n)$ the space of all function f satisfying

$$\|f\|_{L^p_\omega(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx\right)^{1/p} < \infty.$$

In [2], Wilson showed the following weighted L^p boundedness of the intrinsic square functions.

Theorem 1.1. *For $0 < \beta \leq 1, 1 < p < \infty$ and $\omega \in A_p(\mathbb{R}^n)$ (Muckenhoupt weight class), there exists a constant $C > 0$ independent of f such that*

$$\|\mathcal{S}_\beta(f)\|_{L^p_\omega(\mathbb{R}^n)} \leq C\|f\|_{L^p_\omega(\mathbb{R}^n)}.$$

Let b be a locally integrable function on $\mathbb{R}^n$. In this paper, we also consider the commutators generated by b and intrinsic square functions, which are defined by

$$[b, S_\beta](f)(x) = \left(\iint_{\Gamma(x)} \sup_{\varphi\in\mathcal{C}_\beta} \left|\int_{\mathbb{R}^n} [b(x) - b(z)]\varphi_t(y - z)f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

$$[b, \mathcal{G}_\beta](f)(x) = \left(\int_0^\infty \sup_{\varphi\in\mathcal{C}_\beta} \left|\int_{\mathbb{R}^n} [b(x) - b(y)]\varphi_t(x - y)f(y)dy\right|^2 \frac{dt}{t}\right)^{1/2}$$

and

$$[b, \mathcal{G}^*_{\lambda,\beta}](f)(x) =$$
$$\left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t + |x - y|}\right)^{\lambda n} \sup_{\varphi\in\mathcal{C}_\beta} \left|\int_{\mathbb{R}^n} [b(x) - b(z)]\varphi_t(y - z)f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}, \lambda > 1,$$

respectively. Wang in [3] established weighted L^p estimates for the commutators of the intrinsic square functions and proved

Theorem 1.2. *For $0 < \beta \leq 1, 1 < p < \infty, \omega \in A_p(\mathbb{R}^n)$ and $b \in BMO(\mathbb{R}^n)$, the commutators $[b, \mathcal{S}_\beta]$ and $[b, \mathcal{G}_\beta]$ are bounded on $L^p_\omega(\mathbb{R}^n)$. If $\lambda > max\{p, 3\}$, then the commutator $[b, \mathcal{G}^*_{\lambda,\beta}]$ is also bounded on $L^p_\omega(\mathbb{R}^n)$.*

For more L^p estimates of commutators we refer readers to [4] and [5].

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$ for any $k \in \mathbb{Z}$. Denote $\chi_k = \chi_{B_k} - \chi_{B_{k-1}}$ for $k \in \mathbb{Z}$, where χ_{B_k} is the characteristic function of the set B_k. The following weighted Herz space is introduced by Lu and Yang in [6].

Let $\alpha \in \mathbb{R}, 0 < p, q < \infty$ and ω_1, ω_2 be two weight functions on $\mathbb{R}^n$. The homogeneous weighted Herz space $\dot{K}^{\alpha,p}_q(\omega_1, \omega_2)(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_q(\omega_1, \omega_2)(\mathbb{R}^n) = \left\{f \in L^q_{loc}(\mathbb{R}^n \setminus \{0\}, \omega_2) : \|f\|_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{\infty} \omega_1(B_k)^{\alpha p/n} \|f\chi_k\|^p_{L^q_{\omega_2}(\mathbb{R}^n)}\right)^{1/p}.$$

Obviously, if $\alpha = 0$, then $\dot{K}_q^{0,q}(\omega_1,\omega_2)(\mathbb{R}^n) = L^q_{\omega_2}(\mathbb{R}^n)$ for any $0 < q < \infty$. Thus, weighted Herz spaces are generalizations of the weighted Lebesgue spaces. Lu, Yabuta and Yang in [7] proved boundedness results for sublinear operators on weighted Herz-type spaces with general Muckenhoupt weights. Later, Wang in [8] obtained the strong type estimates of intrinsic square functions on the weighted Herz spaces $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)$ with general weights and have the following.

Theorem 1.3. *Let $0 < \beta \le 1, 0 < p < \infty, 1 < q < \infty, \omega_1 \in A_{q_1}(\mathbb{R}^n)$ and $\omega_2 \in A_{q_2}(\mathbb{R}^n)$. Suppose that ω_1 and ω_2 satisfy either of the following conditions*

(i) $\omega_1 = \omega_2, 1 \le q_1 = q_2 \le q$ and $-nq_1/q < \alpha q_1 < n(1 - q_1/q)$;

(ii) $\omega_1 \ne \omega_2, 1 \le q_1 < \infty, 1 \le q_2 \le q$ and $0 < \alpha q_1 < n(1 - q_2/q)$.

Then S_β and $\mathcal{G}_\beta$ are bounded on $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)$. If $\lambda > max\{p, 3\}$, then the operator $\mathcal{G}^*_{\lambda,\beta}$ is also bounded on $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)$.

As a continuation of his work, the purpose of this paper is to investigate the boundedness of commutator generated by intrinsic square functions and $BMO(\mathbb{R}^n)$ function on weighted Herz spaces. Our main results in the paper are presented as follows.

