ON FREE QUADRATIC MODULES OF COMMUTATIVE ALGEBRAS

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Abstract

In this paper, we give a construction of (totally) free quadratic modules of commutative algebras on suitable construction data in terms of free simplicial algebras. Similar freeness results are also explored for reduced quadratic modules and quadratic (chain) complexes of algebras.

1 Introduction

Baues, [11], defined the notion of quadratic module of groups as an algebraic model for homotopy (connected) 3-types and gave a relation between quadratic modules and simplicial groups. This model is also a 2-crossed module structure defined by Conduché (cf. [12]) with additional nilpotent conditions.

In the series of articles [4, 5, 6], Arvasi and Porter have started to see how a study of the links between simplicial commutative algebras and classical constructions of homological algebra can be strengthened by interposing crossed algebraic models for the homotopy types of simplicial algebras. In particular, in [6], they gave a construction of a free 2-crossed module of algebras in terms of the 2-skeleton of a free simplicial algebra. In the group case, a closely related structure to that of 2-crossed module is that of a quadratic module (cf. [7]). Furthermore, in [9], a construction of the (totally) free quadratic module of groups was given. It seems clear that the results of papers [6], [7] and [9] can be extended to an algebra version of quadratic modules. In this work, we continue this process using these methods to give a construction of the (totally) free quadratic module of commutative algebras in terms of the 2-skeleton of a free simplicial algebra.

In [10], an alternative description of the top algebra of the free crossed squares of algebras was given on 2-construction data in terms of tensor product and coproduct of crossed modules of commutative algebras and a link between free crossed squares and Koszul complexes was described. By using the close relationship between crossed squares and quadratic modules, we shall give an alternative construction of the totally free quadratic module. In this context, our result has an advantage; Baues's theory is also related via a quotient functor **q** :(free crossed squares)→ (free quadratic modules) for the case of commutative algebras in terms of the 2-skeleton of a free simplicial algebra. We also interest on reduced quadratic modules of commutative algebras which are special kind of quadratic modules of algebras, describing the 3-types of simply connected CW-complexes constructed with algebras of nilpotency degree 2. In [20], Muro defined a suspension functor from the category of crossed modules of groups to that of reduced quadratic modules of groups which sends a 2-type to the 3-type of its suspension. By using this association, we give a construction of a totally free reduced quadratic module on commutative algebras.

2 (Free) Simplicial Algebras

In what follows 'algebras' will be commutative algebras over an unspecified commutative ring, **k**, but for convenience are not required to have a multiplicative identity. The category of commutative algebras will be denoted by Alg.

A simplicial algebra E consists of a family of algebras $\{E_n\}$ together with face and degeneracy maps $d_i = d_i^n : E_n \to E_{n-1}$, $0 \leqslant i \leqslant n$, $(n \neq 0)$ and $s_i = s_i^n : E_n \to E_{n+1}$, $0 \leqslant i \leqslant n$, satisfying the usual simplicial identities given in [1], [2] or [17]. In fact, it can be completely described as a functor $\mathsf{E} : \Delta^{op} \to \mathsf{Alg}$ where Δ is the category of finite ordinals $[n] = \{0 < 1 < \cdots < n\}$ and increasing maps. We denote the category of simplicial algebras by $\mathsf{SimpAlg}$.

Given a simplicial algebra E, the Moore complex (NE, ∂) of E is the chain complex defined by

$$NE_n = \ker d_0^n \cap \ker d_1^n \cap \cdots \cap \ker d_{n-1}^n$$

with $\partial_n : NE_n \to NE_{n-1}$ induced from d_n^n by restriction.

The nth *homotopy module* $\pi_n(\mathsf{E})$ of E is the nth homology of the Moore complex of E, i.e.

$$\pi_n(\mathsf{E}) \cong H_n(\mathsf{NE}, \partial) = \bigcap_{i=0}^{n} \ker d_i^n / d_{n+1}^{n+1}\left(\bigcap_{i=0}^{n} \ker d_i^{n+1}\right).$$

We say that the Moore complex NE of a simplicial algebra is of length k if $NE_n = 0$ for all $n \geqslant k+1$.

André [1] used simplicial methods to investigate homological propositionerties of commutative algebras and introduced 'step-by-step' construction of a resolution of a commutative algebra. The reader is referred to the book of André [1] and to the article of Arvasi and Porter [4] and the references there.

