A class of finite dimensional modular Lie superalgebras of special type

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Abstract

This paper is concerned with the Lie superalgebra $S(n,m)$ of special type over a field of prime characteristic. We construct the modular Lie superalgebra $S(n,m)$ and discuss some properties of this algebra. Then the derivation superalgebra of $S(n,m)$ is determined.

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1. Introduction

In mathematics, a Lie superalgebra is a generalization of a Lie algebra including a $\mathbb{Z}_2$-grading, where $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ is the residue class ring of integers modulo 2. Lie superalgebras are also important in theoretical physics where they are used to describe the mathematics of supersymmetry [8]. Although many structural features of non-modular Lie superalgebras (see [9, 15, 20, 22]) are well understood, there seem to be few general results on modular Lie superalgebras. The treatment of modular Lie superalgebras necessitates different techniques which are set forth in [11, 13]. In [7] Elduque obtained two new simple modular Lie superalgebras. These Lie superalgebras share the property that their even parts are orthogonal Lie algebras and the odd parts are their spin modules. Since distinctions between non-modular and modular Lie superalgebras are reflected on Lie superalgebras of Cartan type, the structure of Lie superalgebras of Cartan type seems to be important and interesting. In [23], four series of modular graded Lie superalgebras of Cartan type were constructed, which are analogous to the finite dimensional modular Lie algebras of Cartan type [19] or the four series

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of infinite dimensional Lie superalgebras of Cartan type defined by even differential forms over a field of characteristic zero [10]. In [16], the iso-classes of simple restricted modules of the special Lie superalgebra $S(n)$ were classified, and the character formulas of restricted simple modules were given. Recent works on modular Lie superalgebras of Cartan type can also be found in [5, 6, 12, 21] and references therein.

It is well known that derivation techniques are of great importance in the structure and the classification theories of Lie (super)algebras (see [3, 9, 15, 18]). For some classes of modular Lie (super)algebras, the derivation (super)algebras have been well investigated, for example, the derivation algebras of modular Lie algebras of Cartan type [4, 17], the derivation superalgebras of modular Lie superalgebras of Cartan type [2, 13].

The original motivation for this paper comes from the structures of the finite dimensional modular Lie superalgebras $W(n,m)$ and $H(n,m)$, which were first introduced in [1] and [14], respectively. The starting point of our studies is to construct a class of finite dimensional modular Lie superalgebras of special type, which is denoted by $S(n,m)$. A brief summary of the relevant concepts and notations in the finite dimensional modular Lie superalgebras $S(n,m)$ is presented in Section 2. In Section 3, the intrinsic properties of $S(n,m)$ are investigated. In Section 4, the derivation superalgebra of $S(n,m)$ is determined.

2. Preliminaries

Throughout this paper, $\mathbb{F}$ denotes an algebraic closed field of characteristic $p>2$, n is an integer greater than 3. In addition to the standard notation $\mathbb{Z}$, we write $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive integers and non-negative integers, respectively.

Let $\Lambda(n)$ be the Grassmann algebra over $\mathbb{F}$ in n variables $x_1,x_2,\dots,x_n$. Set $\mathbb{B}_k=\{\langle i_1,i_2,\dots,i_k\rangle \mid 1\le i_1<i_2<\cdots<i_k\le n\}$ and $\mathbb{B}(n)=\bigcup_{k=0}^{n}\mathbb{B}_k$, where $\mathbb{B}_0=\emptyset$. For $u=\langle i_1,i_2,\dots,i_k\rangle\in\mathbb{B}_k$, set $|u|=k$, $\{u\}=\{i_1,i_2,\dots,i_k\}$ and $x^u=x_{i_1}x_{i_2}\cdots x_{i_k}$ ($|\emptyset|=0, x^{\emptyset}=1$). Then $\{x^u|u\in\mathbb{B}(n)\}$ is an $\mathbb{F}$-basis of $\Lambda(n)$.

Let Π denote the prime field of $\mathbb{F}$, that is, $\Pi=\{0,1,\dots,p-1\}$. Suppose that the set $\{z_1,z_2,\dots,z_m\}$ is a Π-linearly independent finite subset of $\mathbb{F}$. Let $G=\{\sum_{i=1}^{m}\lambda_i z_i \mid \lambda_i\in\Pi\}$. Then G is an additive subgroup of $\mathbb{F}$. Let $\mathbb{F}[y_1,y_2,\dots,y_m]$ be the truncated polynomial algebra satisfying $y_i^p=1$ for all $i=1,2,\dots,m$. For every element $\lambda=\sum_{i=1}^{m}\lambda_i z_i\in G$, define $y^{\lambda}=y_1^{\lambda_1}y_2^{\lambda_2}\cdots y_m^{\lambda_m}$. Then $y^{\lambda}y^{\eta}=y^{\lambda+\eta}$ for all $\lambda,\eta\in G$. Let $\mathbb{T}(m)$ denote $\mathbb{F}[y_1,y_2,\dots,y_m]$. Then $\mathbb{T}(m)=\{\sum_{\lambda\in G}a_{\lambda}y^{\lambda}\mid a_{\lambda}\in\mathbb{F}\}$. Set $\mathscr{U}=\Lambda(n)\otimes\mathbb{T}(m)$. Then $\mathscr{U}$ is an associative superalgebra with $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of $\mathbb{T}(m)$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(n)$, that is, $\mathscr{U}=\mathscr{U}_{\bar 0}\oplus\mathscr{U}_{\bar 1}$, where $\mathscr{U}_{\bar 0}=\Lambda(n)_{\bar 0}\otimes\mathbb{T}(m)$ and $\mathscr{U}_{\bar 1}=\Lambda(n)_{\bar 1}\otimes\mathbb{T}(m)$.

