Existence and multiplicity of positive almost periodic solutions for a non-autonomous SIR epidemic model

Yaqin Li[a], Tianwei Zhang[b*]
[a]Department of Mathematics, Kunming University
Kunming 650031, China
[b]City College, Kunming University of Science and Technology
Kunming 650051, China

Abstract

In this paper, a non-autonomous SIR epidemic model with almost periodic transmission rate and a constant removal rate is considered. By means of Mawhin's continuous theorem of coincidence degree, some new sufficient conditions for the existence and multiplicity of positive almost periodic solutions to the model are established. Further, the global asymptotical stability of positive almost periodic solution of the model is also investigated by constructing a suitable Lyapunov functional. Finally, some examples and numerical simulations are given to illustrate the main results.

Key words: Almost periodic solution; Multiplicity; Coincidence degree; Epidemic model.
2010 Mathematics Subject Classification: 34K14; 92D25.

1 Introduction

Let $\mathbf{R}$, $\mathbf{Z}$ and $\mathbf{N}^+$ denote the sets of real numbers, integers and positive integers, respectively. Related to a continuous function f, we use the following notations:

$$f^l = \inf_{s\in\mathbf{R}} f(s), \quad f^M = \sup_{s\in\mathbf{R}} f(s), \quad \bar{f} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s)\,\mathrm{d}s.$$

Epidemiology is the branch of biology which deals with the mathematical modeling of spread of diseases. Many problems arising in epidemiology may be described, in a first formulation, by means of differential equations. This means that the models are constructed by averaging some population and keeping only the time variable. To the best of our knowledge the first mathematical model of epidemiology was formulated and solved by Daniel Bernoulli in 1760. Since the time of Kermack and McKendrick [8], the study of mathematical epidemiology has grown rapidly, with a large variety of models having been formulated and applied to infectious diseases [2, 5, 14].

Consider a population which remains constant and which is divide into three classes: the susceptibles, denoted by S, who can catch the disease; the infectives, denoted by I, who are infected and can transmit the disease to the susceptibles, and the removed class, denoted

*The corresponding author. E-mail: zhang@kmust.edu.cn

by R, who had the disease and recovered or died or have developed immunity or have been removed from contact with the other classes. Since from the modeling perspective only the overall state of a person with respect to the disease is relevant, the progress of individuals is schematically described by

$$S \to I \to R.$$

These types of models are known as SIR models. The phenomena of periodic oscillations have been observed in the spread of many infectious diseases, such as influenza, measles, rubella, mumps and chickenpox. It is interesting to study how periodic solutions arise and to determine the number of periodic solutions further in an epidemiological model. In recent years, many scholars pay extensive attention to the dynamic behaviours of SIR epidemic models. We refer the readers to $[1, 9, 12, 13, 16]$.

In [1], Bai and Zhou formulated a non-autonomous SIR epidemic model with saturated incidence rate and constant removal rate by introducing the periodic transmission rate $\beta(t)$ as follows:

$$\begin{cases} \dot{S}(t) = \alpha - \mu S(t) - \frac{\beta(t)S(t)I(t)}{k_1+k_2I(t)}, \\ \dot{I}(t) = \frac{\beta(t)S(t)I(t)}{k_1+k_2I(t)} - (\mu+\gamma)I(t) - h(I), \\ \dot{R}(t) = \gamma I + h(I) - \mu R, \end{cases} \tag{1.1}$$

where α is the recruitment rate, μ is the natural death rate, γ is the recovery rate of the infective, $\beta(t)$ is the transmission rate at time t and h is a treatment function, which is a positive constant σ for $I > 0$, and zero for $I = 0$.

By a simple analysis in [1], Bai and Zhou reduced SIR epidemic model (1.1) to the following model:

$$\begin{cases} \dot{S}(t) = \alpha - \mu S(t) - \frac{\beta(t)S(t)I(t)}{k_1+k_2I(t)}, \\ \dot{I}(t) = \frac{\beta(t)S(t)I(t)}{k_1+k_2I(t)} - (\mu+\gamma)I(t) - \sigma. \end{cases} \tag{1.2}$$

By using the continuation theorem of coincidence degree theory, sufficient conditions for the existence of at least two positive periodic solutions are obtained.

However, any biological or environmental parameters are naturally subject to fluctuation in time and if a model is desired which takes into account such fluctuation it must be nonautonomous, which is, of course, more difficult to study in general. One must of course ascribe some properties to the time dependence of the parameters in the models, for only then can the resulting dynamics to be studied accordingly. For example, one might assume they are periodic, quasi-periodic or almost periodic, etc. Furthermore, in real world phenomenon, the environment varies due to the factors such as seasonal effects of weather, food supplies, mating habits, harvesting. So it is usual to assume the periodicity of parameters in the systems. However, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. For this reason, the assumption of almost periodicity is more realistic, more important and more general when we consider the effects of the environmental factors. In this paper, we consider the following non-autonomous almost periodic epidemic model:

$$\begin{cases} \dot{S}(t) = \alpha(t) - \mu(t)S(t) - \frac{\beta(t)S(t)I(t)}{k_1+k_2I(t)}, \\ \dot{I}(t) = \frac{\beta(t)S(t)I(t)}{k_1+k_2I(t)} - (\mu(t)+\gamma(t))I(t) - \sigma(t), \end{cases} \tag{1.3}$$

where $k_1, k_2 > 0$ and all the coefficients in system (1.3) are nonnegative almost periodic functions. From the epidemiological interpretation our discussion on system (1.3) will be restricted in the following bounded domain

$$\mathcal{D} = \left\{ (S, I)^T : S \geq 0, I > 0, 0 < S + I \leq \frac{\alpha^M}{\mu^l} \right\}.$$

During the last twenty years, many people have been interested in the dynamic behaviours of epidemic models, but most scholars have focused on the permanence and extinction of the disease, global stability and the existence of positive periodic solution [1, 7, 10, 11, 17-19]. There are scarcely any papers on the existence and multiplicity of positive almost periodic solutions for nonlinear epidemic models. On the other hand, the method used to investigate the positive periodic solution of the non-linear biosystem (for example, coincidence degree theory (see [21]) or fixed point theorem of strict-set-contraction (see [22])) is difficult to be used to investigate the almost periodic solution of system (1.3). Motivated by the above statements, the main purpose of this paper is to establish some new sufficient conditions on the existence and multiplicity of positive almost periodic solutions for system (1.3) by using some new analytical techniques and Mawhin's continuous theorem.

The paper is organized as follows. In Section 2, we give some basic definitions and necessary lemmas which will be used in later sections. In Section 3, we obtain some new sufficient conditions for the existence and stability of positive almost periodic solution of system (1.3) by way of Mawhin's continuous theorem and Lyapunov functional. In Section 4, a multiplicity result for system (1.3) is obtained based on some good analytical techniques. Finally, some illustrative examples and numerical simulations are given in Section 5.

2 Preliminaries

Definition 2.1. [3, 6] Let $x \in C(\mathbf{R}) = C(\mathbf{R}, \mathbf{R})$. x is said to be almost periodic on $\mathbf{R}$, if for $\forall \epsilon > 0$, the set

$$T(x, \epsilon) = \{\tau : |x(t + \tau) - x(t)| < \epsilon, \forall t \in \mathbf{R}\}$$

is relatively dense, i.e., for $\forall \epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval length l, there exists a number $\tau = \tau(\epsilon) \in T(x, \epsilon)$ in this interval such that

$$|x(t + \tau) - x(t)| < \epsilon, \quad \forall t \in \mathbf{R}.$$

τ is called to the ϵ-almost period of x, $T(x, \epsilon)$ denotes the set of ϵ-almost periods for x and $l(\epsilon)$ is called to the length of the inclusion interval for $T(x, \epsilon)$.

Let $AP(\mathbf{R})$ denote the set of all real valued almost periodic functions on $\mathbf{R}$ and

$$AP(\mathbf{R}, \mathbf{R}^n) = \left\{ (x_1, x_2, \ldots, x_n)^T : x_i \in AP(\mathbf{R}), i = 1, 2, \ldots, n, n \in \mathbf{N}^+ \right\}.$$

Lemma 2.1. [3, 6] *If $x \in AP(\mathbf{R})$, then x is bounded and uniformly continuous on $\mathbf{R}$.*

Next, we present and prove several useful lemmas which will be used in later section.