We recall briefly the 'step-by-step' construction of a free simplicial algebra from the article [4]. We denote by $\{m, n\}$ the set of increasing surjective maps $[m] \to [n]$ as given in [1], [4] and [19], where $[n] = \{0 < 1 \cdots < n\}$ is an ordered set.

Let E be a simplicial algebra and $k \geqslant 1$ be fixed. Suppose we are given a set Ω of elements $\Omega = \{x_\lambda : \lambda \in \Lambda\}$, $x_\lambda \in \pi_{k-1}(\mathsf{E})$; then we can choose a corresponding set of elements $w_\lambda \in NE_{k-1}$ so that $x_\lambda = w_\lambda + \partial_k(NE_k)$. We want to 'kill' the elements in Ω. It is formed a new simplicial algebra F where F_n is a free E_n-algebra, $F_n = E_n[y_{\lambda,t}]$ with $\lambda \in \Lambda$ and $t \in \{n, k\}$, and for $0 \leqslant i \leqslant n$, the algebra homomorphisms $s_i^n : F_n \to F_{n+1}$ and $d_i^n : F_n \to F_{n-1}$ are obtained from the homomorphism $s_i^n : E_n \to E_{n+1}$ and $d_i^n : E_n \to E_{n-1}$ respectively together with the relations given in [4].

Thus from the 'step-by-step' construction we can give the definition of a free simplicial algebra as follows.

Let E be a simplicial algebra and $k \geqslant 1$, k-skeletal be fixed. A simplicial algebra F is called a free if

(i) $F_n = E_n$ for $n < k$,

(ii) $F_k =$ a free E_k-algebra over a set of non- degenerate indeterminates, all of whose faces are zero except the kth,

(iii) F_n is a free E_n-algebra over the degenerate elements for $n > k$.

A variant of the 'step-by-step' construction gives: if A is a simplicial algebra, then there exists a free simplicial algebra E and an epimorphism $\mathsf{E} \to \mathsf{A}$ which induces isomorphisms on all homotopy modules. The details are omitted as they are well-known.

3 Quadratic Modules of Algebras

Baues in [11] defined the quadratic module of groups as an algebraic model of connected 3-types. In [8, Section 5], the second author with Arvasi defined the notion of quadratic module for commutative algebras and constructed a functor from 2-crossed modules to quadratic modules of algebras.

First we give some basic definitions about the structure.

3.1 Nil(n)-Modules for Algebras

For an algebra C, C/C^2 is the quotient of the algebra C by its ideal of squares. Then, there is a functor from the category of k-algebras to the category of k-modules. This functor goes from C to C/C^2, plays the role of abelianization in the category of k-algebras. As modules are often called singular algebras (e.g., in the theory of singular extensions) we shall call this functor "singularisation".

A *pre-crossed module* of algebras is a homomorphism of algebras $\partial : C \to R$ together with an action of R on C written $c \cdot r$ for $r \in R$ and $c \in C$ satisfying the condition $\partial(c \cdot r) = \partial(c)r$ for all $r \in R$ and $c \in C$. Let $\partial : C \to R$ be a pre-crossed module, $P_1(\partial) = C$, let $P_2(\partial)$ be the Peiffer ideal of C generated by elements of the form

$$\langle x, y\rangle = xy - x \cdot \partial y$$

which is called the *Peiffer element* for $x, y \in C$.

The pre-crossed modules in which all Peiffer elements are trivial, precisely crossed modules. Namely, a crossed module (cf. [21, 22]) is a pre-crossed module $\partial : C \to R$ satisfying the extra condition

$$x \cdot \partial y = xy \text{ for all } x, y \in C.$$

We will denote the category of crossed modules by XMod. Let $\partial : C \to R$ be a pre-crossed module. Then, the homomorphism

$$\partial^{cr} : C^{cr} = C/P_2(\partial) \longrightarrow R$$

defined by $\partial^{cr}(xP_2) = \partial(x)$ is the crossed module associated to the pre-crossed module $\partial : C \to R$. Associating the crossed module ∂^{cr} to a pre-crossed module ∂ gives a functor

$$(-)^{cr} : \mathsf{PXMod} \to \mathsf{XMod}$$

from the category of pre-crossed modules to that of crossed modules. That is, a morphism (g, f) of pre-crossed modules yields a morphism (g^{cr}, f) of crossed modules, and this association satisfies the usual functorial rules.