For $f\in\Lambda(n)$ and $\alpha\in\mathbb{T}(m)$, we abbreviate $f\otimes\alpha$ as $f\alpha$. Then the elements $x^u y^{\lambda}$ with $u\in\mathbb{B}(n)$ and $\lambda\in G$ form an $\mathbb{F}$-basis of $\mathscr{U}$. It is easy to see that $\mathscr{U}=\bigoplus_{i=0}^{n}\mathscr{U}_i$ is a $\mathbb{Z}$-graded superalgebra, where $\mathscr{U}_i=\mathrm{span}_{\mathbb{F}}\{x^u y^{\lambda}\mid u\in\mathbb{B}(n), |u|=i, \lambda\in G\}$. In particular, $\mathscr{U}_0=\mathbb{T}(m)$ and $\mathscr{U}_n=\mathrm{span}_{\mathbb{F}}\{x^{\pi}y^{\lambda}\mid\lambda\in G\}$, where $\pi:=\langle 1,2,\dots,n\rangle\in\mathbb{B}(n)$.

In this paper, if $A=A_{\bar 0}\oplus A_{\bar 1}$ is a superalgebra (or $\mathbb{Z}_2$-graded linear space), then let $\mathrm{hg}(A)=A_{\bar 0}\cup A_{\bar 1}$ is the set of all $\mathbb{Z}_2$-homogeneous elements of A. If $\mathrm{deg}x$ occurs in some expression, we regard x as a $\mathbb{Z}_2$-homogeneous element and $\mathrm{deg}x$ as the $\mathbb{Z}_2$-degree of x. Let $A=\bigoplus_{i=-r}^{n}A_i$ be a $\mathbb{Z}$-graded superalgebra. If $x\in A_i$, then we call x a $\mathbb{Z}$-homogeneous element, i the $\mathbb{Z}$-degree of x and set $\mathrm{zd}(x)=i$.

Let $\mathrm{pl}(A)=\mathrm{pl}_{\bar 0}(A)\oplus\mathrm{pl}_{\bar 1}(A)$ denote the general linear Lie superalgebra of the $\mathbb{Z}_2$-graded space A. For $\varphi\in\mathrm{pl}_{\theta}(A)$ with $\theta\in\mathbb{Z}_2$, if

$$\varphi(xy)=\varphi(x)y+(-1)^{\theta degx}x\varphi(y)$$

for all $x \in \mathrm{hg}(A)$ and $y \in A$, then φ is called a derivation of A with $\mathbb{Z}_2$-degree θ. Let $\mathrm{Der}_\theta A$ denote the set of all derivations of A with $\mathbb{Z}_2$-degree θ. Then $\mathrm{Der} A = \mathrm{Der}_{\bar{0}} A \oplus \mathrm{Der}_{\bar{1}} A$, which is called the derivation superalgebra of A, is a subalgebra of $\mathrm{pl}(A)$ (see [15]).

Set $\mathrm{Y} = \{1, 2, \ldots, n\}$. Given $i \in \mathrm{Y}$, let $\partial/\partial x_i$ be the partial derivative on $\Lambda(n)$ with respect to x_i. For $i \in Y$, let D_i be the linear transformation on $\mathscr{U}$ such that $D_i(x^u y^\lambda) = (\partial x^u/\partial x_i) y^\lambda$ for all $u \in \mathbb{B}(n)$ and $\lambda \in G$. Then $D_i \in \mathrm{Der}_{\bar{1}} \mathscr{U}$ for all $i \in \mathrm{Y}$ since $\partial/\partial x_i \in \mathrm{Der}_{\bar{1}}(\Lambda(n))$.

Suppose that $u \in \mathbb{B}_k \subseteq \mathbb{B}(n)$ and $i \in \mathrm{Y}$. When $i \in \{u\}$, we denote the uniquely determined element of $\mathbb{B}_{k-1}$ satisfying $\{u - \langle i \rangle\} = \{u\} \setminus \{i\}$ by $u - \langle i \rangle$, and denote the number of integers less than i in $\{u\}$ by $\tau(u, i)$. When $i \notin \{u\}$, we set $\tau(u, i) = 0$ and $x^{u - \langle i \rangle} = 0$. Therefore, $D_i(x^u) = (-1)^{\tau(u,i)} x^{u - \langle i \rangle}$ for all $i \in \mathrm{Y}$ and $u \in \mathbb{B}(n)$.