Theorem 1.4. *Let $0 < \beta \le 1, 0 < p < \infty, 1 < q < \infty, \omega_1 \in A_{q_1}(\mathbb{R}^n)$ and $\omega_2 \in A_{q_2}(\mathbb{R}^n)$. Suppose $b \in BMO(\mathbb{R}^n)$, then $[b, S_\beta]$ is bounded on $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)$ provided that ω_1 and ω_2 satisfy either of the following*

(i) $\omega_1 = \omega_2, 1 \le q_1 = q_2 \le q$ and $-nq_1/q < \alpha q_1 < n(1 - q_1/q)$;

(ii) $\omega_1 \ne \omega_2, 1 \le q_1 < \infty, 1 \le q_2 \le q$ and $0 < \alpha q_1 < n(1 - q_2/q)$.

Theorem 1.5. *Let $0 < \beta \le 1, 0 < p < \infty, 1 < q < \infty, \omega_1 \in A_{q_1}(\mathbb{R}^n)$ and $\omega_2 \in A_{q_2}(\mathbb{R}^n)$. Suppose $b \in BMO(\mathbb{R}^n)$, then $[b, \mathcal{G}_\beta]$ is bounded on $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)$ provided that ω_1 and ω_2 satisfy either of the following*

(i) $\omega_1 = \omega_2, 1 \le q_1 = q_2 \le q$ and $-nq_1/q < \alpha q_1 < n(1 - q_1/q)$;

(ii) $\omega_1 \ne \omega_2, 1 \le q_1 < \infty, 1 \le q_2 \le q$ and $0 < \alpha q_1 < n(1 - q_2/q)$.

Theorem 1.6. *Let $0 < \beta \le 1, 0 < p < \infty, 1 < q < \infty, \omega_1 \in A_{q_1}(\mathbb{R}^n)$ and $\omega_2 \in A_{q_2}(\mathbb{R}^n)$. If $b \in BMO(\mathbb{R}^n)$ and $\lambda > \max\{p, 3\}$, then $[b, \mathcal{G}^*_{\lambda,\beta}]$ is bounded on $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)$ provided that ω_1 and ω_2 satisfy either of the following*

(i) $\omega_1 = \omega_2, 1 \le q_1 = q_2 \le q$ and $-nq_1/q < \alpha q_1 < n(1 - q_1/q)$;

(ii) $\omega_1 \ne \omega_2, 1 \le q_1 < \infty, 1 \le q_2 \le q$ and $0 < \alpha q_1 < n(1 - q_2/q)$.

The paper is organized as follows. In Section 2, we give some basic notation and definitions, and recall some preliminaries. In Section 3, we prove Theorem 1.4. In Section 4, we give the proofs of Theorems 1.5 and Theorems 1.6.

Throughout this paper, without special mention made, C will be used to denote a positive constant that is not necessarily the same at each occurrence.

2 Definitions and preliminaries

We begin with some properties of A_p weights which play a great role in the proofs of our main results.

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B. For any ball B and $\lambda >, \lambda B$ denotes the ball concentric with B whose radius is λ times as long. For a given weight function ω and a measurable set E, we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $\omega(E) = \int_E \omega(x)dx$.

A weight ω is said to belong to $A_p(\mathbb{R}^n)$ for $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{1-p'}dx\right)^{p-1} \leq C, \tag{2.1}$$

where s' is the dual of s such that $1/s + 1/s' = 1$. The class $A_1(\mathbb{R}^n)$ is defined by replacing the above inequality with

$$\frac{1}{|B|}\int_B w(y)dy \leq C \cdot \operatorname*{ess\,inf}_{x\in B} w(x) \quad \text{for every ball } B \subset \mathbb{R}^n. \tag{2.2}$$

A weight ω is said to belong to $A_\infty(\mathbb{R}^n)$ if there are positive numbers C and δ so that

$$\frac{\omega(E)}{\omega(B)} \leq C\left(\frac{|E|}{|B|}\right)^\delta \tag{2.3}$$

for all balls B and all measurable $E \subset B$. It is well known that

$$A_\infty(\mathbb{R}^n) = \bigcup_{1\leq p<\infty} A_p(\mathbb{R}^n). \tag{2.4}$$

By (2.1), we have

$$\left(\int_B \omega(x)dx\right)\left(\int_B \omega(x)^{1-p'}dx\right)^{p-1} \leq C|B|^p \tag{2.5}$$

for $1 < p < \infty$.

The classical $A_p(\mathbb{R}^n)$ weight theory was first introduced by Muckenhoupt in the study of weighted L^p-boundedness of Hardy-Littlewood maximal function in [9].

Lemma 2.1. (*see*[*9*][*10*]) *Suppose $\omega \in A_p(\mathbb{R}^n)$ and the following statements hold.*

(i) For any $1 \leq p < \infty$, there are positive numbers C and δ such that

$$\frac{\omega(B_k)}{\omega(B_j)} \leq C2^{np(k-j)} \quad \text{for } \ k > j \tag{2.6}$$

and

$$\frac{\omega(B_k)}{\omega(B_j)} \leq C2^{np\delta(k-j)} \quad \text{for } \ k < j; \tag{2.7}$$

(ii) $A_{p_1}(\mathbb{R}^n) \subset A_{p_2}(\mathbb{R}^n)$ for any $1 \leq p_1 < p_2 \leq \infty$;

(iii) For any $1 < p < \infty$, one has $\omega^{1-p'} \in A_{p'}(\mathbb{R}^n)$.