For any pre-crossed module $\partial : C \to R$, let $P_n(\partial)$ be the ideal of C generated by Peiffer elements of weight n. We say that ∂ is a *nil(n)-module* of algebras, if $P_{n+1}(\partial) = 0$. Thus, by a similar reasoning the related nil(2)-module to the pre-crossed module ∂ is $\partial^{nil} : C^{nil} = C/P_3(\partial) \to R$ where ∂^{nil} is given by $\partial^{nil}(xP_3) = \partial(x)$ for all $x \in C$. The construction of the nil(2)-module ∂^{nil} from the pre-crossed module of algebras ∂ gives a functor from the category of pre-crossed modules to that of nil(2)-modules:

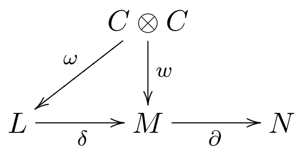
$$(-)^{nil} : \mathsf{PXMod} \to \mathsf{Nil}(2).$$

Similarly the related nil(n)-module to a pre-crossed module can be defined. This gives a functor $\mathsf{PXMod} \to \mathsf{Nil}(n)$ from the category of pre-crossed modules to that of nil(n)-modules of algebras.

3.2 The Definition of Quadratic Modules of Algebras

We can give the definition of a quadratic module of algebras from [8].

Definition 3.1 *A quadratic module $(\omega, \delta, \partial)$ of algebras is a diagram*



of homomorphisms of algebras such that the following axioms are satisfied.

QM1)- *The homomorphism $\partial : M \to N$ is a nil(2)-module and the quotient map $M \twoheadrightarrow C = M^{cr}/(M^{cr})^2$ is given by $x \mapsto [x]$, where $[x] \in C$ denotes the class represented by $x \in M$. The map w is defined by Peiffer multiplication, i.e., $w([x] \otimes [y]) = xy - x \cdot \partial(y)$.*

QM2)- *The homomorphisms δ and ∂ satisfy $\delta\partial = 0$ and the quadratic map ω is a lift of the map w, that is $\delta\omega = w$ or equivalently*

$$\delta\omega([x] \otimes [y]) = w([x] \otimes [y]) = xy - x \cdot \partial(y)$$

for $x, y \in M$.

QM3)- *L is a N-algebra and all homomorphisms of the diagram are equivariant with respect to the action of N. Moreover, the action of N on L satisfies the following equality*

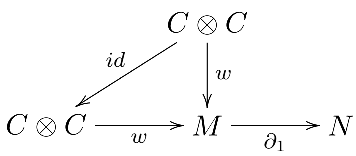
$$a \cdot \partial(x) = \omega([\delta a] \otimes [x] + [x] \otimes [\delta a])$$

for $a \in L, x \in N$.

QM4)- *For $a, b \in L$,*

$$\omega([\delta a] \otimes [\delta b]) = ab.$$

Example 3.2 *Any nil(2)-module $\partial_1 : M \to N$ gives a quadratic module. Since (M, N, ∂_1) is a nil(2)-module, if we add $L = C \otimes C$ and the resulting diagram*



with the obvious actions is a quadratic module.

4 Free Quadratic Modules of Algebras

In this section, we will define the notion of a totally free quadratic module of algebras and we will construct it by using 2-skeleton of a free simplicial algebra.

Definition 4.1 *Let*

$$\begin{array}{ccccc} & & C\otimes C & & \\ & \overset{\omega}{\swarrow} & \downarrow w & & \\ C_2 & \xrightarrow[\partial_2]{} & C_1 & \xrightarrow[\partial_1]{} & C_0. \end{array}$$

be a quadratic module and let $\vartheta : Y \to C_2$ *be the function. Then* $(\omega, \partial_2, \partial_1)$ *is said to be free quadratic module with basis* ϑ*, or alternatively on the function* $\partial_2\vartheta : Y \to C_1$*, if for any quadratic module* $(\omega', \partial_2', \partial_1)$ *and* $\vartheta' : Y' \to C_2'$ *such that* $\partial_2'\vartheta' = \partial_2\vartheta$*, there is a unique morphism* $\Phi : C_2 \to C_2'$ *such that* $\partial_2'\Phi = \partial_2$*. We say that a free quadratic module* $(\omega, \partial_2, \partial_1)$ *is a totally free quadratic module if* ∂_1 *is a free nil*(2)*-module.*

Now, we construct the totally free quadratic module of algebras. To give it, we need to recall the 2-skeleton of a free simplicial algebra given by

$$\mathsf{E}^{(2)} : \cdots R[s_0X, s_1X][Y] \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow}\; R[X] \;\substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow}\; R$$

with the simplicial structure defined as in Section 3 of [4].