We define $(fD)(g) = fD(g)$ for $f, g \in \mathrm{hg}(\mathscr{U})$ and $D \in \mathrm{hg}(\mathrm{Der}\mathscr{U})$. Since the multiplication of $\mathscr{U}$ is supercommutative, fD is a derivation of $\mathscr{U}$. Let

$$W(n,m) = \mathrm{span}_{\mathbb{F}}\{x^u y^\lambda D_i \mid u \in \mathbb{B}(n), \lambda \in G, i \in \mathrm{Y}\}.$$

Then $W(n,m)$ is a finite dimensional Lie superalgebra contained in $\mathrm{Der}\mathscr{U}$. A direct computation shows that

$$[fD_i, gD_j] = fD_i(g)D_j - (-1)^{\deg(fD_i)\deg(gD_j)} gD_j(f)D_i, \tag{2.1}$$

where $f, g \in \mathrm{hg}(\mathscr{U})$ and $i, j \in \mathrm{Y}$.

Let $D_{r_1 r_2} : \mathscr{U} \longrightarrow W(n,m)$ be the linear mapping such that for every $f \in \mathrm{hg}(\mathscr{U})$ and $r_1, r_2 \in \mathrm{Y}$,

$$D_{r_1 r_2}(f) = \sum_{i=1}^{2} f_{r_i} D_{r_i}, \tag{2.2}$$

where $f_{r_1} = -D_{r_2}(f)$ and $f_{r_2} = -D_{r_1}(f)$. It is easy to see that $D_{r_1 r_2}$ is an even linear mapping. Let $S(n,m) = \{D_{ij}(f) \mid f \in \mathscr{U}, i, j \in \mathrm{Y}\}$. Then $S(n,m)$ is a finite dimensional Lie superalgebra with a $\mathbb{Z}$-gradation $S(n,m) = \bigoplus_{r=-1}^{n-2} S_r(n,m)$, where $S_r(n,m) = \{D_{ij}(x^u y^\lambda) \mid u \in \mathbb{B}(n), |u| = r+2, \lambda \in G, i, j \in \mathrm{Y}\}$. In this paper, $S(n,m)$ is called the Lie superalgebra of special type.

By the definition of linear mapping $D_{r_1 r_2}$, the following equalities are easy to verify.

$$D_{ii}(f) = -2D_i(f)D_i, \tag{2.3}$$

$$D_{ij}(f) = D_{ji}(f), \tag{2.4}$$

$$[D_k, D_{ij}(f)] = -D_{ij}(D_k(f)), \tag{2.5}$$

$$[D_{s_1 s_2}(f), D_{r_1 r_2}(g)] = \sum_{i,j=1}^{2} (-1)^{\deg f} D_{s_i r_j}(f_{s_i} g_{r_j}), \tag{2.6}$$

where $f, g \in \mathrm{hg}(\mathscr{U})$, $i, j, k \in \mathrm{Y}$ and f_{s_i}, g_{r_j} as in (2.2). The equality (2.6) shows that $S(n,m)$ is a subalgebra of $W(n,m)$.

3. Some basic properties of $S(n,m)$

Hereafter, $W(n,m)$, $S(n,m)$ and $S_i(n,m)$ will be simply denoted by W, S and S_i, respectively.

The linear mapping $\mathrm{div}\colon W \to \mathscr{U}$ defined by $\mathrm{div}(fD_i) = (-1)^{\deg f} D_i(f)$ for all $f \in \mathrm{hg}(\mathscr{U})$ and $i \in \mathrm{Y}$ is called the *divergence*.

A direct calculation shows that div is a derivation of W with values in $\mathscr{U}$ (see [9] or [23]), that is

$$\mathrm{div}[D,E] = D\mathrm{div}(E) - (-1)^{\mathrm{deg}D\mathrm{deg}E}E\mathrm{div}(D) \text{ for all } D,E \in W. \tag{3.1}$$

Following [23], put $\overline{S}(n,m) := \{D \in W \mid \mathrm{div}(D) = 0\}$. We also denote $\overline{S}(n,m)$ by $\overline{S}$ for short.

Suppose that D_1 and D_2 are arbitrary elements of $\overline{S}$, then $\mathrm{div}(D_1) = 0$ and $\mathrm{div}(D_2) = 0$. By the equality (3.1), we have

$$\mathrm{div}[D_1,D_2] = D_1(\mathrm{div}(D_2)) - (-1)^{\mathrm{deg}D_1\mathrm{deg}D_2}D_2(\mathrm{div}(D_1)) = 0$$

Therefore, $\overline{S}$ is a subalgebra of W.

Proposition 3.1. $S = \overline{S}$.

Proof. A direct calculation shows that

$$\begin{aligned}
\mathrm{div}(D_{ij}(f)) &= \mathrm{div}(-D_i(f)D_j - D_j(f)D_i) \\
&= -(-1)^{\mathrm{deg}f+1}D_jD_i(f) - (-1)^{\mathrm{deg}f+1}D_iD_j(f) \\
&= (-1)^{\mathrm{deg}f}D_jD_i(f) - (-1)^{\mathrm{deg}f}D_jD_i(f) \\
&= 0,
\end{aligned}$$

where $f \in \mathrm{hg}(\mathscr{U})$ and $i,j \in \mathrm{Y}$. Hence $D_{ij}(f) \in \overline{S}$. This implies that S is contained in $\overline{S}$.