A locally integrable function b is said to be in $BMO(\mathbb{R}^n)$ if

$$\sup_{B\subset\mathbb{R}^n}\frac{1}{|B|}\int_B|b(x)-b_B|dx=\|b\|_*<\infty,$$

where $b_B=|B|^{-1}\int_B b(y)dy$.

Lemma 2.2. *(John-Nirenberg inequality, see[11]) Let $b\in BMO(\mathbb{R}^n)$. Then for any ball $B\subset\mathbb{R}^n$, there exist positive constants C_1 and C_2 such that for all $\lambda>0$,*

$$|\{x\in B:|b(x)-b_B|>\lambda\}|\le C_1|B|exp(-C_2\lambda/\|b\|_*).$$

Lemma 2.3. [12] *Let $\omega\in A_\infty(\mathbb{R}^n)$. Then the norm of $BMO(\omega,\mathbb{R}^n)$ is equivalent to the norm of $BMO(\mathbb{R}^n)$, where*

$$BMO(\omega,\mathbb{R}^n)=\left\{b:\|b\|_{*,\omega}=\sup_{B\subset\mathbb{R}^n}\frac{1}{\omega(B)}\int_B|b(x)-b_{B,\omega}|\omega(x)dx\right\},$$

and

$$b_{B,\omega}=\frac{1}{\omega(B)}\int_B b(z)\omega(z)dz.$$

Lemma 2.4. *Suppose $\omega\in A_\infty(\mathbb{R}^n)$ and $b\in BMO(\mathbb{R}^n)$. Then for any $p\ge1$ we have*

$$\left(\frac{1}{\omega(B)}\int_B|b(x)-b_{B,\omega}|^p\omega(x)dx\right)^{1/p}\le C\|b\|_*. \tag{2.8}$$

Proof Since $\omega(x)\in A_\infty(\mathbb{R}^n)$, (2.3) and Lemma2.2 imply that

$$\omega\left(\{x\in B:\ |b(x)-b_B|>\lambda\}\right)\le C\omega(B)exp(-C_2\delta\lambda/\|b\|_*).$$

Therefore,

$$\begin{aligned}\int_B|b(x)-b_B|^p\omega(x)dx&\le p\int_0^\infty\lambda^{p-1}\omega\left(\{x\in B:|b(x)-b_B|>\lambda\}\right)d\lambda\\&\le C\omega(B)\int_0^\infty\lambda^{p-1}exp(-C_2\delta\lambda\|b\|_*)d\lambda\le\ C\omega(B)\|b\|_*^p.\end{aligned}$$

Hence

$$\frac{1}{\omega(B)}\int_B|b(x)-b_{B,\omega}|^p\omega(x)dx\le\frac{C}{\omega(B)}\int_B|b(x)-b_B|^p\omega(x)dx\le C\|b\|_*^p.$$

3 Proof of Theorem 1.4

Let $f\in\dot{K}_p^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)$ and $b\in BMO(\mathbb{R}^n)$, then

$$\begin{aligned}&\left\|[b,\mathcal{S}_\beta]f\right\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)}\\ \le\ & C\sum_{j=-\infty}^{\infty}(\omega_1(B_j))^{\frac{\alpha p}{n}}\left(\sum_{k=-\infty}^{j-2}\left\|([b,\mathcal{S}_\beta](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)}\right)^p\\ &+\ C\sum_{j=-\infty}^{\infty}(\omega_1(B_j))^{\frac{\alpha p}{n}}\left(\sum_{k=j-1}^{j+1}\left\|([b,\mathcal{S}_\beta](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)}\right)^p\\ &+\ C\sum_{j=-\infty}^{\infty}(\omega_1(B_j))^{\frac{\alpha p}{n}}\left(\sum_{k=j+2}^{\infty}\left\|([b,\mathcal{S}_\beta](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)}\right)^p\\ =\ & E_1+E_2+E_3.\end{aligned}$$

Using the fact that $[b,\mathcal{S}_\beta]$ is a bounded operator on $L^q_{\omega_2}(\mathbb{R}^n)$, we obtain

$$
\begin{aligned}
E_2 &\le C\|b\|_*^p \sum_{j=-\infty}^{\infty} (\omega_1(B_j))^{\frac{\alpha p}{n}} \left(\sum_{k=j-1}^{j+1} \left\| f\chi_k \right\|_{L^q_{\omega_2}(\mathbb{R}^n)} \right)^p \\
&\le C\|b\|_*^p \sum_{j=-\infty}^{\infty} (\omega_1(B_j))^{\frac{\alpha p}{n}} \left\| f\chi_j \right\|^p_{L^q_{\omega_2}(\mathbb{R}^n)} \\
&\le C\|b\|_*^p \|f\|^p_{\dot{K}^{\alpha,p}_p(\omega_1,\omega_2)(\mathbb{R}^n)}.
\end{aligned}
$$