Analysis of this 2-dimensional construction data shows that it consists of some 1-dimensional data, namely the function $\vartheta : X \to R$, that is used to induce $d_1 : R[X] \to R$, together with strictly 2-dimensional construction data consisting of the function $\psi : Y \to R^+[X]$ and this function is used to induce $d_2 : R[s_0X, s_1X][Y] \to R[X]$. We will denote this 2-dimensional construction data by (ϑ, ψ, R).

Theorem 4.2 *A totally free quadratic module of algebras exists on the* 2*-dimensional construction data* (ϑ, ψ, R)*.*

Proof: Suppose that given a 2-dimensional construction data (ϑ, ψ, R), i.e. given a function $\vartheta : X \to R$ and $\psi : Y \to R^+[X]$, with the obvious action of R on $R^+[X]$, we have a free pre-crossed module $\partial : R^+[X] \to R$ with basis $\vartheta : X \to R$. Here $R^+[X] = NE_1^{(1)} = \ker d_0$ is the positively degree part of $R[X]$. Now let P_3 be the ideal of $R^+[X]$ generated by the Peiffer elements of length 3 in $R^+[X]$. We can define the quotient morphism $q_1 : R^+[X] \to R^+[X]/P_3 = M$ and then, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & R^+[X] \\ {\scriptstyle\vartheta}\downarrow & & \downarrow{\scriptstyle q_1} \\ R & \xleftarrow[\partial_1]{} & R^+[X]/P_3 \end{array}$$

where i is the inclusion.

Since $\partial : R^{+}[X] \to R$ is a pre-crossed module we have $\partial(P_3) = 0$ and then the map $\partial_1 : R^{+}[X]/P_3 \to R$ given by $x + P_3 \mapsto \partial(x)$ for $x \in R^{+}[X]$ is a well-defined homomorphism. Therefore we have a free nil(2)-module $\partial_1 : M = R^{+}[X]/P_3 \to R$ with basis $\vartheta : X \to R$, or on the function $\partial_1 q_1 i$, since the triple Peiffer elements are trivial in M.

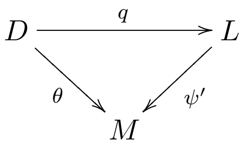
Now, take $D = NE_2^{(2)} = R[s_0X]^{+}[s_1X, Y] \cap ((s_0 - s_1)(X))$. Then, as 2-dimensional construction data, the function

$$q_1\psi : Y \to M = R^{+}[X]/P_3$$

induces a morphism of algebras

$$\theta : R[s_0X]^{+}[s_1X, Y] \cap ((s_0 - s_1)(X)) \to R^{+}[X]/P_3$$

given by $\theta(y) = q_1\psi(y)$. Let $P' = \partial_3(NE_3^{(2)}) \subset D$ be the Peiffer ideal. Then, $q_1\psi(P') = 0$ as all generator elements of P' in $\ker d_2$. By taking the factor algebra $L = D/P'$ then there is a morphism $\psi' : L \to M$ such that the diagram



commutes, that is, $\psi' q = q_1\psi = \theta$. The Peiffer elements in $R^{+}[X]$ are given by

$$\langle X_i, X_j \rangle = X_iX_j - \partial(X_i)X_j$$

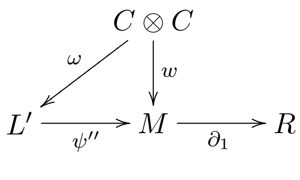
for $X_i, X_j \in R^{+}[X]$. Let P_3' be the ideal of L generated by elements of the form

$$s_1(\langle X_i, X_j \rangle)s_1(X_k) - s_1(\langle X_i, X_j \rangle)s_0(X_k)$$

and

$$s_1(X_i)s_1(\langle X_j, X_k \rangle) - s_1(X_i)s_0(\langle X_j, X_k \rangle)$$

for all $X_i, X_j, X_k \in R^{+}[X]$. Let $L' = L/P_3'$ and let $q_2 : L \to L' = L/P_3'$ be the quotient morphism. We have $\psi(P_3') = P_3$. Hence, we define a morphism ψ'' from L' to M such that $\psi'' q_2 = q_1\psi$. Then, the diagram



becomes a totally free quadratic module of algebras, where $C = M^{cr}/(M^{cr})^2$ and the quadratic map ω can be given by

$$\omega(\{q_1X_i\} \otimes \{q_1X_j\}) \;=\; q_2(s_1X_i(s_1X_j - s_0X_j) + P')$$

for $X_i \in R^+[X]$, $q_1(X_i) \in M = R^+[X]/P_3$ and $\{q_1 X_i\} \in C = M^{cr}/(M^{cr})^2$.