Lemma 2.2. *Assume that $x \in AP(\mathbf{R}) \cap C^1(\mathbf{R})$ with $\dot{x} \in C(\mathbf{R})$. For arbitrary interval $[a, b]$ with $b - a = \omega > 0$, let $\xi, \eta \in [a, b]$ and*

$$\Xi^{\dot{x}}_{[\xi,b]} = \left\{ s \in [\xi, b] : \dot{x}(s) \geq 0 \right\}, \quad \Pi^{\dot{x}}_{[\eta,b]} = \left\{ s \in [\eta, b] : \dot{x}(s) \leq 0 \right\}.$$

Then

$$x(t) \le x(\xi) + \int_{\Xi^{\dot{x}}_{[\xi,b]}} \dot{x}(s)\,\mathrm{d}s, \quad \forall t \in [\xi, b],$$

$$x(t) \ge x(\eta) + \int_{\Pi^{\dot{x}}_{[\eta,b]}} \dot{x}(s)\,\mathrm{d}s, \quad \forall t \in [\eta, b].$$

Proof. For $t \in [\xi, b]$, let $\Xi^{\dot{x}}_{[\xi,t]} = \big\{s \in [\xi, t] : \dot{x}(s) \ge 0\big\}$. It follows that

$$x(t) = x(\xi) + \int_{\xi}^{t} \dot{x}(s)\,\mathrm{d}s \le x(\xi) + \int_{\Xi^{\dot{x}}_{[\xi,t]}} \dot{x}(s)\,\mathrm{d}s \le x(\xi) + \int_{\Xi^{\dot{x}}_{[\xi,b]}} \dot{x}(s)\,\mathrm{d}s, \ \ \forall t \in [\xi, b].$$

On the other hand, for $t \in [\eta, b]$, let $\Pi^{\dot{x}}_{[\eta,t]} = \big\{s \in [\eta, t] : \dot{x}(s) \le 0\big\}$. It follows that

$$x(t) = x(\eta) + \int_{\eta}^{t} \dot{x}(s)\,\mathrm{d}s \ge x(\eta) + \int_{\Pi^{\dot{x}}_{[\eta,t]}} \dot{x}(s)\,\mathrm{d}s \ge x(\eta) + \int_{\Pi^{\dot{x}}_{[\eta,b]}} \dot{x}(s)\,\mathrm{d}s, \ \ \forall t \in [\xi, b].$$

This completes the proof. ■

Lemma 2.3. *Assume that $x \in AP(\mathbf{R})$, then for arbitrary interval $[a, b]$ with $b - a = \omega > 0$, there exist $\xi_0 \in [a, b]$, $\xi_1 \in (-\infty, a]$ and $\xi_2 \in [\xi_1 + \omega, +\infty)$ such that*

$$x(\xi_1) = x(\xi_2) \quad \textit{and} \quad x(\xi_0) \le x(s), \ \forall s \in [\xi_1, \xi_2].$$

Proof. Without loss of generality, we consider $[a, b]$ as $[0, \omega]$. We shall present three cases to prove this lemma.

(C_1) $x(0) = x(\omega)$. Let $\xi_0 \in [0, \omega]$ such that $x(\xi_0) = \min_{s\in[0,\omega]} x(s)$, $\xi_1 = 0$ and $\xi_2 = \omega$. So Lemma 2.3 holds.

(C_2) $x(0) > x(\omega)$. Let $x_* = \inf_{s\in\mathbf{R}} x(s)$. From Lemma 2.1, $-\infty < x_* \le x(\omega) < x(0)$.

(a) If $x(\omega) > x_*$, then we claim that there exists $\omega_1 \in (-\infty, 0]$ such that $x(\omega_1) = x(\omega)$ and

$$x(s) \ge x(\omega_1) = x(\omega), \ \forall s \in [\omega_1, 0]. \tag{2.1}$$

In fact, if it is not true, then

$$x(s) > x(\omega), \ \forall s \in (-\infty, 0]. \tag{2.2}$$

By the definition of x_* and the continuity of x, there must exist $t_0 \in \mathbf{R}$ such that $x(t_0) = \frac{x(\omega)+x_*}{2} \in (x_*, x(\omega))$. Since $x \in AP(\mathbf{R})$, then for $\epsilon = x(\omega) - x(t_0) = \frac{x(\omega)-x_*}{2} > 0$, there exists a number $\tau \in T(x, \epsilon) \cap (-\infty, -t_0]$ $(t_0 + \tau \le 0)$ such that

$$|x(t + \tau) - x(t)| < \epsilon = x(\omega) - x(t_0), \ \forall t \in \mathbf{R},$$

which implies from (2.2) that

$$|x(t_0 + \tau) - x(t_0)| = x(t_0 + \tau) - x(t_0) < x(\omega) - x(t_0) \Longrightarrow x(t_0 + \tau) < x(\omega),$$

which leads to a contradiction with (2.2). Therefore, our claim is valid. From (2.1), $\min_{s\in[\omega_1,\omega]} x(s) = \min_{s\in[0,\omega]} x(s)$. So we can choose $\xi_0 \in [0, \omega]$ such that $x(\xi_0) = \min_{s\in[0,\omega]} x(s)$, $\xi_1 = \omega_1$ and $\xi_2 = \omega$, we obtain from (2.1) that

$$x(\xi_1) = x(\xi_2) \quad \text{and} \quad x(\xi_0) \le x(s), \ \forall s \in [\xi_1, \xi_2].$$

Therefore, Lemma 2.3 holds.

(b) If $x(\omega) = x_*$, then we claim that there exist $\omega_1 \in (-\infty, 0)$ and $\omega_2 \in (\omega, +\infty)$ such that

$$x(\omega_1) = x(\omega_2) = \frac{x(0) + x(\omega)}{2} \in (x(\omega), x(0)). \tag{2.3}$$

First, we prove that there exist $\omega_1 \in (-\infty, 0)$ such that

$$x(\omega_1) = \frac{x(0) + x(\omega)}{2}. \tag{2.4}$$

By the continuity of x, there must exist $t_1 \in \mathbf{R}$ such that $x(t_1) = \frac{x(0)+x(\omega)}{2} \in (x(\omega), x(0))$. If (2.4) is not true, then

$$x(s) > x(t_1) = \frac{x(0) + x(\omega)}{2} > x(\omega), \;\; \forall s \in (-\infty, 0]. \tag{2.5}$$

Since $x \in AP(\mathbf{R})$, then for $\epsilon = x(t_1) - x(\omega) = \frac{x(0)-x(\omega)}{2} > 0$, there exists a number $\tau \in T(x, \epsilon) \cap (-\infty, -\omega]$ $(\omega + \tau \leq 0)$ such that

$$|x(t+\tau) - x(t)| < \epsilon = x(t_1) - x(\omega), \quad \forall t \in \mathbf{R},$$

which implies from (2.5) that

$$|x(\omega+\tau) - x(\omega)| = x(\omega+\tau) - x(\omega) < x(t_1) - x(\omega) \Longrightarrow x(\omega+\tau) < x(t_1),$$

which leads to a contradiction with (2.5). Therefore, (2.4) holds. Similar to the above argument, it is not difficult to prove that there exists $\omega_2 \in (\omega, +\infty)$ such that $x(\omega_2) = \frac{x(0)+x(\omega)}{2} = x(\omega_1)$. Hence, (2.3) is satisfied, which implies that Lemma 2.3 holds by choosing $\xi_0 = \omega$, $\xi_1 = \omega_1$ and $\xi_2 = \omega_2$.

(C_3) $x(0) < x(\omega)$. Similar to the argument as that in (C_2), it is not difficult to verify that Lemma 2.3 holds. So we omit the proof of this case.

This completes the proof of Lemma 2.3. ∎

Similar to the argument as that in Lemma 2.3, we can easily show that

Lemma 2.4. *Assume that $x \in AP(\mathbf{R})$, then for arbitrary interval $[a, b]$ with $b - a = \omega > 0$, there exist $\eta_0 \in [a, b]$, $\eta_1 \in (-\infty, a]$ and $\eta_2 \in [\eta_1 + \omega, +\infty)$ such that*

$$x(\eta_1) = x(\eta_2) \quad \textit{and} \quad x(\eta_0) \geq x(s), \;\; \forall s \in [\eta_1, \eta_2].$$

Lemma 2.5. [3, 6] *Assume that $f \in AP(\mathbf{R})$ and $\bar{f} = m(f) > 0$, then for $\forall a \in \mathbf{R}$, there exists a positive constant T_0 independent of a such that*

$$\frac{1}{T}\int_a^{a+T} f(s)\,\mathrm{d}s \in \left[\frac{\bar{f}}{2}, \frac{3\bar{f}}{2}\right], \quad \forall T \geq T_0.$$

3 Existence of at least one almost periodic solution

The method to be used in this paper involves the applications of the continuation theorem of coincidence degree. This requires us to introduce a few concepts and results from Gaines and Mawhin [4].