We now estimate E_1. Obviously,

$$
\begin{aligned}
&\left\| [b,S_\beta](f\chi_k)\chi_j(x) \right\|_{L^q_{\omega_2}(\mathbb{R}^n)} \\
&\le C \left(\int_{C_j} |(b(x)-b_{B_k})\mathcal{S}_\beta(f\chi_k)(x)|^q\, \omega_2(x)dx \right)^{1/q} \\
&\quad + \; C \left(\int_{C_j} |\mathcal{S}_\beta((b(\cdot)-b_{B_k})f\chi_k)(x)|^q\, \omega_2(x)dx \right)^{1/q} \\
&= F_1 + F_2.
\end{aligned}
$$

First, we estimate F_1. For any $\varphi \in \mathcal{C}_\beta$ and $(y,t) \in \Gamma(x)$, we have

$$
\left| \int_{C_k} \varphi_t(y-z)f(z)dz \right| \le Ct^{-n} \int_{C_k \cap \{z:|y-z|\le t\}} |f(z)|dz. \tag{3.1}
$$

For any $x \in C_j$, $(y,t) \in \Gamma(x)$ and $z \in C_k \cap \{z : |y-z| \le t\}$ with $j \ge k+2$, by a direct computation, one easily see that

$$
2t \ge |x-y| + |y-z| \ge |x-z| \ge |x| - |z| \ge |x|/2.
$$

Thus, by using the above inequality and (3.1), we deduce

$$
\begin{aligned}
\mathcal{S}_\beta(f\chi_k)(x) = &\left(\iint_{\Gamma(x)} \sup_{\varphi\in\mathcal{C}_\beta} \left| \int_{C_k} \varphi_t(y-z)f(z)dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \\
\le \; & C \left(\int_{|x|/4}^{\infty} \int_{|x-y|<t} \left| t^{-n} \int_{C_k} |f(z)|dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \\
\le \; & C \left(\int_{C_k} |f(z)|dz \right) \left(\int_{|x|/4}^{\infty} \frac{dt}{t^{2n+1}} \right)^{1/2} \\
\le \; & C|x|^{-n} \left(\int_{C_k} |f(z)|dz \right) \le C2^{-jn} \|f\chi_k\|_{L^1(\mathbb{R}^n)}.
\end{aligned} \tag{3.2}
$$

Then

$$
\begin{aligned}
F_1 &\le C2^{-jn}\|f\chi_k\|_{L^1(\mathbb{R}^n)} \left(\int_{B_j} |b(x)-b_{B_k}|^q \omega_2(x)dx \right)^{1/q} \\
&\le C2^{-jn}\|f\chi_k\|_{L^1(\mathbb{R}^n)} \Big(\big(\int_{B_j} |b(x)-b_{B_j,\omega_2}|^q \omega_2(x)dx \big)^{1/q} \\
&\quad + \big(|b_{B_j}-b_{B_j,\omega_2}| + |b_{B_j}-b_{B_k}| \big) \big(\int_{B_j} \omega_2(x)dx \big)^{1/q} \Big).
\end{aligned}
$$

By Lemma 2.4

$$\Big(\int_{B_j}|b(x)-b_{B_j,\omega_2}|^q\omega_2(x)dx\Big)^{1/q}\le C\|b\|_*(\int_{B_j}\omega_2(x)dx)^{1/q}. \tag{3.3}$$

Similar to the proof of Lemma 2.4, we deduce

$$|b_{B_j}-b_{B_j,\omega_2}|\le C\|b\|_*. \tag{3.4}$$

From the definition of $BMO(\mathbb{R}^n)$, it is easy to that

$$|b_{B_j}-b_{B_k}|\le C|j-k|\|b\|_*. \tag{3.5}$$

By Hölder's inequality,

$$\|f\chi_k\|_{L^1(\mathbb{R}^n)}\le\ C\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}. \tag{3.6}$$

Thus

$$F_1\le C(j-k)2^{-jn}\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(\int_{B_j}\omega_2(x)dx)^{1/q}(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}.$$

We now turn to estimate F_2. For any $\varphi\in\mathcal{C}_\beta$ and $(y,t)\in\Gamma(x)$, we have

$$\left|\int_{C_k}(b(z)-b_{B_k})\varphi_t(y-z)f(z)dz\right|\le Ct^{-n}\int_{C_k\cap\{z:|y-z|\le t\}}|b(z)-b_{B_k}||f(z)|dz \tag{3.7}$$

So, similar to the estimate of (3.2), we deduce

$$\begin{aligned}&\mathcal{S}_\beta((b(\cdot)-b_{B_k})f\chi_k)(x)\\ &=\left(\iint_{\Gamma(x)}\sup_{\varphi\in\mathcal{C}_\beta}\left|\int_{C_k}(b(z)-b_{B_k})\varphi_t(y-z)f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\ &\le C2^{-jn}\left(\int_{C_k}|b(z)-b_{B_k}||f(z)|dz\right).\end{aligned} \tag{3.8}$$

Since $\omega_2\in A_{q_2}(\mathbb{R}^n)\subset A_q(\mathbb{R}^n)$, by Lemma 2.1 we know $\omega_2^{1-q'}\in A_{q'}(\mathbb{R}^n)$. From Lemma 2.4, we have

$$\left(\int_{B_k}|b(x)-b_{B_k}|^{q'}\omega_2(x)^{1-q'}dx\right)^{1/q'}\le C\|b\|_*(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}. \tag{3.9}$$