Conversely, since $\overline{S} \in W$ and $\mathrm{div}(x^{u-\langle i\rangle}y^\lambda D_i) = 0$, we suppose that an arbitrary element of $\overline{S}$ has the form $x^{u-\langle i\rangle}y^\lambda D_i$, where $u \in \mathbb{B}(n)$, $\lambda \in G$ and $i \in \mathrm{Y}$. The equality (2.4) shows that $x^{u-\langle i\rangle}y^\lambda D_i = (-1)^{\tau(u,i)}D_i(x^uy^\lambda)D_i = (-1)^{\tau(u,i)+1}2^{-1}D_{ii}(x^uy^\lambda) \in S$. Hence $\overline{S} \subseteq S$. Consequently, $S = \overline{S}$. □

Adopting the notion of [15, Definition 2, p73], the Lie superalgebra L is called *transitive* if $\{A \in L_i \mid [A, L_{-1}] = 0\} = 0$ for all $i \in \mathbb{N}_0$.

Proposition 3.2. *The Lie superalgebra S is transitive.*

Proof. Let $D = D_{ij}(f) \in S_t$, where $t \in \mathbb{N}_0$ and $i,j \in \mathrm{Y}$. Then $f \in \mathscr{U}_{t+2}$. If $[D,S_{-1}] = 0$, then $[D_{ij}(f), y^\lambda D_k] = 0$ for all $k \in \mathrm{Y}$ and $\lambda \in G$. The equality (2.5) shows that $D_{ij}(D_k(y^\lambda f)) = 0$, that is $D_k(y^\lambda f) \in \mathscr{U}_0 = \mathbb{T}(m)$. Then $f \in \mathscr{U}_1$ which implies that $f \in \mathscr{U}_1 \cap \mathscr{U}_{t+2}$ for $t \in \mathbb{N}_0$. Hence $f = 0$. It follows that $D = 0$. Therefore, S is transitive. □

Let $\mathcal{I} = \{\sum_{\lambda\in G} a_\lambda y^\lambda \mid \sum_{\lambda\in G} a_\lambda = 0, a_\lambda \in \mathbb{F}\}$ and $\Gamma = \{x^u\alpha \mid u \in \mathbb{B}(n), \alpha \in \mathcal{I}\}$. By [14, Lemma 4.2 and 4.3], $\mathcal{I}$ and Γ are ideals of $\mathbb{T}(m)$ and $\mathscr{U}$, respectively. Furthermore, $\mathbb{T}(m) = \mathcal{I} \oplus \mathbb{F}1$. By definition of S, we have $S(n,0) = \{D_{ij}(x^u) \mid u \in \mathbb{B}(n)\}$, which is isomorphic to the special Lie superalgebra $S(n)$ (see [16] or [23] for definition).

Proposition 3.3. *Let $\mathfrak{J} = \{D_{ij}(x^u\alpha) \mid x^u\alpha \in \Gamma, i,j \in \mathrm{Y}\}$. Then $\mathfrak{J}$ is an ideal of S and $S \cong S(n) \oplus \mathfrak{J}$.*

Proof. Suppose that $x^u\alpha \in \Gamma$ and $x^v\beta \in \mathscr{U}$, where $\alpha \in \mathcal{I}$ and $\beta \in \mathbb{T}(m)$. Since $\mathcal{I}$ is an ideal of $\mathbb{T}(m)$ and $D_{s_1s_2}(x^u\alpha) \in \mathfrak{J}$, by equality (2.6), we have

$$\begin{aligned}
[D_{s_1s_2}(x^u\alpha), D_{r_1r_2}(x^v\beta)] &= (-1)^{|u|}D_{s_1r_1}(D_{s_2}(x^u\alpha)D_{r_2}(x^v\beta)) \\
&\quad +(-1)^{|u|}D_{s_1r_2}(D_{s_2}(x^u\alpha)D_{r_1}(x^v\beta))
\end{aligned}$$

$$
\begin{aligned}
&+(-1)^{|u|}D_{s_2r_1}(D_{s_1}(x^u\alpha)D_{r_2}(x^v\beta))\\
&+(-1)^{|u|}D_{s_2r_2}(D_{s_1}(x^u\alpha)D_{r_1}(x^v\beta))\\
=\ &(-1)^{|u|}D_{s_1r_1}(D_{s_2}(x^u)D_{r_2}(x^v)\alpha\beta)\\
&+(-1)^{|u|}D_{s_1r_2}(D_{s_2}(x^u)D_{r_1}(x^v)\alpha\beta)\\
&+(-1)^{|u|}D_{s_2r_1}(D_{s_1}(x^u)D_{r_2}(x^v)\alpha\beta)\\
&+(-1)^{|u|}D_{s_2r_2}(D_{s_1}(x^u)D_{r_1}(x^v)\alpha\beta) \in \mathfrak{J},
\end{aligned}
$$

where $s_1, s_2, r_1, r_2 \in \mathrm{Y}$. Thus $\mathfrak{J}$ is an ideal of S.

Suppose that $D_{ij}(x^u\beta)$ is an arbitrary element of S. Since $\mathbb{T}(m) = \mathcal{I} \oplus \mathbb{F}1$, we may write $\beta = \alpha + a1$, where $\alpha \in \mathcal{I}$ and $a \in \mathbb{F}$. Then $D_{ij}(x^u\beta) = D_{ij}(x^u\alpha) + D_{ij}(ax^u)$ for all $i, j \in \mathrm{Y}$. It follows from $S(n, 0) \cap \mathfrak{J} = 0$ that $S = S(n, 0) \oplus \mathfrak{J}$. As $S(n, 0)$ is isomorphic to $S(n)$, the desired result follows immediately. □

Remark. *It was shown in [23] that $S(n)$ is a simple Lie superalgebra. Then the results of Proposition 3.3 imply that the ideal $\mathfrak{J}$ of S is maximal.*

Let $S(m, n, \underline{1})$ denote the restricted special Lie superalgebra as definition in [23]. Then the following proposition is easy to obtain.