Let $\mathbb{X}$ and $\mathbb{Y}$ be real Banach spaces, $L : \mathrm{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}$ be a linear mapping and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\mathrm{Im}L$ is closed in $\mathbb{Y}$ and $\dim\mathrm{Ker}L = \mathrm{codim}\mathrm{Im}L < +\infty$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I - Q)$. It follows that $L|_{\mathrm{Dom}L\cap\mathrm{Ker}P} : (I - P)\mathbb{X} \to \mathrm{Im}L$ is invertible and its inverse is denoted by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to \mathbb{X}$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Lemma 3.1. [4] *Let $\Omega \subseteq \mathbb{X}$ be an open bounded set, L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. If all the following conditions hold:*

(*a*) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap \mathrm{Dom}L$, $\lambda \in (0, 1)$;

(*b*) $QNx \neq 0$, $\forall x \in \partial\Omega \cap \mathrm{Ker}L$;

(*c*) $\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$, *where* $J : \mathrm{Im}Q \to \mathrm{Ker}L$ *is an isomorphism.*

Then $Lx = Nx$ has a solution on $\bar{\Omega} \cap \mathrm{Dom}L$.

Our first observation is that under the invariant transformation $(S, I)^T = (e^x, e^y)^T$, system (1.3) reduces to

$$\begin{cases} \dot{x}(t) = \alpha(t)e^{-x(t)} - \mu(t) - \frac{\beta(t)e^{y(t)}}{k_1+k_2e^{y(t)}}, \\ \dot{y}(t) = \frac{\beta(t)e^{x(t)}}{k_1+k_2e^{y(t)}} - (\mu(t) + \gamma(t)) - \sigma(t)e^{-y(t)}. \end{cases} \tag{3.1}$$

For $x \in AP(\mathbf{R})$, we denote by

$$\bar{x} = m(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(s)\,\mathrm{d}s,$$

$$a(x, \varpi) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(s)e^{-\mathrm{i}\varpi s}\,\mathrm{d}s,$$

$$\Lambda(x) = \left\{\varpi \in \mathbf{R} : \lim_{T\to\infty} \frac{1}{T} \int_0^T x(s)e^{-\mathrm{i}\varpi s}\mathrm{d}s \neq 0\right\}$$

the mean value, the Bohr transform and the set of Fourier exponents of x, respectively.

Set $\mathbb{X} = \mathbb{Y} = \mathbb{V}_1 \bigoplus \mathbb{V}_2$, where

$$\begin{aligned} \mathbb{V}_1 = \Big\{w = (x, y)^T \in AP(\mathbf{R}, \mathbf{R}^2) :\ & \mathrm{mod}(x) \subseteq \mathrm{mod}(L_x), \\ & \mathrm{mod}(y) \subseteq \mathrm{mod}(L_y), \\ & \forall \varpi \in \Lambda(x) \cup \Lambda(y), |\varpi| \geq \theta_0\Big\}, \end{aligned}$$

$$\mathbb{V}_2 = \big\{w = (x, y)^T \equiv (k_1, k_2)^T, k_1, k_2 \in \mathbb{R}\big\},$$

where

$$L_x = L_x(t,\varphi) = \alpha(t)e^{-\varphi_1} - \mu(t) - \frac{\beta(t)e^{\varphi_2}}{k_1 + k_2 e^{\varphi_2}},$$

$$L_y = L_y(t,\varphi) = \frac{\beta(t)e^{\varphi_1}}{k_1 + k_2 e^{\varphi_2}} - (\mu(t) + \gamma(t)) - \sigma(t)e^{-\varphi_2},$$

$\varphi = (\varphi_1, \varphi_2)^T \in \mathbf{R}^2$, θ_0 is a given positive constant. Define the norm

$$\|w\|_{\mathbb{X}} = \max\left\{\sup_{s\in\mathbf{R}} |x(s)|, \sup_{s\in\mathbf{R}} |y(s)|\right\}, \quad \forall w = (x,y)^T \in \mathbb{X} = \mathbb{Y}.$$

Similar to the proof as that in [15], it follows that

Lemma 3.2. *$\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces endowed with $\|\cdot\|$.*

Lemma 3.3. *Let $L : \mathbb{X} \to \mathbb{Y}$, $Lw = L(x,y)^T = (\dot{x},\dot{y})^T$, then L is a Fredholm mapping of index zero.*

Lemma 3.4. *Define $N : \mathbb{X} \to \mathbb{Y}$, $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ by*

$$Nw = N\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha(t)e^{-x(t)} - \mu(t) - \frac{\beta(t)e^{y(t)}}{k_1+k_2e^{y(t)}} \\ \frac{\beta(t)e^{x(t)}}{k_1+k_2e^{y(t)}} - (\mu(t)+\gamma(t)) - \sigma(t)e^{-y(t)} \end{pmatrix},$$

$$Pw = P\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m(x) \\ m(y) \end{pmatrix} = \lim_{T\to\infty}\frac{1}{T}\begin{pmatrix} \int_0^T x(s)\,\mathrm{d}s \\ \int_0^T y(s)\,\mathrm{d}s \end{pmatrix} = Qw, \ \ \forall w = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{X} = \mathbb{Y}.$$

Then N is L-compact on $\bar{\Omega}$(Ω is an open and bounded subset of $\mathbb{X}$).

Now we are in the position to present and prove our result on the existence of at least one almost periodic solution for system (1.3).

First of all, we introduce a assumption:

(H_1) $\bar{\alpha} > 0$, $\bar{\beta} > 0$ and $\bar{\sigma} > 0$.

Theorem 3.1. *Assume that (H_1) holds, then system* (1.3) *admits at least one positive almost periodic solution.*

Proof. It is easy to see that if system (3.1) has one almost periodic solution $(\bar{x},\bar{y})^T$, then $(\bar{S},\bar{I})^T = (e^{\bar{x}}, e^{\bar{y}})^T$ is a positive almost periodic solution of system (1.3). Therefore, to completes the proof it suffices to show that system (3.1) has one almost periodic solution.

In order to use Lemma 3.1, we set the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ as those in Lemma 3.2 and L, N, P, Q the same as those defined in Lemmas 3.3 and 3.4, respectively. It remains to search for an appropriate open and bounded subset $\Omega \subseteq \mathbb{X}$.

Corresponding to the operator equation $Lw = \lambda w$, $\lambda \in (0,1)$, we have

$$\begin{cases} \dot{x}(t) = \lambda\left[\alpha(t)e^{-x(t)} - \mu(t) - \frac{\beta(t)e^{y(t)}}{k_1+k_2e^{y(t)}}\right], \\ \dot{y}(t) = \lambda\left[\frac{\beta(t)e^{x(t)}}{k_1+k_2e^{y(t)}} - (\mu(t)+\gamma(t)) - \sigma(t)e^{-y(t)}\right]. \end{cases} \tag{3.2}$$

Suppose that $(x,y)^T \in \mathrm{Dom}L \subseteq \mathbb{X}$ is a solution of system (3.1) for some $\lambda \in (0,1)$, where $\mathrm{Dom}L = \{w = (x,y)^T \in \mathbb{X} : x, y \in C^1(\mathbf{R}), \dot{x}, \dot{y} \in C(\mathbf{R})\}$. By the almost periodicity of x and y, there exist two sequences $\{T_n : n \in \mathbf{N}^+\}$ and $\{P_n : n \in \mathbf{N}^+\}$ such that

$$x(T_n) \in [x^* - \frac{1}{n}, x^*], \ x^* = \sup_{s\in\mathbf{R}} x(s), \ n \in \mathbf{N}^+, \tag{3.3}$$

$$y(P_n)\in[y^*-\frac{1}{n},y^*],\ \ y^*=\sup_{s\in\mathbf{R}}y(s),\ \ n\in\mathbf{N}^+. \tag{3.4}$$

From (H_1) and Lemma 2.5, for $\forall a\in\mathbf{R}$, there exists a constant $\omega_0\in(0,+\infty)$ independent of a such that

$$\begin{cases}\frac{1}{T}\int_a^{a+T}\alpha(s)\,\mathrm{d}s\in[\frac{\bar{\alpha}}{2},\frac{3\bar{\alpha}}{2}],\\ \frac{1}{T}\int_a^{a+T}\sigma(s)\,\mathrm{d}s\in[\frac{\bar{\sigma}}{2},\frac{3\bar{\sigma}}{2}],\quad \forall T\geq\omega_0.\end{cases} \tag{3.5}$$

Consider $[T_{n_0}-\omega_0,T_{n_0}]$ and $[P_{n_0}-\omega_0,P_{n_0}]$ for $\forall n_0\in\mathbf{N}^+$. By Lemma 2.3, there exist $\xi_x^{n_0}\in[T_{n_0}-\omega_0,T_{n_0}]$, $\xi_1^{n_0}\in(-\infty,T_{n_0}-\omega_0]$ and $\xi_2^{n_0}\in[\xi_1^{n_0}+\omega_0,+\infty)$ such that

$$x(\xi_1^{n_0})=x(\xi_2^{n_0})\quad\text{and}\quad x(\xi_x^{n_0})\leq x(s),\ \ \forall s\in[\xi_1^{n_0},\xi_2^{n_0}]. \tag{3.6}$$