Using Hölder's inequality and (3.9),

$$\begin{aligned}F_2&\le C2^{-jn}\int_{B_k}|b(y)-b_{B_k}||f\chi_k(y)|dy\big(\int_{B_j}\omega_2(x)dx\big)^{1/q}\\ &\le C2^{-jn}\left(\int_{B_k}|b(y)-b_{B_k}|^{q'}\omega_2(y)^{1-q'}dy\right)^{1/q'}\|f\chi_k\|_{L^q(\omega_2)}\big(\int_{B_j}\omega_2(x)dx\big)^{1/q}\\ &\le C\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}2^{-jn}(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}\big(\int_{B_j}\omega_2(x)dx\big)^{1/q}.\end{aligned} \tag{3.10}$$

Since $\omega_2\in A_q(\mathbb{R}^n)$, by (2.5) and (2.6),

$$\begin{aligned}&(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}\big(\int_{B_j}\omega_2(x)dx\big)^{1/q}\\ &=(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}\big(\int_{B_j}\omega_2(x)dx\big)^{1/q}\left(\frac{\omega_2(B_j)}{\omega_2(B_k)}\right)^{1/q}\\ &\le C2^{kn+(j-k)nq_2/q}.\end{aligned} \tag{3.11}$$

Summarizing the above estimates, we have that for $j \geq k+2$,

$$\left\|[b, S_\beta](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)} \leq C\|b\|_*\|f\chi_k\|_{L^q(\omega_2)}(j-k)2^{(j-k)n(q_2/q-1)}. \tag{3.12}$$

Then note that $\omega_1 \in A_{q_1}(\mathbb{R}^n)$ and get

$$\begin{aligned}
E_1 &= C\sum_{j=-\infty}^{\infty}(\omega_1(B_j))^{\frac{\alpha p}{n}}\left(\sum_{k=-\infty}^{j-2}\left\|[b,\mathcal{S}_\beta](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)}\right)^p \\
&\leq C\|b\|_*^p\sum_{j=-\infty}^{\infty}\left(\sum_{k=-\infty}^{j-2}(\omega_1(B_j))^{\frac{\alpha}{n}}\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(j-k)2^{(j-k)n(q_2/q-1)}\right)^p \\
&\leq C\|b\|_*^p\sum_{j=-\infty}^{\infty}\left(\sum_{k=-\infty}^{j-2}(\omega_1(B_k))^{\frac{\alpha}{n}}\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(j-k)2^{(j-k)(\alpha q_1+q_2n/q-n)}\right)^p.
\end{aligned}$$

When $0 < p \leq 1$, we have

$$\begin{aligned}
E_1 &\leq C\|b\|_*^p\sum_{k=-\infty}^{\infty}(\omega_1(B_k))^{\frac{\alpha p}{n}}\|f\chi_k\|^p_{L^q_{\omega_2}(\mathbb{R}^n)}\sum_{j=k+2}^{\infty}(j-k)^p 2^{(j-k)p(\alpha q_1+q_2n/q-n)} \\
&\leq C\|b\|_*^p\sum_{k=-\infty}^{\infty}(\omega_1(B_k))^{\frac{\alpha p}{n}}\|f\chi_k\|^p_{L^q_{\omega_2}(\mathbb{R}^n)} \\
&\leq C\|b\|_*^p\|f\|^p_{\dot K^{\alpha,p}_q(\omega_1,\omega_2)(\mathbb{R}^n)}.
\end{aligned}$$

When $p > 1$, by Hölder's inequality we have

$$\begin{aligned}
E_1 &\leq C\|b\|_*^p\sum_{j=-\infty}^{\infty}\left(\sum_{k=-\infty}^{j-2}(\omega_1(B_k))^{\frac{\alpha p}{n}}\|f\chi_k\|^p_{L^q_{\omega_2}(\mathbb{R}^n)}(j-k)2^{(j-k)(\alpha q_1+q_2n/q-n)}\right) \\
&\qquad\left(\sum_{k=-\infty}^{j-2}(j-k)2^{(j-k)(\alpha q_1+q_2n/q-n)}\right)^{p/p'} \\
&\leq C\|b\|_*^p\sum_{k=-\infty}^{\infty}(\omega_1(B_k))^{\frac{\alpha p}{n}}\|f\chi_k\|^p_{L^q_{\omega_2}(\mathbb{R}^n)} \\
&\leq C\|b\|_*^p\|f\|^p_{\dot K^{\alpha,p}_q(\omega_1,\omega_2)(\mathbb{R}^n)}.
\end{aligned}$$