Proposition 3.4. *The Lie superalgebra S is isomorphic to a subalgebra of $S(m, n, \underline{1})$.*

4. The derivation superalgebra of $S(n, m)$

In this section we will investigate the question of the derivation superalgebra of S.

Let $\Theta := \mathbb{T}(m)^m = \mathbb{T}(m) \times \cdots \times \mathbb{T}(m)$. Then

$$\Theta = \{\theta = (h_1(y), \ldots, h_m(y)) \mid h_i(y) \in \mathbb{T}(m), i = 1, 2, \ldots, m\}.$$

For every $\theta = (h_1(y), \ldots, h_m(y)) \in \Theta$, we define $\widetilde{\theta}\colon G \to \mathbb{T}(m)$ by $\widetilde{\theta}(\lambda) = \sum_{j=1}^m \lambda_j h_j(y)$ for $\lambda = \sum_{j=1}^m \lambda_j z_j \in G$. It is easy to check that $\widetilde{\theta}(\lambda + \eta) = \widetilde{\theta}(\lambda) + \widetilde{\theta}(\eta)$ for $\lambda, \eta \in G$. For every $\theta \in \Theta$, let $D_\theta\colon S \to S$ be the linear mapping given by $D_\theta D_{ij}(x^u y^\lambda) = \widetilde{\theta}(\lambda) D_{ij}(x^u y^\lambda)$ for $x^u y^\lambda \in \mathscr{U}$ and $i, j \in \mathrm{Y}$. A direct verification shows that $D_\theta \in \mathrm{Der}_{\bar{0}}(S)$ for all $\theta \in \Theta$.

The derivation D_θ is called the derivation of Θ-type. Put $\Omega := \{D_\theta \mid \theta \in \Theta\}$.

If $\widetilde{\theta}(\lambda) = \sum_{i=1}^m \lambda_i h_{\theta i}(y)$ and $\widetilde{\vartheta}(\lambda) = \sum_{i=1}^m \lambda_i h_{\vartheta i}(y)$, where $\theta, \vartheta \in \Theta$, then

$$\widetilde{\theta}(\lambda) + \widetilde{\vartheta}(\lambda) = \sum_{i=1}^m \lambda_i (h_{\theta i}(y) + h_{\vartheta i}(y)) = \widetilde{\theta + \vartheta}(\lambda).$$

The equality above shows that $D_\theta + D_\vartheta = D_{\theta+\vartheta}$ and $aD_\theta = D_{a\theta}$ for $a \in \mathbb{F}$. Hence Ω is a subspace of $\mathrm{Der}_{\bar{0}}(S)$ and $\dim \Omega = \dim \Theta = (p^m)^m$.

Analogous to [14, Lemma 3.3], we have the following lemma.

Lemma 4.1. *If $\varphi \in \mathrm{hg}(\mathrm{Der}(W))$ and $\varphi(D_j) = 0$ for all $j \in \mathrm{Y}$, then there exists $\theta \in \Theta$ such that $\varphi(y^\lambda D_j) = D_\theta(y^\lambda D_j)$ for all $\lambda \in G$.*

Lemma 4.2. *Let $h_{k\lambda} = x_k y^\lambda D_k$, where $k \in \mathrm{Y}$ and $\lambda \in G$. Then $h_{k\lambda} \in \mathrm{Nor}_W(S)$.*

Proof. For arbitrary $i,j\in \Upsilon$, $u\in\mathbb{B}(n)$ and $\eta\in G$, we have

$$[h_{k\lambda}, D_{ij}(x^u y^\eta)] = -[x_k y^\lambda D_k, D_i(x^u y^\eta)D_j] - [x_k y^\lambda D_k, D_j(x^u y^\eta)D_i].$$

If $k\in \Upsilon\setminus\{i,j\}$, then

$$\begin{aligned}[h_{k\lambda}, D_{ij}(x^u y^\eta)] &= -x_k y^\lambda D_k D_i(x^u y^\eta)D_j - D_i(x^u y^\eta)D_j(x_k)y^\lambda D_k \\ &\quad -x_k y^\lambda D_k D_j(x^u y^\eta)D_i - D_j(x^u y^\eta)D_i(x_k)y^\lambda D_k \\ &= -D_j(x^u y^{\lambda+\eta})D_i - D_i(x^u y^{\lambda+\eta})D_j \\ &= D_{ij}(x^u y^{\lambda+\eta}).\end{aligned}$$

In the case of $k=i\neq j$ we have

$$\begin{aligned}[h_{k\lambda}, D_{ij}(x^u y^\eta)] &= -[x_k y^\lambda D_k, D_j(x^u y^\eta)D_k + D_k(x^u y^\eta)D_j] \\ &= -D_j(x^u y^{\lambda+\eta})D_k + D_j(x^u y^{\lambda+\eta})D_k \\ &= 0.\end{aligned}$$

When $k=j\neq i$, we may obtain $[h_{k\lambda}, D_{ij}(x^u y^\eta)]=0$ similarly.
If $k=i=j$, then $[h_{k\lambda}, D_{ij}(x^u y^\eta)] = -D_{kk}(x^u y^{\lambda+\eta})$.
In conclusion, $h_{k\lambda}\in \mathrm{Nor}_W(S)$. □

Lemma 4.3. *Let $\varphi\in \mathrm{Der}_t(S)$ and $t\in\mathbb{Z}$. Suppose that $\varphi(S_i)=0$ for $i=-1,0,\ldots,j$ and $j\geq -1$. If $j+t\geq -1$, then $\varphi=0$.*

Proof. Suppose that $q\geq j$. To prove $\varphi(S_q)=0$, we use induction on q.