Integrating the first equation of system (3.1) from $\xi_1^{n_0}$ to $\xi_2^{n_0}$ leads to

$$\int_{\xi_1^{n_0}}^{\xi_2^{n_0}}\left[\alpha(s)e^{-x(s)}-\mu(s)-\frac{\beta(s)e^{y(s)}}{k_1+k_2e^{y(s)}}\right]\mathrm{d}s=0,$$

which yields from (3.5)-(3.6) that

$$\alpha^Me^{-x(\xi_x^{n_0})}(\xi_1^{n_0}-\xi_2^{n_0})\geq\int_{\xi_1^{n_0}}^{\xi_2^{n_0}}\alpha(s)e^{-x(s)}\,\mathrm{d}s\geq\int_{\xi_1^{n_0}}^{\xi_2^{n_0}}\mu(s)\,\mathrm{d}s\geq\frac{\bar{\mu}}{2}(\xi_1^{n_0}-\xi_2^{n_0}).$$

Then

$$e^{x(\xi_x^{n_0})}\leq\frac{2\alpha^M}{\bar{\mu}}. \tag{3.7}$$

Let $\mathbb{I}=[\xi_x^{n_0},T_{n_0}]$ and $\Xi_{\mathbb{I}}^{(e^x)'}=\{s\in\mathbb{I}:(e^{x(s)})'\geq 0\}$. It follows from system (3.1) that

$$\begin{aligned}\int_{\Xi_{\mathbb{I}}^{(e^x)'}}\frac{\mathrm{d}e^{x(s)}}{\mathrm{d}s}\,\mathrm{d}s=\int_{\Xi_{\mathbb{I}}^{(e^x)'}}(e^{x(s)})'\,\mathrm{d}s&=\int_{\Xi_{\mathbb{I}}^{(e^x)'}}\lambda\left[\alpha(s)-\mu(s)e^{x(s)}-\frac{\beta(s)e^{x(s)}e^{y(s)}}{k_1+k_2e^{y(s)}}\right]\mathrm{d}s\\&\leq\int_{\Xi_{\mathbb{I}}^{(e^x)'}}\lambda\alpha(s)\,\mathrm{d}s\leq\int_{T_{n_0}-\omega_0}^{T_{n_0}}\alpha(s)\,\mathrm{d}s\\&\leq\alpha^M\omega_0.\end{aligned} \tag{3.8}$$

By Lemma 2.2, it follows from (3.7)-(3.8) that

$$e^{x(t)}\leq e^{x(\xi_x^{n_0})}+\int_{\Xi_{\mathbb{I}}^{(e^x)'}}\frac{\mathrm{d}e^{x(s)}}{\mathrm{d}s}\,\mathrm{d}s\leq\frac{2\alpha^M}{\bar{\mu}}+\alpha^M\omega_0,\quad\forall t\in[\xi_x^{n_0},T_{n_0}],$$

which implies that

$$x(T_{n_0})\leq\ln\left[\frac{2\alpha^M}{\bar{\mu}}+\alpha^M\omega_0\right]:=\rho_1.$$

In view of (3.3), letting $n_0\to+\infty$ in the above inequality leads to

$$x^*=\lim_{n_0\to+\infty}x(T_{n_0})\leq\rho_1. \tag{3.9}$$

Also, there exist $\xi_y^{n_0}\in[P_{n_0}-\omega_0,P_{n_0}]$, $\xi_3^{n_0}\in(-\infty,P_{n_0}-\omega_0]$ and $\xi_4^{n_0}\in[\xi_3^{n_0}+\omega_0,+\infty)$ such that

$$y(\xi_3^{n_0})=y(\xi_4^{n_0})\quad\text{and}\quad y(\xi_y^{n_0})\leq y(s),\ \ \forall s\in[\xi_3^{n_0},\xi_4^{n_0}]. \tag{3.10}$$

Integrating (3.2) from $\xi_3^{n_0}$ to $\xi_4^{n_0}$ leads to

$$\int_{\xi_3^{n_0}}^{\xi_4^{n_0}} \left[\frac{\beta(s)e^{x(s)}}{k_1 + k_2 e^{y(s)}} - (\mu(s) + \gamma(s)) - \sigma(s)e^{-y(s)}\right] \mathrm{d}s = 0, \tag{3.11}$$

which yields from (3.10) and (3.5) that

$$\frac{\beta^M e^{\rho_1} e^{-y(\xi_y^{n_0})}(\xi_4^{n_0} - \xi_3^{n_0})}{k_2} \geq \int_{\xi_3^{n_0}}^{\xi_4^{n_0}} \frac{\beta(s)e^{x(s)}}{k_1 + k_2 e^{y(s)}} \mathrm{d}s \geq \int_{\xi_3^{n_0}}^{\xi_4^{n_0}} (\mu(s)+\gamma(s)) \, \mathrm{d}s \geq (\frac{\bar{\mu}}{2}+\gamma^l)(\xi_4^{n_0}-\xi_3^{n_0}),$$

which implies that

$$e^{y(\xi_y^{n_0})} \leq \frac{2\beta^M e^{\rho_1}}{k_2(\bar{\mu} + 2\gamma^l)} \Longleftrightarrow y(\xi_y^{n_0}) \leq \ln\left[\frac{2\beta^M e^{\rho_1}}{k_2(\bar{\mu} + 2\gamma^l)}\right]. \tag{3.12}$$

Let $\hat{\mathbb{I}} = [\xi_y^{n_0}, P_{n_0}]$ and $\Xi_{\hat{\mathbb{I}}}^{\dot{y}} = \{s \in \hat{\mathbb{I}} : \dot{y}(s) \geq 0\}$. Similar to the argument as that in (3.8), it follows from the second equation of system (3.2) that

$$\int_{\Xi_{\hat{\mathbb{I}}}^{\dot{y}}} \dot{y}(s) \, \mathrm{d}s \leq \int_{\Xi_{\hat{\mathbb{I}}}^{\dot{y}}} \frac{\beta(s)e^{x(s)}}{k_1 + k_2 e^{y(s)}} \mathrm{d}s \leq \int_{P_{n_0}-\omega_0}^{P_{n_0}} \frac{\beta(s)e^{x(s)}}{k_1 + k_2 e^{y(s)}} \mathrm{d}s \leq \frac{\beta^M e^{\rho_1}\omega_0}{k_1}. \tag{3.13}$$

By Lemma 2.2, it follows from (3.12)-(3.13) that

$$\begin{aligned} y(t) &\leq y(\xi_y^{n_0}) + \int_{\Xi_{\hat{\mathbb{I}}}^{\dot{y}}} \dot{y}(s) \, \mathrm{d}s \\ &\leq \ln\left[\frac{2\beta^M e^{\rho_1}}{k_2(\bar{\mu} + 2\gamma^l)}\right] + \frac{\beta^M e^{\rho_1}\omega_0}{k_1} := \rho_2, \quad \forall t \in [\xi_y^{n_0}, P_{n_0}], \end{aligned}$$

which implies that

$$y(P_{n_0}) \leq \rho_2.$$

In view of (3.4), letting $n_0 \to +\infty$ in the above inequality leads to

$$y^* = \lim_{n_0 \to +\infty} y(P_{n_0}) \leq \rho_2. \tag{3.14}$$

For $\forall n_0 \in \mathbf{Z}$, by Lemma 2.4, there exist $\eta_x^{n_0} \in [n_0\hat{\omega}, n_0\omega_0 + \omega_0]$, $\eta_1^{n_0} \in (-\infty, n_0\omega_0]$ and $\eta_2^{n_0} \in [\eta_1^{n_0} + \omega_0, +\infty)$ such that

$$x(\eta_1^{n_0}) = x(\eta_2^{n_0}) \quad \text{and} \quad x(\eta_x^{n_0}) \geq x(s), \ \forall s \in [\eta_1^{n_0}, \eta_2^{n_0}].$$

Integrating the first equation of system (3.2) from $\eta_1^{n_0}$ to $\eta_2^{n_0}$ leads to

$$\begin{aligned} e^{-x(\eta_x^{n_0})} \int_{\eta_1^{n_0}}^{\eta_2^{n_0}} \alpha(s) \, \mathrm{d}s \leq \int_{\eta_1^{n_0}}^{\eta_2^{n_0}} \alpha(s)e^{-x(s)} \, \mathrm{d}s &= \int_{\eta_1^{n_0}}^{\eta_2^{n_0}} \left[\mu(s) + \frac{\beta(s)e^{y(s)}}{k_1 + k_2 e^{y(s)}}\right] \mathrm{d}s \\ &\leq \left[\mu^M + \frac{\beta^M e^{\rho_2}}{k_1 + k_2 e^{\rho_2}}\right](\eta_2^{n_0} - \eta_1^{n_0}), \end{aligned}$$

which implies from (3.5) that

$$e^{x(\eta_x^{n_0})} \geq \left(\frac{2}{\bar{\alpha}}\left[\mu^M + \frac{\beta^M e^{\rho_2}}{k_1 + k_2 e^{\rho_2}}\right]\right)^{-1} := \Delta_1. \tag{3.15}$$