Let us now turn to estimate the last term E_3. In this case, for any $x \in C_j$, $(y,t) \in \Gamma(x)$ and $z \in C_k \cap \{z : |y-z| \leq t\}$ with $j \leq k-2$, then by a direct computation, we easily see that

$$2t \geq |x-y| + |y-z| \geq |x-z| \geq |z| - |x| \geq |z|/2.$$

Thus, by using the above inequality and (3.1), we deduce

$$\begin{aligned}
\mathcal{S}_\beta(f\chi_k)(x) &= \left(\iint_{\Gamma(x)}\sup_{\varphi\in\mathcal{C}_\beta}\left|\int_{C_k}\varphi_t(y-z)f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\
&\leq C\left(\int_{|z|/4}^{\infty}\int_{|x-y|<t}\left|t^{-n}\int_{C_k}|f(z)|dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}
\end{aligned}$$

$$\leq C\left(\int_{C_k}|f(z)|dz\right)\left(\int_{|z|/4}^{\infty}\frac{dt}{t^{2n+1}}\right)^{1/2}$$

$$\leq C|z|^{-n}\left(\int_{C_k}|f(z)|dz\right)\leq C2^{-kn}\|f\chi_k\|_{L^1(\mathbb{R}^n)}. \tag{3.13}$$

By (3.6),(2.8),(2.5) and (2.7),

$$F_1\leq 2^{-kn}\|f\chi_k\|_{L^1(\mathbb{R}^n)}\left(\int_{B_j}|b(x)-b_{B_k}|^q\omega_2(x)dx\right)^{1/q}$$

$$\leq C2^{-kn}(k-j)\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(\int_{B_k}\omega_2(x)dx)^{1/q}(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}\left(\frac{\omega_2(B_j)}{\omega_2(B_k)}\right)^{1/q}$$

$$\leq C\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(k-j)2^{(j-k)\delta_2/q}. \tag{3.14}$$

Similar to the proof of estimate for E_1,

$$F_2\leq C2^{-kn}\int_{B_k}|b(y)-b_{B_k}||f\chi_k(y)|dy(\int_{B_j}\omega_2(x)dx)^{1/q}$$

$$\leq C\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}2^{-kn}(\int_{B_k}\omega_2(x)^{1-q'}dx)^{1/q'}(\int_{B_j}\omega_2(x)dx)^{1/q}$$

$$\leq C\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}2^{(j-k)\delta_2/q}. \tag{3.15}$$

Hence

$$\left\|[b,S_\beta](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)}\leq C\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(k-j)2^{(j-k)\delta_2/q}. \tag{3.16}$$

When $0<p\leq 1$, we have

$$E_3\leq C\|b\|_*^p\sum_{j=-\infty}^{\infty}\omega_1(B_j)^{\frac{\alpha p}{n}}\sum_{k=j+2}^{\infty}\|f\chi_k\|^p_{L^q_{\omega_2}(\mathbb{R}^n)}(k-j)^p2^{(j-k)p\delta_2/q}$$

$$\leq C\|b\|_*^p\sum_{k=-\infty}^{\infty}(\omega_1(B_k))^{\frac{\alpha p}{n}}\|f\chi_k\|^p_{L^q_{\omega_2}(\mathbb{R}^n)}\sum_{j=-\infty}^{k-2}\left(\frac{\omega_1(B_j)}{\omega_1(B_k)}\right)^{\frac{\alpha p}{n}}(k-j)^p2^{(j-k)p\delta_2/q}$$

$$\leq C\|b\|_*^p\sum_{k=-\infty}^{\infty}(\omega_1(B_k))^{\frac{\alpha p}{n}}\|f\chi_k\|^p_{L^q_{\omega_2}(\mathbb{R}^n)}\sum_{j=-\infty}^{k-2}(k-j)^p2^{(j-k)p(\delta_1\alpha/n+\delta_2/q)}$$

$$\leq C\|b\|_*^p\|f\|^p_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)(\mathbb{R}^n)}.$$

When $q>1$, by Hölder's inequality again, we get

$$E_3\leq C\|b\|_*^p\sum_{j=-\infty}^{\infty}\left(\sum_{k=j+2}^{\infty}(\omega_1(B_k))^{\frac{\alpha}{n}}\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(k-j)2^{(j-k)(\delta_1\alpha/n+\delta_2/q)}\right)^p$$

$$\leq C\|b\|_*^p\sum_{j=-\infty}^{\infty}\left(\sum_{k=j+2}^{\infty}(\omega_1(B_k))^{\frac{\alpha p}{n}}\|f\chi_k\|^p_{L^q_{\omega_2}(\mathbb{R}^n)}(k-j)2^{(j-k)(\delta_1\alpha/n+\delta_2/q)}\right)$$

$$\left(\sum_{k=j+2}^{\infty}(k-j)2^{(j-k)(\delta_1\alpha/n+\delta_2/q)}\right)^{p/p\prime}$$

$$\le C\|b\|_*^p \sum_{k=-\infty}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|^p_{L^q_{\omega_2}(\mathbb{R}^n)} \sum_{j=-\infty}^{k-2} (k-j)2^{(j-k)(\delta_1\alpha/n+\delta_2/q)}$$
$$\le C\|b\|_*^p \|f\|^p_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)(\mathbb{R}^n)}.$$

Combining the above estimates for E_1, E_2 and E_3, we have completed the proof of Theorem 1.4.

4 Proofs of Theorems 1.5 and Theorems 1.6

Let $b \in BMO(\mathbb{R}^n)$ and T be a sublinear operator. Assumes that T and $[b,T]$ are bounded on $L^q_{\omega_2}(\mathbb{R}^n)$. According to the proof of Theorem 1.4, in order to prove the boundedness of $[b,T]$ on $\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)(\mathbb{R}^n)$, we only need to prove

$$\left\|[b,T](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)} \le C\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(j-k)2^{(j-k)n(q_2/q-1)} \tag{4.1}$$

for $j \ge k+2$ and

$$\left\|[b,T](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)} \le C\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(k-j)2^{(j-k)\delta_2/q} \tag{4.2}$$

for $j \le k-2$.