Note that $\varphi(S_j)=0$. Let $q>j$ and z be an arbitrary element of S_q. Suppose that $[z, y^\lambda D_l]=z_l$ for all $l\in\Upsilon$ and $\lambda\in G$, then $z_l\in S_{q-1}$. The induction hypothesis yields $\varphi(z_l)=0$. Let $\varphi(z)=\sum_{k\in\Upsilon} f_k D_k$, where $f_k\in\mathscr{U}$. Applying φ to the equality $[z,y^\lambda D_l]=z_l$, we have

$$\varphi([z,y^\lambda D_l]) = [\varphi(z), y^\lambda D_l] + (-1)^{\mathrm{deg}\varphi\mathrm{deg}z}[z,\varphi(y^\lambda D_l)] = 0.$$

It follows from $\varphi(y^\lambda D_l)=0$ that $[\varphi(z), y^\lambda D_l]=0$. Hence $[\sum_{k\in\Upsilon} f_k D_k, y^\lambda D_l]=0$, that is $\sum_{k\in\Upsilon} y^\lambda D_l(f_k)D_k=0$. It follows that $y^\lambda D_l(f_k)=0$ for all $l\in\Upsilon$ and $\lambda\in G$. This implies $f_k\in\mathbb{T}(m)$ for $k\in\Upsilon$. Thus $\varphi(z)\in S_{-1}$. On the other hand, $z\in S_q$ and $\varphi\in\mathrm{Der}_t(S)$. So $\varphi(z)\in S_{q+t}$. Since $q+t>j+t\geq -1$, we have $\varphi(z)\in S_{-1}\cap S_{q+t}=0$, that is $\varphi(S_q)=0$.

The above considerations show that $\varphi(S)=0$. Therefore, the desired result $\varphi=0$ follows. □

Theorem 4.4. *Suppose that $t\in\mathbb{N}_0$ and $N=\mathrm{span}_{\mathbb{F}}\{h_{k\lambda}\mid k\in\Upsilon,\lambda\in G, h_{k\lambda}=x_k y^\lambda D_k\}$, then $\mathrm{Der}_t(S)=\mathrm{ad}(S+N)_t+\Omega$.*

Proof. By virtue of Lemma 4.2, we have $[N,S]\subseteq S$. It is easy to show that $\mathrm{ad}(S+N)+\Omega\subseteq\mathrm{Der}(S)$. Hence $\mathrm{ad}(S+N)_t+\Omega\subseteq\mathrm{Der}_t(S)$. Note that $\mathrm{div}(h_{k\lambda})\neq 0$ and $S=\overline{S}$. Then $N\not\subseteq S$.

Next we will consider the converse inclusion.

Suppose that for every $k\in\Upsilon$ and $\varphi_t\in\mathrm{Der}_t(S)$, $\varphi_t(D_k)=\sum_{i=1}^n f_{ik}D_i$, where $f_{ik}\in\mathscr{U}$. Applying φ_t to $[D_k,D_l]=0$, where $k\neq l\in\Upsilon$, we have $D_l(f_{ik})=-D_k(f_{il})$ for all $i\in\Upsilon$. Then [14, Lemma 3.1] shows that there exist g_i for $i=1,\ldots,n$ such that $D_k(g_i)=f_{ik}$ for all $k\in\Upsilon$.

Let $z=-(-1)^{\mathrm{deg}\varphi_t}\sum_{i=1}^n g_i D_i\in W_t$. Then $\varphi_t(D_k)=\mathrm{ad}z(D_k)$ for all $k\in\Upsilon$. Consequently, by Lemma 4.1, there exists $\theta\in\Theta$ such that $(\varphi_t-\mathrm{ad}z)(D_{kj}(x_k y^\lambda))=D_\theta(D_{kj}(x_k y^\lambda))$ for all distinct elements j,k of Υ and $\lambda\in G$.

Set $\phi_t = \varphi_t - \mathrm{ad}z - D_\theta$. We use induction on j to show that $\phi_t(S_j) = 0$ for $j = -1, 0, \ldots, n-2$. A direct calculation shows that $\phi_t(S_{-1}) = 0$. Suppose that $j \geq 0$. For arbitrary element ξ of S_j , the induction hypothesis yields $[\phi_t(\xi), D_i] = 0$ for all $i \in \mathrm{Y}$. Therefore, $\phi_t(\xi) \in S_{-1} \cap S_{t+j}$. Noting that $t + j \geq 0$, we have $\phi_t(\xi) = 0$, hence $\phi_t(S_j) = 0$. Consequently, $\varphi_t = \mathrm{ad}z + D_\theta$.