Further, we obtain from the first equation of system (3.2) that

$$\int_{n_0\omega_0}^{n_0\omega_0+\omega_0} \left|\frac{\mathrm{d}e^{x(s)}}{\mathrm{d}s}\right| \mathrm{d}s = \int_{n_0\omega_0}^{n_0\omega_0+\omega_0} \lambda \left|\alpha(s) - \mu(s)e^{x(s)} - \frac{\beta(s)e^{x(s)}e^{y(s)}}{k_1 + k_2 e^{y(s)}}\right| \mathrm{d}s$$

$$\leq \left[\alpha^M + \mu^M e^{\rho_1} + \frac{\beta^M e^{\rho_1} e^{\rho_2}}{k_1 + k_2 e^{\rho_2}}\right]\omega_0 := \Theta_1. \tag{3.16}$$

It follows from (3.15)-(3.16) that

$$e^{x(t)} \geq e^{x(\eta_x^{n_0})} - \int_{n_0\omega_0}^{n_0\omega_0+\omega_0} \left|\frac{\mathrm{d}e^{x(s)}}{\mathrm{d}s}\right| \mathrm{d}s \geq \Delta_1 - \Theta_1 := \rho_3, \quad \forall t \in [n_0\omega_0, n_0\omega_0 + \omega_0]. \tag{3.17}$$

Obviously, ρ_3 is a constant independent of n_0. So it follows from (3.17) that

$$x_* = \inf_{s\in\mathbf{R}} x(s) = \inf_{n_0\in\mathbf{Z}} \{ \min_{s\in[n_0\omega_0, n_0\omega_0+\omega_0]} x(s)\} \geq \inf_{n_0\in\mathbf{Z}} \{\rho_3\} = \rho_3. \tag{3.18}$$

Also, there exist $\eta_y^{n_0} \in [n_0\omega_0, n_0\omega_0 + \omega_0]$, $\eta_3^{n_0} \in (-\infty, n_0\omega_0]$ and $\eta_4^{n_0} \in [\eta_3^{n_0} + \omega_0, +\infty)$ such that

$$y(\eta_3^{n_0}) = y(\eta_4^{n_0}) \quad \text{and} \quad y(\eta_y^{n_0}) \geq y(s), \ \forall s \in [\eta_3^{n_0}, \eta_4^{n_0}].$$

Integrating the second equation of system (3.2) from $\eta_3^{n_0}$ to $\eta_4^{n_0}$, it leads to

$$\begin{aligned} e^{-y(\eta_y^{n_0})} \int_{\eta_3^{n_0}}^{\eta_4^{n_0}} \sigma(s)\,\mathrm{d}s \leq \int_{\eta_3^{n_0}}^{\eta_4^{n_0}} \sigma(s) e^{-y(s)}\,\mathrm{d}s &\leq \int_{\eta_3^{n_0}}^{\eta_4^{n_0}} \frac{\beta(s)e^{x(s)}}{k_1 + k_2 e^{y(s)}}\,\mathrm{d}s \\ &\leq \frac{\beta^M e^{\rho_1}}{k_1}(\eta_4^{n_0} - \eta_3^{n_0}). \end{aligned}$$

Then

$$e^{y(\eta_y^{n_0})} \geq \left[\frac{2\beta^M e^{\rho_1}}{k_1\bar{\sigma}}\right]^{-1} := \Delta_2. \tag{3.19}$$

In view of the second equation of system (3.2), it follows that

$$\begin{aligned} \int_{n_0\omega_0}^{n_0\omega_0+\omega_0} \left|\frac{\mathrm{d}e^{y(s)}}{\mathrm{d}s}\right| \mathrm{d}s &= \int_{n_0\omega_0}^{n_0\omega_0+\omega_0} \lambda\left|\frac{\beta(s)e^{x(s)}e^{y(s)}}{k_1 + k_2 e^{y(s)}} - (\mu(s) + \gamma(s))e^{y(s)} - \sigma(s)\right| \mathrm{d}s \\ &\leq \left[\frac{\beta^M e^{\rho_1} e^{\rho_2}}{k_1 + k_2 e^{\rho_2}} + (\mu^M + \gamma^M)e^{\rho_2} + \sigma^M\right]\omega_0 := \Theta_2. \end{aligned} \tag{3.20}$$

Similar to the argument as that in (3.18), we obtain from (3.19)-(3.20) that

$$y_* \geq \Delta_2 + \Theta_2 := \rho_4. \tag{3.21}$$

Set $K = |\rho_1| + |\rho_2| + |\rho_3| + |\rho_4| + 1$, then $\|w\| = \|(x, y)^T\| < K$. Clearly, K is independent of $\lambda \in (0, 1)$. Consider the algebraic equations $QNw_0 = 0$ for $w_0 = (x_0, y_0)^T \in \mathbf{R}^2$ as follows:

$$\begin{cases} m(\alpha)e^{-x_0} - m(\mu) - \frac{m(\beta)e^{y_0}}{k_1+k_2e^{y_0}} = 0, \\ \frac{m(\beta)e^{x_0}}{k_1+k_2e^{y_0}} - m(\mu + \gamma) - m(\sigma)e^{-y_0} = 0. \end{cases}$$

Similar to the arguments as that in (3.9), (3.14), (3.18) and (3.21), we can easily obtain that

$$\rho_3 \leq x_* \leq x^* \leq \rho_1, \quad \rho_4 \leq y_* \leq y^* \leq \rho_2.$$

Then $\|w_0\| = |x_0| + |y_0| < K$. Let $\Omega = \{w \in \mathbb{X} : \|w\| < K\}$, then Ω satisfies conditions (a) and (b) of Lemma 3.1.

Finally, we will show that condition (c) of Lemma 3.1 is satisfied. Let us consider the homotopy

$$H(\iota, w) = \iota QNw + (1 - \iota)Fw, \ \ (\iota, w) \in [0, 1] \times \mathbf{R}^2,$$

where

$$Fw = F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m(\alpha)e^{-x} - m(\mu) \\ -m(\mu+\gamma) - m(\sigma)e^{-y} \end{pmatrix}.$$

From the above discussion it is easy to verify that $H(\iota, w) \neq 0$ on $\partial\Omega \cap \mathrm{Ker}L$, $\forall \iota \in [0,1]$. By the invariance property of homotopy, direct calculation produces

$$\deg\big(JQN, \Omega\cap \mathrm{Ker}L, 0\big) = \deg\big(QN, \Omega\cap \mathrm{Ker}L, 0\big) = \deg\big(F, \Omega\cap \mathrm{Ker}L, 0\big) \neq 0,$$

where $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree and J is the identity mapping since $\mathrm{Im}Q = \mathrm{Ker}L$. Obviously, all the conditions of Lemma 3.1 are satisfied. Therefore, system (3.1) has one almost periodic solution, that is, system (1.3) has at least one positive almost periodic solution. This completes the proof. ∎

Next, we shall investigate the stability of positive almost periodic solution of system (1.3).

Lemma 3.5. [23] *Assume that for $y(t) > 0$, $t \geq t_0$, it holds that*

$$\dot{y}(t) \leq a - by(t),$$

with initial condition $y(t_0) \geq 0$, where a, b are positive constants. Then

$$\lim_{t\to+\infty} \sup y(t) \leq \frac{a}{b}.$$

Lemma 3.6. *For any positive solution $(S(t), I(t))^T$ of system* (1.3) *satisfies*

$$\limsup_{t\to\infty} S(t) \leq \frac{\alpha^M}{\mu^l} := S_0.$$

Proof. In view the first equation of system (1.3), we have

$$\dot{S}(t) \leq \alpha^M - \mu^l S(t),$$

which implies from Lemma 3.5 that

$$\lim_{t\to+\infty} \sup S(t) \leq \frac{\alpha^M}{\mu^l}.$$

This completes the proof. ∎

Theorem 3.2. *Assume that (H_1) holds. Suppose further that*

(H_2) $k_2\mu^l > 2\beta^M$ and $k_1(\mu^l + \gamma^l) > 2\beta^M S_0$, where $S_0 := \frac{\alpha^M}{\mu^l}$.

Then the almost periodic solution of system (1.3) *is globally asymptotically stable.*

Proof. From Theorem 3.1, we know that system (1.3) has at least one positive almost periodic solution $(S, I)^T$. Suppose that $(\bar{S}, \bar{I})^T$ is another positive solution of system (1.3).