Proof of Theorem 1.5 In [1], Wilson showed that for any $0 < \beta \le 1$, the function $\mathcal{S}_\beta(f)(x)$ and $\mathcal{G}_\beta(f)(x)$ are pointwise comparable, with comparability constants depending only on β and n.

If $j \ge k+2$, then from (3.2) and (3.8) we have

$$\begin{aligned}
&\|[b,\mathcal{G}_\beta](f\chi_k)\chi_j(x)\|_{L^q_{\omega_2}(\mathbb{R}^n)} \\
&\le C\left(\int_{C_j} |(b(x)-b_{B_k})\mathcal{G}_\beta(f\chi_k)(x)|^q\omega_2(x)dx\right)^{1/q} \\
&\quad + C\left(\int_{C_j} |\mathcal{G}_\beta((b(\cdot)-b_{B_k})f\chi_k)(x)|^q\omega_2(x)dx\right)^{1/q} \\
&\le C2^{-jn}\left(\|f\chi_k\|_{L^1(\mathbb{R}^n)} + \int_{B_k} |b(y)-b_{B_k}||f\chi_k(y)|dy\right).
\end{aligned}$$

Using the inequalities (3.3)-(3.6) and (3.9)-(3.11), we can deduce

$$\|[b,\mathcal{G}_\beta](f\chi_k)\chi_j(x)\|_{L^q_{\omega_2}(\mathbb{R}^n)} \le C\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(j-k)2^{(j-k)n(q_2/q-1)}.$$

If $j \le k-2$, according to (3.13)-(3.15), we obtain

$$\begin{aligned}
&\|[b,\mathcal{G}_\beta](f\chi_k)\chi_j(x)\|_{L^q_{\omega_2}(\mathbb{R}^n)} \\
&\le C2^{-kn}\left(\|f\chi_k\|_{L^1(\mathbb{R}^n)} + \int_{B_k} |b(y)-b_{B_k}||f\chi_k(y)|dy\right) \\
&\le C\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(k-j)2^{(j-k)\delta_2/q}
\end{aligned}$$

and the conclusion follows.

Proof of Theorem 1.6 For any $i \in \mathbb{Z}_+, x \in C_j, (y,t) \in \Gamma_{2^i}(x)$ and $z \in C_k \cap B(y,t)$ with

$j \geq k+2$, we can easily deduce

$$t + 2^i t \geq |x-y| + |y-z| \geq |x-z| \geq |x| - |z| \geq |x|/2.$$

Thus, in virtue of the above inequality and (3.1) we have

$$\begin{aligned}
&\left(\iint_{\Gamma_{2^i}(x)} \sup_{\varphi\in\mathcal{C}_\beta} \left|\int_{C_k} \varphi_t(y-z) f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} \\
\leq\ & C\left(\int_{\frac{|x|}{2^{i+2}}}^\infty \int_{|x-y|<2^i t} \left| t^{-n} \int_{C_k} |f(z)|dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} \\
\leq\ & C2^{in} \left(\int_{C_k} |f(z)|dz\right) \left(\int_{\frac{|x|}{2^{i+2}}}^\infty \frac{dt}{t^{2n+1}}\right)^{1/2} \\
\leq\ & C2^{3in/2}|x|^{-n} \left(\int_{C_k} |f(z)|dz\right) \\
\leq\ & C2^{3in/2} 2^{-jn} \|f\chi_k\|_{L^1(\mathbb{R}^n)}. \qquad (4.3)
\end{aligned}$$

By the same method, we get

$$\begin{aligned}
&\left(\iint_{\Gamma_{2^i}(x)} \sup_{\varphi\in\mathcal{C}_\beta} \left|\int_{C_k} (b(z)-b_{B_k})\varphi_t(y-z) f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} \\
\leq\ & C2^{3in/2} 2^{-jn} \int_{C_k} |b(z)-b_{B_k}||f(z)|dz. \qquad (4.4)
\end{aligned}$$

Similar to that of Theorem 1.4

$$\begin{aligned}
&\left\|[b, S_{\beta,2^i}](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)} \\
\leq\ & C\left(\int_{C_j} \left((b(x)-b_{B_k})\left(\iint_{\Gamma_{2^i}(x)} \sup_{\varphi\in\mathcal{C}_\beta} \left|\int_{C_k} \varphi_t(y-z) f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}\right)^q \omega_2(x)dx\right)^{1/q} \\
&+\ C\left(\int_{C_j} \left(\iint_{\Gamma_{2^i}(x)} \sup_{\varphi\in\mathcal{C}_\beta} \left|\int_{C_k} (b(z)-b_{B_k})\varphi_t(y-z) f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{q/2} \omega_2(x)dx\right)^{1/q} \\
=\ & N_1 + N_2.
\end{aligned}$$