Since $\mathrm{ad}z(D_k) = \varphi_t(D_k) \in S$ and $D_k \in S$ for $k \in \mathrm{Y}$, we have $[D_k, z] \in S$. Thus $\mathrm{div}[D_k, z] = 0$ and $\mathrm{div}(D_k) = 0$. The equality (3.1) shows that $\mathrm{div}(z) \in \mathbb{T}(m)$. Then $\mathrm{div}(z - \mathrm{div}(z)x_iD_i) = 0$ for all $i \in \mathrm{Y}$. It follows from $z - \mathrm{div}(z)x_iD_i \in S$ that $z \in S+N$. This implies that $\mathrm{ad}z \in \mathrm{ad}(S+N)$. The above considerations show that $\mathrm{ad}z + D_\theta \in \mathrm{ad}(S+N)+\Omega$. Therefore, the inclusion relation $\mathrm{Der}_t(S) \subseteq \mathrm{ad}(S+N)_t + \Omega$ holds.

In conclusion, $\mathrm{Der}_t(S) = \mathrm{ad}(S+N)_t + \Omega$. □

Lemma 4.5. $\mathrm{ad}(S+N) \cap \Omega = \{0\}$.

Proof. Firstly, we consider the set $\mathrm{ad}S \cap \Omega$. For any $x \in \mathrm{ad}S \cap \Omega$, we suppose that $x = \mathrm{ad}E = D_\theta$, where $E \in S$ and $\theta \in \Theta$. Then $\mathrm{ad}E(D_j) = D_\theta(D_j) = 0$ for all $j \in \mathrm{Y}$. It follows that $E \in S_{-1}$. Without loss of generality, we may suppose that $E = \sum_{k=1}^{n}\sum_{\lambda\in G} a_{k\lambda} y^\lambda D_k$, where $a_{k\lambda} \in \mathbb{F}$. For all $i, j \in \mathrm{Y}$, we have $\mathrm{ad}E(D_{ij}(x_ix_j)) = D_\theta(D_{ij}(x_ix_j)) = 0$. This implies that $a_{k\lambda} = 0$ for all $k \in \mathrm{Y}$. Then $E = 0$, that is $x = 0$. Therefore, $\mathrm{ad}S \cap \Omega = \{0\}$.

Next we will prove that $\mathrm{ad}N \cap \Omega = \{0\}$. Suppose that $x \in \mathrm{ad}N \cap \Omega$ such that $x = \mathrm{ad}E = D_\theta$, where $E \in N$ and $\theta \in \Theta$. Using the similar discussion above, the $\mathbb{Z}$-graded degree of E is -1. It is contradiction to $\mathrm{zd}(N) = 0$. Therefore, there does not exist a non-zero element belongs to $\mathrm{ad}N \cap \Omega$, that is $\mathrm{ad}N \cap \Omega = \{0\}$.

Consequently, $\mathrm{ad}(S+N) \cap \Omega = \{0\}$ and the proof is completed. □

Theorem 4.6. $\mathrm{Der}_{-1}(S) = \mathrm{ad}S_{-1}$.

Proof. It suffices to show that $\mathrm{Der}_{-1}(S) \subseteq \mathrm{ad}S_{-1}$. According to the definition of S_i, we may easily obtain

$$S_0 = \mathrm{span}_{\mathbb{F}}\{D_{ij}(x_ix_jy^\lambda), x_iy^\lambda D_j \mid i, j \in \mathrm{Y}, i \neq j, \lambda \in G\}.$$

Let $\varphi \in \mathrm{Der}_{-1}(S)$. Suppose that $\varphi(D_{ij}(x_ix_jy^\lambda)) = \sum_{t\in\mathrm{Y}} \alpha_{ijt\lambda} D_t$, where $\alpha_{ijt\lambda} \in \mathbb{T}(m)$. For $k \in \mathrm{Y} \setminus \{i, j\}$, we obtain that $\varphi(D_{jk}(x_jx_ky^\eta)) = \sum_{t\in\mathrm{Y}} \alpha_{jkt\eta} D_t$, where $\alpha_{jkt\eta} \in \mathbb{T}(m)$ and $\eta \in G$. Applying φ to the equality $[D_{ij}(x_ix_jy^\lambda), D_{jk}(x_jx_ky^\eta)] = 0$, we have

$$\alpha_{ijj\lambda} D_j - \alpha_{ijk\lambda} D_k - \alpha_{jki\eta} D_i + \alpha_{jkj\eta} D_j = 0.$$

In particular, $\alpha_{ijk\lambda} = 0$ for all $k \in \mathrm{Y} \setminus \{i, j\}$. Therefore,

$$\varphi(D_{ij}(x_ix_jy^\lambda)) = \alpha_{iji\lambda} D_i + \alpha_{ijj\lambda} D_j, \tag{4.1}$$

where $\alpha_{iji\lambda}, \alpha_{ijj\lambda} \in \mathbb{T}(m)$.