By Lemma 3.6 and (H_2), for $\forall\epsilon > 0$ small enough, there exist positive constants T_0 and θ such that

$$\max\{S(t), \bar{S}(t)\} \leq S_0 + \epsilon, \quad , \quad \forall t \geq T_0$$

and

$$\min\left[\mu^l - \frac{2\beta^M}{k_2}, \mu^l + \gamma^l - \frac{2\beta^M(S_0+\epsilon)}{k_1}\right] > \theta.$$

Define

$$V(t) = |S(t) - \bar{S}(t)| + |I(t) - \bar{I}(t)|, \quad t \geq T_0.$$

By calculating the upper right derivative of V along the solution of system (1.3), it follows from the mean value theorem for multivariate function that

$$\begin{aligned}
D^+V(t) &= \operatorname{sgn}[S(t) - \bar{S}(t)][\dot{S}(t) - \dot{\bar{S}}(t)] + \operatorname{sgn}[I(t) - \bar{I}(t)][\dot{I}(t) - \dot{\bar{I}}(t)] \\
&\leq \Big[- \mu^l |S(t) - \bar{S}(t)| + \frac{\beta^M}{k_2} |S(t) - \bar{S}(t)| + \frac{\beta^M (S_0 + \epsilon)}{k_1} |I(t) - \bar{I}(t)| \Big] \\
&\quad + \Big[- (\mu^l + \gamma^l) |I(t) - \bar{I}(t)| + \frac{\beta^M}{k_2} |S(t) - \bar{S}(t)| + \frac{\beta^M (S_0 + \epsilon)}{k_1} |I(t) - \bar{I}(t)| \Big] \\
&= - \Big[\mu^l - \frac{2\beta^M}{k_2} \Big] |S(t) - \bar{S}(t)| - \Big[\mu^l + \gamma^l - \frac{2\beta^M (S_0 + \epsilon)}{k_1} \Big] |I(t) - \bar{I}(t)| \\
&\leq -\theta \{ |S(t) - \bar{S}(t)| + |I(t) - \bar{I}(t)| \}, \quad t \geq T_0.
\end{aligned}$$

Therefore, V is non-increasing. Integrating of the last inequality from T_0 to t leads to

$$V(t) + \theta \int_{T_0}^{t} |S(t) - \bar{S}(t)| \, \mathrm{d}t + \theta \int_{T_0}^{t} |I(t) - \bar{I}(t)| \, \mathrm{d}t \leq V(T_0) < +\infty, \;\; \forall t \geq T_0,$$

that is,

$$\int_{T_0}^{+\infty} |S(t) - \bar{S}(t)| \, \mathrm{d}t < +\infty, \;\; \int_{T_0}^{+\infty} |I(t) - \bar{I}(t)| \, \mathrm{d}t < +\infty,$$

which implies that

$$\lim_{t \to +\infty} |S(t) - \bar{S}(t)| = \lim_{t \to +\infty} |I(t) - \bar{I}(t)| = 0.$$

This completes the proof. ■

4 A multiplicity result

Now we present and prove our main result of this section on the existence of at least two positive almost periodic solutions for system (1.3).

Lemma 4.1. [20] *Assume that $x \in AP(\mathbf{R}) \cap C^1(\mathbf{R})$ with $\dot{x} \in C(\mathbf{R})$, for $\forall \epsilon > 0$, we have the following conclusions:*

(I) there is a point $\xi_\epsilon \in [0, +\infty)$ such that $x(\xi_\epsilon) \in [x^* - \epsilon, x^*]$ and $\dot{x}(\xi_\epsilon) = 0$;

(II) there is a point $\eta_\epsilon \in [0, +\infty)$ such that $x(\eta_\epsilon) \in [x_*, x_* + \epsilon]$ and $\dot{x}(\eta_\epsilon) = 0$.

Let

$$o_1 := \ln \frac{\alpha^M}{\mu^l}, \quad \varsigma = \beta^l e^{o_1} - k_1 \kappa - k_2 \sigma^M, \quad \kappa = \mu^M + \gamma^M.$$

We introduce some assumptions and two numbers as follows:

(P_1) $\mu^l > 0$ and $\beta^M > 0$.

(P_2) $\frac{\beta^M e^{o_1} - (\mu^l + \gamma^l)}{k_2 (\mu^l + \gamma^l)} < \frac{\alpha^M}{\mu^l}$.

(P_3) $\varsigma > 2\sqrt{k_1 k_2 \sigma^M \kappa}$.

$$l_{\pm} = \frac{\varsigma \pm \sqrt{\varsigma^2 - 4k_1k_2\sigma^M\kappa}}{2k_2\kappa}.$$

Theorem 4.1. *Assume that* (P_1)*-*(P_3) *hold, then system* (1.3) *admits at least two positive almost periodic solutions.*

Proof. Similar to Theorem 3.1, we consider the operator equation (3.2). Suppose that $(x,y)^T \in \mathrm{Dom}L \subseteq \mathbb{X}$ is a solution of system (3.2) for some $\lambda \in (0,1)$, where $\mathrm{Dom}L = \{(x,y)^T \in \mathbb{X} : x, y \in C^1(\mathbf{R}), \dot{x}, \dot{y} \in C(\mathbf{R})\}$. By Lemma 4.1, for $\forall \epsilon \in (0,1)$, there are four points $\xi_\epsilon^{(1)}, \eta_\epsilon^{(1)}, \xi_\epsilon^{(2)}, \eta_\epsilon^{(2)} \in [0, +\infty)$ such that

$$\dot{x}(\xi_\epsilon^{(1)}) = 0,\ x(\xi_\epsilon^{(1)}) \in [x^* - \epsilon, x^*];\quad \dot{y}(\xi_\epsilon^{(2)}) = 0,\ y(\xi_\epsilon^{(2)}) \in [y^* - \epsilon, y^*], \tag{4.1}$$

$$\dot{x}(\eta_\epsilon^{(1)}) = 0,\ x(\eta_\epsilon^{(1)}) \in [x_*, x_* + \epsilon];\quad \dot{y}(\eta_\epsilon^{(2)}) = 0,\ y(\eta_\epsilon^{(2)}) \in [y_*, y_* + \epsilon], \tag{4.2}$$

where $x^* = \sup_{s\in\mathbf{R}} x(s)$, $x_* = \inf_{s\in\mathbf{R}} x(s)$, $y^* = \sup_{s\in\mathbf{R}} y(s)$, $y_* = \inf_{s\in\mathbf{R}} y(s)$. In view of (3.2), it follows from (4.1)-(4.2) that

$$0 = \alpha(\xi_\epsilon^{(1)})e^{-x(\xi_\epsilon^{(1)})} - \mu(\xi_\epsilon^{(1)}) - \frac{\beta(\xi_\epsilon^{(1)})e^{y(\xi_\epsilon^{(1)})}}{k_1 + k_2e^{y(\xi_\epsilon^{(1)})}}, \tag{4.3}$$

$$0 = \frac{\beta(\xi_\epsilon^{(2)})e^{x(\xi_\epsilon^{(2)})}}{k_1 + k_2e^{y(\xi_\epsilon^{(2)})}} - (\mu(\xi_\epsilon^{(2)}) + \gamma(\xi_\epsilon^{(2)})) - \sigma(\xi_\epsilon^{(2)})e^{-y(\xi_\epsilon^{(2)})}, \tag{4.4}$$

$$0 = \alpha(\eta_\epsilon^{(1)})e^{-x(\eta_\epsilon^{(1)})} - \mu(\eta_\epsilon^{(1)}) - \frac{\beta(\eta_\epsilon^{(1)})e^{y(\eta_\epsilon^{(1)})}}{k_1 + k_2e^{y(\eta_\epsilon^{(1)})}}, \tag{4.5}$$

$$0 = \frac{\beta(\eta_\epsilon^{(2)})e^{x(\eta_\epsilon^{(2)})}}{k_1 + k_2e^{y(\eta_\epsilon^{(2)})}} - (\mu(\eta_\epsilon^{(2)}) + \gamma(\eta_\epsilon^{(2)})) - \sigma(\eta_\epsilon^{(2)})e^{-y(\eta_\epsilon^{(2)})}. \tag{4.6}$$

From (4.3) and (4.5), we obtain that

$$x(\xi_\epsilon^{(1)}) < \ln\frac{\alpha^M}{\mu^l} := o_1 \quad \text{and} \quad x(\eta_\epsilon^{(1)}) > \ln\frac{\alpha^l}{\mu^M + \beta^M/k_2} := o_2.$$