Since $j \geq k+2$, by (3.3)-(3.6) and (4.3), we have

$$\begin{aligned}
N_1 &\leq C2^{3in/2} 2^{-jn} \|f\chi_k\|_{L^1(\mathbb{R}^n)} \left(\int_{B_j} |b(x)-b_{B_k}|^q \omega_2(x)dx\right)^{1/q} \\
&\leq C2^{3in/2} \|b\|_* \|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)} (j-k) 2^{(j-k)n(q_2/q-1)}.
\end{aligned}$$

By (4.4) and (3.9)-(3.11),

$$\begin{aligned}
N_2 &\leq C2^{3in/2} 2^{-jn} \int_{C_k} |b(z)-b_{B_k}||f(z)|dz \left(\int_{B_j} \omega_2(x)dx\right)^{1/q} \\
&\leq C2^{3in/2} \|b\|_* \|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)} (j-k) 2^{(j-k)n(q_2/q-1)}.
\end{aligned}$$

Hence, for $j \geq k+2$, we get

$$\left\|[b, S_{\beta,2^i}](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)} \leq C2^{3in/2} \|b\|_* \|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)} (j-k) 2^{(j-k)n(q_2/q-1)}. \qquad (4.5)$$

From (3.12),(4.5) and $\lambda > 3$, it follows that

$$\begin{aligned}
&\left\|[b,\mathcal{G}^*_{\lambda,\beta}](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)}\\
&\le C\left\|[b,S_\beta](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)}+C\sum_{i=1}^{\infty}2^{-i\lambda n/2}\left\|[b,S_{\beta,2^i}](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)}\\
&\le C\left(1+\sum_{i=1}^{\infty}2^{(3-\lambda)in/2}\right)\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(j-k)2^{(j-k)n(q_2/q-1)}\\
&\le C\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(j-k)2^{(j-k)n(q_2/q-1)}.
\end{aligned}$$

Thus, we have obtained (4.1) for $T=\mathcal{G}^*_{\lambda,\beta}$.

We now pay attention to the inequality (4.2). For any $i\in\mathbb{Z}_+, x\in C_j, (y,t)\in\Gamma_{2^i}(x)$ and $z\in C_k\cap B(y,t)$ with $j\le k-2$, we deduce

$$t+2^it\ge|x-y|+|y-z|\ge|x-z|\ge|z|-|x|\ge|z|/2.$$

Similar to the estimates of (4.3) and (4.4), we have

$$\begin{aligned}
&\left(\iint_{\Gamma_{2^i}(x)}\sup_{\varphi\in\mathcal{C}_\beta}\left|\int_{C_k}\varphi_t(y-z)f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\
\le\;& C\left(\int_{\frac{|x|}{2^{i+2}}}^{\infty}\int_{|x-y|<2^it}\left|t^{-n}\int_{C_k}|f(z)|dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\
\le\;& C2^{3in/2}2^{-kn}\|f\chi_k\|_{L^1(\mathbb{R}^n)}
\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}
&\left(\iint_{\Gamma_{2^i}(x)}\sup_{\varphi\in\mathcal{C}_\beta}\left|\int_{C_k}(b(z)-b_{B_k})\varphi_t(y-z)f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\
&\le C2^{3in/2}2^{-kn}\int_{C_k}|b(z)-b_{B_k}||f(z)|dz.
\end{aligned}\tag{4.7}$$

By (4.4) and (4.5), similar to the estimates of (3.14) and (3.15), we get

$$\begin{aligned}
N_1&\le C2^{3in/2}2^{-kn}\|f\chi_k\|_{L^1(\mathbb{R}^n)}\left(\int_{B_j}|b(x)-b_{B_k}|^q\omega_2(x)dx\right)^{1/q}\\
&\le C2^{3in/2}\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(k-j)2^{(j-k)\delta_2/q},
\end{aligned}$$

and

$$\begin{aligned}
N_2&\le C2^{3in/2}2^{-kn}\int_{C_k}|b(z)-b_{B_k}||f(z)|dz\left(\int_{B_j}\omega_2(x)dx\right)^{1/q}\\
&\le C2^{3in/2}\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(k-j)2^{(j-k)\delta_2/q}.
\end{aligned}$$

Hence, for $j\le k-2$, we get

$$\left\|[b,S_{\beta,2^i}](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)}\le C2^{3in/2}\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(k-j)2^{(j-k)\delta_2/q}.\tag{4.8}$$

From (3.16),(4.8) and $\lambda > 3$, we have

$$\begin{aligned}
&\left\|[b,\mathcal{G}^*_{\lambda,\beta}](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)}\\
&\leq C\left\|[b,S_\beta](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)}+C\sum_{i=1}^{\infty}2^{-i\lambda n/2}\left\|[b,S_{\beta,2^i}](f\chi_k)\chi_j(x)\right\|_{L^q_{\omega_2}(\mathbb{R}^n)}\\
&\leq C\left(1+\sum_{i=1}^{\infty}2^{(3-\lambda)in/2}\right)\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(k-j)2^{(j-k)\delta_2/q}\\
&\leq C\|b\|_*\|f\chi_k\|_{L^q_{\omega_2}(\mathbb{R}^n)}(k-j)2^{(j-k)\delta_2/q}.
\end{aligned}$$

Consequently, (4.2) has been proved for $T=\mathcal{G}^*_{\lambda,\beta}$ and Theorem 1.6 holds.

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