We may also suppose that $\varphi(x_iy^\lambda D_j) = \sum_{t\in\mathrm{Y}} \beta_{ijt\lambda} D_t$, where $\beta_{ijt\lambda} \in \mathbb{T}(m)$. Applying φ to $[D_{ij}(x_ix_j), x_iy^\lambda D_j] = 2x_iy^\lambda D_j$, by equality (4.1), we have

$$\alpha_{ijj0} y^\lambda D_j - \beta_{iji\lambda} D_i + \beta_{ijj\lambda} D_j = 2\sum_{t\in\mathrm{Y}} \beta_{ijt\lambda} D_t.$$

It follows that $\beta_{ijt\lambda} = 0$ for all $t \in \mathrm{Y} \setminus \{i, j\}$. Therefore,

$$\varphi(x_iy^\lambda D_j) = \beta_{iji\lambda} D_i + \beta_{ijj\lambda} D_j,$$

By equality (4.1), we may suppose that $\varphi(D_{lj}(x_lx_j)) = \alpha_{ljl0}D_l + \alpha_{ljj0}D_i$, where $\alpha_{ljl0}, \alpha_{ljj0} \in \mathbb{T}(m)$ and $l \in \mathrm{Y} \setminus \{i,j\}$. A direct calculation shows that $[D_{lj}(x_lx_j), x_iy^\lambda D_j] = x_iy^\lambda D_j$. Applying φ to this equality yields that $\beta_{iji\lambda} = 0$. Hence $\varphi(x_iy^\lambda D_j) = \beta_{ijj\lambda}D_j$, where $\beta_{ijj\lambda} \in \mathbb{T}(m)$. Without loss of generality, we may suppose that $\varphi(x_iy^\lambda D_{i+1}) = \gamma_{i\lambda}D_{i+1}$ and $\varphi(x_ny^\lambda D_1) = \gamma_{n\lambda}D_1$, where $\gamma_{i\lambda}, \gamma_{n\lambda} \in \mathbb{T}(m)$ and $i \in \mathrm{Y} \setminus \{n\}$.

Let $\phi = \varphi - \sum_{i\in\mathrm{Y}} \gamma_{i\lambda}(\mathrm{ad}D_i)$. Then $\phi(x_iy^\lambda D_{i+1}) = \phi(x_ny^\lambda D_1) = 0$ for all $i \in \mathrm{Y} \setminus \{n\}$. Suppose that M is generated by $\{x_iy^\lambda D_{i+1}, x_ny^\lambda D_1 \mid i \in \mathrm{Y} \setminus \{n\}, \lambda \in G\}$, then M is a subalgebra of S. It is obvious that $D_{ij}(x_ix_jy^\lambda), x_iy^\lambda D_j \in M$, where i,j are distinct elements of Y. Then $M = S_0$. Considering the $\mathbb{Z}$-graded degree of ϕ we have $\phi(S_{-1}) = 0$. Lemma 4.3 shows that $\phi = 0$. Hence $\varphi = \sum_{i\in\mathrm{Y}} \gamma_{i\lambda}(\mathrm{ad}D_i) \in \mathrm{ad}S_{-1}$. This implies that $\mathrm{Der}_{-1}(S) \subseteq \mathrm{ad}S_{-1}$. □

Theorem 4.7. *Suppose that* $t > 1$, *then* $\mathrm{Der}_{-t}(S) = 0$.

Proof. Let $\varphi \in \mathrm{Der}_{-t}(S)$. By Lemma 4.3, it suffices to prove that $\varphi(S_{t-1}) = 0$. Suppose that $D_{ij}(x^uy^\lambda)$ with $u \in \mathbb{B}(n)$ and $\lambda \in G$ is a basis element of S_{t-1} and $\varphi(D_{ij}(x^uy^\lambda)) = \sum_{l\in\mathrm{Y}} \alpha_{ijl\lambda}D_l$, where $\alpha_{ijl\lambda} \in \mathbb{T}(m)$ and $i,j \in \mathrm{Y}$.

Next we will prove that $\varphi(D_{ij}(x^uy^\lambda)) = 0$ for the different cases of i and j.

It is obvious that $\varphi(D_{ij}(x^uy^\lambda)) = 0$ for $i,j \in \mathrm{Y} \setminus \{u\}$.

If i and j are distinct elements of $\{u\}$, then $[D_{ij}(x_ix_j), D_{ij}(x^uy^\lambda)] = 0$. For $k \in \{u\}\setminus\{i,j\}$, we have

$$[D_{jj}(x_kx_j), D_{ij}(x^uy^\lambda)] = 0.$$

Applying φ to these equalities yield $\alpha_{iji\lambda} = \alpha_{ijj\lambda} = \alpha_{ijk\lambda} = 0$. Thus $\varphi(D_{ij}(x^uy^\lambda)) = 0$.

When $i \in \{u\}$ and $j \in \mathrm{Y} \setminus \{u\}$, by the similar argument above, we may obtain $\alpha_{ijl\lambda} = 0$ for all $l \in \mathrm{Y}$. Hence $\varphi(D_{ij}(x^uy^\lambda)) = 0$ in this case.

By equality (2.4), $\varphi(D_{ij}(x^uy^\lambda)) = 0$ for $j \in \{u\}$ and $i \in \mathrm{Y} \setminus \{u\}$.

In conclusion, $\varphi(S_{t-1}) = 0$. Now Lemma 4.3 ensures that $\varphi = 0$, as desired. □

Theorem 4.8. $\mathrm{Der}(S) = \mathrm{ad}(S + N) \oplus \Omega$.

Proof. This is a direct consequence of Theorem 4.4, 4.7, 4.6 and Lemma 4.5. □

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