Let

$$C \otimes C \xrightarrow{\omega'} A \xrightarrow{\partial_2'} M \xrightarrow{\partial_1} R$$

be any quadratic module of algebras and let $\vartheta' : Y \to A$ be the function. Then there exists a unique morphism $\Phi : L' \to A$ given by $\Phi(q_2(y + P')) = \vartheta'(y)$ such that $\partial_2' \Phi = \psi''$. Thus $(\omega, \psi'', \partial_1)$ is the required totally free quadratic module with basis $q_1\psi : Y \to M$. □

4.1 From Free Crossed Squares to Free Quadratic Modules

Crossed squares were initially defined by Guin-Waléry and Loday in [16]. The commutative algebra analogue of crossed squares has been studied by Ellis (cf. [13]).

A *crossed square* of algebras is a commutative diagram of morphisms of algebras

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \mu} \\ N & \xrightarrow[\nu]{} & R \end{array}$$

together with the actions of R on L, M and N and a function

$$h : M \times N \to L$$

satisfying a number of axioms which we do not give in full here.

Ellis, [14], presented the notion of a free crossed square for the case of groups, and in the context of CW-complexes gave a neat description of the top term L in a free crossed square of groups. He also constructed a free quadratic module of groups from a free crossed square by using the quotient functor $\mathbf{q}$. In [10], a construction of the top algebra L in a free crossed square of algebras was given on the 2-dimensional construction data in terms of tensor and coproduct in the category of crossed modules of algebras.

Now, we recall from [10] the definition of a free crossed square of algebras on a pair of functions (f_2, f_3):

Let $\mathsf{S}_1, \mathsf{S}_2$ and S_3 be sets and $f_2 : \mathsf{S}_2 \to R$ is a function from the set S_2 to the free algebra R on S_1. Let $\partial : M \to R$ be the free pre-crossed module on the function f_2. Consider the semi-direct product $M \rtimes R$ and the inclusion $\mu : M \to M \rtimes R$ and the ideals M and $\overline{M}$ of $M \rtimes R$ where $\overline{M} = \{(m, -\partial m) : m \in M\}$ with the inclusion $\mu' : \overline{M} \to M \rtimes R$. Assume given a function $f_3 : \mathsf{S}_3 \to M$, which is to satisfy $\partial f_3 = 0$. Then there is a corresponding function $\overline{f_3} : \mathsf{S}_3 \to \overline{M}$ given by $y \mapsto (f_3(y), 0)$.

We say a crossed square

$$\begin{array}{ccc} L & \xrightarrow{\partial_2'} & \overline{M} \\ {\scriptstyle \partial_2}\downarrow & & \downarrow{\scriptstyle \mu'} \\ M & \xrightarrow[\mu]{} & M \rtimes R \end{array}$$

is *totally free* on the pair of functions (f_2, f_3) if

(i) (M, R, ∂) is the free pre-crossed module on f_2,

(ii) S_3 is a subset of L with f_3 and $\overline{f_3}$ the restrictions of ∂_2 and ∂_2' respectively,

(iii) given any crossed square $(L', M, \overline{M}, M \rtimes R)$ and function $\nu : \mathsf{S}_3 \to L'$ there is a unique morphism $\phi : L \to L'$ such that $\phi\nu' = \nu$ where $\nu' : S_3 \to L$ is the inclusion.

Proposition 4.3 ([10]) Let

$$\begin{array}{ccc} L & \longrightarrow & \overline{M} \\ \downarrow & & \downarrow \\ M & \longrightarrow & M \rtimes R \end{array}$$

be the free crossed square of algebras. Let $\partial : C \to M \rtimes R$ be the free crossed module on $\mathsf{S}_3 \to M \rtimes R$. Form the crossed module $M \otimes \overline{M} \to M \rtimes R$. Then

$$L \cong \{(M \otimes \overline{M}) \sqcup C\}/\sim$$

where $\sim$ corresponds to the relations

(i) $i_{M \otimes \overline{M}}(\partial c \otimes \overline{n}) \sim j(c) - j(\overline{n} \cdot c)$,

(ii) $i_{M \otimes \overline{M}}(m \otimes \partial c) \sim j(m \cdot c) - j(c)$

for $c \in C, m \in M$ and $\overline{n} \in \overline{M}$.