It follows (4.1)-(4.2) that

$$x^* < o_1 + \epsilon \quad \text{and} \quad x_* > o_2 - \epsilon. \tag{4.7}$$

Letting $\epsilon \to 0$ in (4.7) leads to

$$x^* \le o_1 \quad \text{and} \quad x_* \ge o_2. \tag{4.8}$$

From (4.4) and (4.6), we obtain that

$$e^{y(\xi_\epsilon^{(2)})} < \frac{\beta(\xi_\epsilon^{(2)})e^{x(\xi_\epsilon^{(2)})}e^{y(\xi_\epsilon^{(2)})} - (\mu(\xi_\epsilon^{(2)}) + \gamma(\xi_\epsilon^{(2)}))}{(k_1 + k_2e^{y(\xi_\epsilon^{(2)})})(\mu(\xi_\epsilon^{(2)}) + \gamma(\xi_\epsilon^{(2)}))} < \frac{\beta^M e^{o_1} - (\mu^l + \gamma^l)}{k_2(\mu^l + \gamma^l)} \tag{4.9}$$

and

$$\sigma(\eta_\epsilon^{(2)})e^{-y(\eta_\epsilon^{(2)})} < \frac{\beta(\eta_\epsilon^{(2)})e^{x(\eta_\epsilon^{(2)})}}{k_1 + k_2e^{y(\eta_\epsilon^{(2)})}} < \frac{\beta^M e^{o_1}}{k_1}, \tag{4.10}$$

which implies that

$$y(\xi_\epsilon^{(2)}) < \ln \frac{\beta^M e^{o_1} - (\mu^l + \gamma^l)}{k_2(\mu^l + \gamma^l)} := o_3 \quad \text{and} \quad y(\eta_\epsilon^{(2)}) > \ln \frac{k_1 \sigma^l}{\beta^M e^{o_1}} := o_4.$$

It follows (4.1)-(4.2) that

$$y^* < o_3 + \epsilon \quad \text{and} \quad y_* > o_4 - \epsilon. \tag{4.11}$$

Letting $\epsilon \to 0$ in (4.11) leads to

$$y^* \leq o_3 \quad \text{and} \quad y_* \geq o_4. \tag{4.12}$$

By (4.4), we have

$$0 = \frac{\beta(\xi_\epsilon^{(2)}) e^{x(\xi_\epsilon^{(2)}) + y(\xi_\epsilon^{(2)})}}{k_1 + k_2 e^{y(\xi_\epsilon^{(2)})}} - (\mu(\xi_\epsilon^{(2)}) + \gamma(\xi_\epsilon^{(2)})) e^{y(\xi_\epsilon^{(2)})} - \sigma(\xi_\epsilon^{(2)}), \tag{4.13}$$

which yields that

$$k_2(\mu^M + \gamma^M) e^{2y(\xi_\epsilon^{(2)})} - \left[\beta^l e^{o_1} - k_1(\mu^M + \gamma^M) - k_2 \sigma^M\right] e^{y(\xi_\epsilon^{(2)})} + k_1 \sigma^M > 0. \tag{4.14}$$

Under the hypothesis of (P_3), we have

$$y(\xi_\epsilon^{(2)}) > \ln l_+ \quad \text{or} \quad y(\xi_\epsilon^{(2)}) < \ln l_-.$$

It follows (4.1) that

$$y^* > \ln l_+ \quad \text{or} \quad y^* < \ln l_- + \epsilon. \tag{4.15}$$

Letting $\epsilon \to 0$ in (4.15) leads to

$$y^* \geq \ln l_+ \quad \text{or} \quad y^* \leq \ln l_-. \tag{4.16}$$

Similarly, from (4.6), we obtain a parallel argument to (4.16)

$$y_* \geq \ln l_+ \quad \text{or} \quad y_* \leq \ln l_-. \tag{4.17}$$

Therefore, it follows from (4.16)-(4.17) that

$$o_4 \leq y(t) \leq \ln l_- \quad \text{or} \quad \ln l_+ \leq y(t) \leq o_3, \ \ \forall t \in \mathbf{R}. \tag{4.18}$$

Obviously, $\ln l_\pm$, o_1, o_2, o_3 and o_4 are independent of λ. Let $\varepsilon = \frac{\ln l_+ - \ln l_-}{4}$ and

$$\Omega_1 = \left\{ w = (x, y)^T \in \mathbb{X} : o_2 - 1 < x(t) < o_1 + 1, o_4 - 1 < y(t) < \ln l_- + \varepsilon, \forall t \in \mathbf{R} \right\},$$

$$\Omega_2 = \left\{ w = (x, y)^T \in \mathbb{X} : o_2 - 1 < x(t) < o_1 + 1, \varepsilon - \ln l_+ < y(t) < o_3 + 1, \forall t \in \mathbf{R} \right\}.$$

Then Ω_1 and Ω_2 are bounded open subsets of $\mathbb{X}$, $\Omega_1 \cap \Omega_2 = \emptyset$. Therefore, Ω_1 and Ω_2 satisfies condition (a) of Lemma 3.1.

Now we show that condition (b) of Lemma 3.1 holds, i.e., we prove that $QNw \neq 0$ for all $w = (x, y)^T \in \partial\Omega_i \cap \mathrm{Ker} L = \partial\Omega_i \cap \mathbf{R}^2$, $i = 1, 2$. If it is not true, then there exists at least one constant vector $w_0 = (x_0, y_0)^T \in \partial\Omega_i \, (i = 1, 2)$ such that

$$\begin{cases} m(\alpha) e^{-x_0} - m(\mu) - \frac{m(\beta) e^{y_0}}{k_1 + k_2 e^{y_0}} = 0, \\ \frac{m(\beta) e^{x_0}}{k_1 + k_2 e^{y_0}} - m(\mu + \gamma) - m(\sigma) e^{-y_0} = 0. \end{cases}$$

Similar to the arguments as that in (4.8), (4.12) and (4.18), we can easily obtain that

$$o_2 < x(t) < o_1 \quad \text{and} \quad o_4 < y(t) < \ln l_- \quad \text{or} \quad \ln l_+ < y(t) < o_3, \ \ \forall t \in \mathbf{R}.$$

Then $w_0 \in \Omega_1 \cap \mathbf{R}^2$ or $w_0 \in \Omega_2 \cap \mathbf{R}^2$. This contradicts the fact that $w_0 \in \partial\Omega_i\,(i = 1, 2)$. This proves that condition (b) of Lemma 3.1 holds.

Finally, we show that condition (c) of Lemma 3.1 holds. Let us consider the homotopy

$$H(\iota, w) = \iota QNw + (1 - \iota)Fw, \quad (\iota, w) \in [0, 1] \times \mathbf{R}^2,$$

where

$$Fw = F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m(\alpha)e^{-x} - m(\mu) - \frac{m(\beta)}{k_2} \\ \big(m(\beta)e^{x} - m(\mu+\gamma)\big)e^{y} - k_2 m(\mu+\gamma)e^{2y} - m(\sigma) \end{pmatrix}.$$

By a parallel argument to Theorem 3.1 in paper [1], we can obtain that Ω_1 and Ω_2 satisfy condition (c) of Lemma 3.1. Obviously, all the conditions of Lemma 3.1 are satisfied. Therefore, system (3.1) has two different almost periodic solutions, that is, system (1.3) has at least two different positive almost periodic solutions. This completes the proof. ∎

Remark 4.1. In system (1.3), let $\alpha(t) \equiv \alpha$, $\mu(t) \equiv \mu$, $\gamma(t) \equiv \gamma$, $\sigma(t) \equiv \sigma$ and β be periodic, then Theorem 4.1 in this section changes to Theorem 3.1 in paper [1]. Therefore, our result extends the result obtained in paper [1].

5 Two examples and numerical simulations

Example 5.1. Consider the following almost periodic system:

$$\begin{cases} \dot{S}(t) = |\sin\sqrt{10}t| - \sin^2(\sqrt{2}t)S(t) - \frac{5\cos^2(\sqrt{3}t)S(t)I(t)}{1+I(t)}, \\ \dot{I}(t) = \frac{5\cos^2(\sqrt{3}t)S(t)I(t)}{1+I(t)} - (\sin^2(\sqrt{2}t) + 10)I(t) - |\cos\sqrt{9}t|. \end{cases} \tag{5.1}$$

Corresponding to system (1.3), we have that $\bar{\alpha} > 0$, $\bar{\mu} > 0$ and $\bar{\sigma} > 0$. Therefore, all the conditions of Theorem 3.1 are satisfied. By Theorem 3.1, system (5.1) admits at least one positive almost periodic solution(see Figure 1).

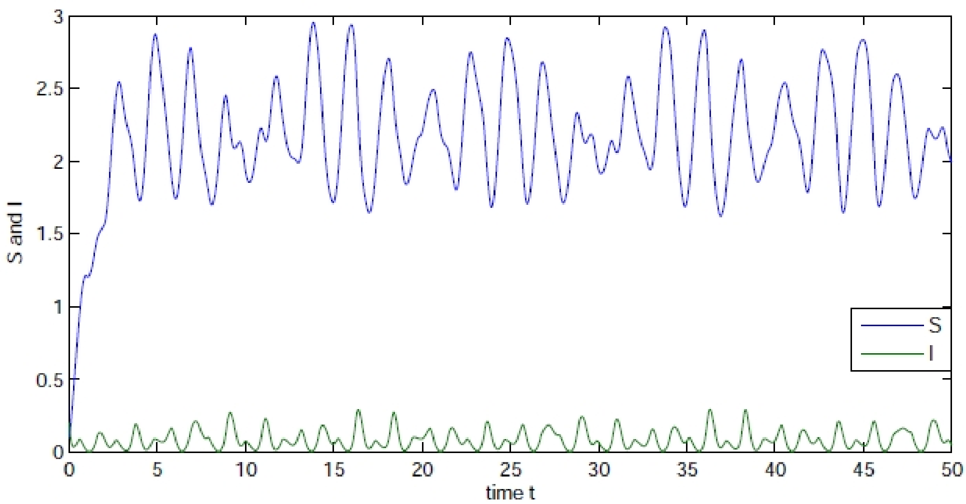

Figure 1 Almost periodicity of state variables $(S, I)^T$ of system (5.1)

Example 5.2. Consider the following almost periodic system:

$$\begin{cases} \dot{S}(t) = |\sin\sqrt{10}t| - 0.001[\sin^2(\sqrt{2}t) + 1]S(t) - \frac{0.001S(t)I(t)}{1+I(t)}, \\ \dot{I}(t) = \frac{0.001S(t)I(t)}{1+I(t)} - [0.001\sin^2(\sqrt{2}t) + 0.002]I(t) - 0.009|\cos\sqrt{9}t|. \end{cases} \tag{5.2}$$

Corresponding to system (1.3), we have $\alpha^M = 1$, $\mu^l = 0.001$, $\mu^M = 0.002$, $\beta^l = \beta^M = 0.001$, $k_1 = k_2 = 1$, $\gamma^l = \gamma^M = 0.001$, $\sigma^l = 0$ and $\sigma^M = 0.009$. By a simple computation, we can easily verify that (P_1)-(P_3) of Theorem 4.1 are satisfied. By Theorem 4.1, system (5.2) admits at least two positive almost periodic solutions(see Figure 2).

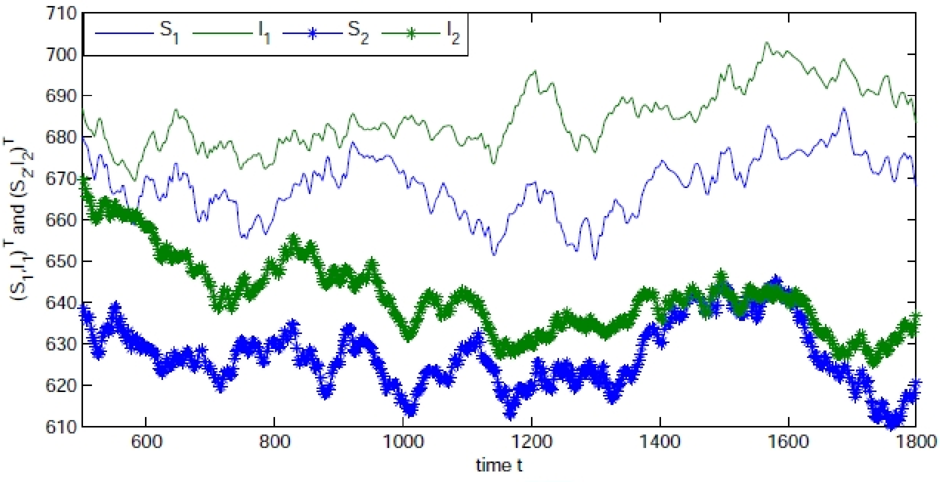

Figure 2 Two positive almost periodic solutions $(S_1, I_1)^T$ and $(S_2, I_2)^T$ of system (5.2)

6 Discussion

In this paper we have obtained the existence, multiplicity and stability of positive almost periodic solution for a non-autonomous SIR model with almost periodic transmission rate and a constant removal rate. The approach is based on the continuation theorem of coincidence degree theory. And Lemma 4.1 in Section 4 is critical to study the multiplicity of positive almost periodic solution of the model. It is important to notice that the approach used in this paper can be extended to other types of epidemics model such as SEIR; SIRS and other similar models of first order [24]. Future work will include models based on impulsive differential equations and biological dynamic systems on time scales [25].

References

[1] Z.G. Bai and Y.C. Zhou, Existence of two periodic solutions for a non-autonomous SIR epidemic model, *Appl. Math. Modelling* **35** (2011) 382-391.

[2] O. Diekmann, J.A.P. Heesterbeek and J.A.J. Metz, On the definition and computation of the basic reproductive ratio in models for infectious diseases in heterogeneous population, *J. Math. Biol.* **28** (1990) 365-382.

[3] A.M. Fink, *Almost Periodic Differential Equations*, Springer, Berlin, 1974.

[4] R.E. Gaines and J.L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Springer, Berlin, 1977.

[5] H.W. Hethcote, The mathematics of infectious diseases, *Siam Review* **42** (2000) 599-653.

[6] C.Y. He, *Almost Periodic Differential Equations*, Higher Education Publishing House, Beijing, 1992 (Chinese).

[7] S. Iwami and T. Hara, Global stability of a generalized epidemic model, *J. Math. Anal. Appl.* **362** (2010) 286-300.

[8] W.O. Kermack and A.G. McKendrick, Contribution to the mathematical theory of epidemics, *Proc. Roy. Soc. A* **115** (1927) 700-721.

[9] S.Y. Liu, Y.Z. Pei, C.G. Li and L.S. Chen, Three kinds of TVS in a SIR epidemic model with saturated infectious force and vertical transmission, *Appl. Math. Modelling* **33** (2009) 1923-1932.

[10] J.L. Liu and T.L. Zhang, Analysis of a nonautonomous epidemic model with density dependent birth rate, *Appl. Math. Modelling* **34** (2010) 866-877.

[11] X.G. Niu, T.L. Zhang and Z.D. Teng, The asymptotic behavior of a nonautonomous eco-epidemic model with disease in the prey, *Appl. Math. Modelling* **35** (2011) 457-470.

[12] Y.Z. Pei, S.P. Li, C.G. Li and S.Z. Chen, The effect of constant and pulse vaccination on an SIR epidemic model with infectious period, *Appl. Math. Modelling* **35** (2011) 3866-3878.

[13] L. Stone, B. Shulgin and Z. Agur, Theoretical examination of the pulse vaccination policy in the SIR epidemic model, *Math. Comput. Modelling* **31** (2000) 207-215.

[14] R.W. West and J.R. Thompson, Models for the simple epidemic, *Math. Biosci.* **141** (1997) 29-39.

[15] Y. Xie, X.G. Li, Almost periodic solutions of single population model with hereditary effects, *Appl. Math. Comput.* **203** (2008) 690-697.

[16] R. Xu and Z.E. Ma, Global stability of a SIR epidemic model with nonlinear incidence rate and time delay, *Nonlinear Anal.: RWA* **10** (2009) 3175-3189.

[17] Y.P. Yang and Y.N. Xiao, Threshold dynamics for compartmental epidemic models with impulses, *Nonlinear Anal.: RWA* **13** (2012) 224-234.

[18] T.L. Zhang and Z.D. Teng, Permanence and extinction for a nonautonomous SIRS epidemic model with time delay, *Appl. Math. Modelling* **33** (2009) 1058-1071.

[19] L.H. Zhou and M. Fan, Dynamics of an SIR epidemic model with limited medical resources revisited, *Nonlinear Anal.: RWA* **13** (2012) 312-324.

[20] T.W. Zhang, Multiplicity of positive almost periodic solutions in a delayed Hassell-Varley-type predator-prey model with harvesting on prey, *Math. Meth. Appl. Sci.* (2013): in press.

[21] T.W. Zhang and Y.K. Li, Anti-periodic solutions for a class of fourth-order nonlinear differential equations with variable coefficients, *Electron. J. Qual. Theory Differ. Equ.* **12** (2011) 1-10.

[22] T.W. Zhang and Y.K. Li, Positive periodic solutions for a generalized impulsive n-species Gilpin-Ayala competition system with continuously distributed delays on time scales, *Int. J. Biomath.* **4** (2011) 23-34.

[23] M. Hu, L.L. Wang, Dynamic inequalities on time scales with applications in permanence of predator-prey system, *Discrete Dyn. Nat. Soc.* **2012**, Article ID 281052 (2012).

[24] G. Zaman, Qualitative behavior of giving up smoking models, *Bull. Malays. Math. Sci. Soc.* **34** (2011) 403-415.

[25] M. Hu, P.L. Xie, L.L. Wang, Almost periodic solutions of neutral delay functional differential equations on time scales, *Bull. Malays. Math. Sci. Soc.* (2), accepted.