The homomorphisms $L \to M$ and $L \to \overline{M}$ are given by the homomorphisms $\lambda : M \otimes \overline{M} \to M$, $(m \otimes \overline{n}) \mapsto m\overline{n} - \partial(\overline{n})m$ and $\lambda' : M \otimes \overline{M} \to \overline{M}$, $(m \otimes \overline{n}) \mapsto m\overline{n} - \partial(m)\overline{n}$ and $\partial : C \to M \cap \overline{M}$. The h-map is given by $h(m, \overline{n}) = i(m \otimes \overline{n})$.

Ellis [14] gave a construction of a free quadratic module of groups from a free crossed square. Now, we adapt the Ellis's construction of a free quadratic module of groups to the algebra context:

Suppose that

$$\begin{array}{ccc} L & \xrightarrow{\lambda'} & \overline{M} \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \mu'} \\ M & \xrightarrow[\mu]{} & M \rtimes R \end{array}$$

is a free crossed square of algebras. We denote by $h(M, M, M)$ the ideal of L generated by elements of the form $h(m, \overline{\langle m', m'' \rangle})$ and $h(\langle m, m' \rangle, \overline{m''})$ for $m, m', m'' \in M$. (We are using the notation $\overline{m} = (-m, \delta m)$ for $m \in M$.) Consider the quotient algebra $L/h(M,M,M)$ and the quotient morphism $q_2 : L \to L/h(M,M,M)$.

Let $\langle M,M,M\rangle$ be the ideal of M generated by elements of the form $\langle\langle m,m'\rangle,m''\rangle$ or $\langle m,\langle m',m''\rangle\rangle$ for $m,m',m'' \in M$. Consider the quotient algebra $M/\langle M,M,M\rangle$ and the quotient morphism $q_1 : M \to M/\langle M,M,M\rangle$.

From the h-map axioms, we have $\lambda h(M,M,M) = \langle M,M,M\rangle$ and then the map $\overline{\lambda}$: $L/h(M,M,M) \to M/\langle M,M,M\rangle$ given by $\overline{\lambda}(l + h(M,M,M)) = \lambda(l) + \langle M,M,M\rangle$ is a well-defined homomorphism. Similarly, since $\partial : M \to R$ is a pre-crossed module, we have $\partial(\langle M,M,M\rangle) = 0$ and then the map $\partial' : M/\langle M,M,M\rangle \to R$ given by $\partial'(m + \langle M,M,M\rangle) = \partial(m)$ is well defined. Let $M' = M/\langle M,M,M\rangle$ and $C = (M'^{cr}/((M'^{cr})^2$. Then

$$\begin{array}{ccccc} & & C\otimes C & & \\ & \swarrow^{\omega} & \downarrow^{w} & & \\ \sigma : \quad \dfrac{L}{h(M,M,M)} & \xrightarrow{\overline{\lambda}} & \dfrac{M}{\langle M,M,M\rangle} & \xrightarrow{\partial'} & R \end{array}$$

is a free quadratic module of algebras. The h-map $h : M \times \overline{M} \to L$ induces the quadratic map $\omega : C \otimes C \to L/h(M,M,M)$.

5 Free Reduced Quadratic Modules

Definition 5.1 *A reduced quadratic module (ω, δ) is a following diagram,*

$$C \otimes C \xrightarrow{\omega} C_2 \xrightarrow{\delta} C_1$$

of algebras such that the following axioms are satisfied.

1. *The algebra C_1 is a nil(2)-algebra and $C = C_1/(C_1)^2$ is the singularization of algebra C_1. The quotient map $q : C_1 \to C$ is given by $x \longrightarrow [x]$.*
2. *For $x, y \in C_1$, $\delta\omega([x] \otimes [y]) = xy$.*
3. *For $a \in C_2, x \in C_1$, $\omega([\delta a] \otimes [x] + [x] \otimes [\delta a]) = 0$.*
4. *For $a, b \in C_2$, $\omega([\delta a] \otimes [\delta b]) = ab$.*

We denote the category of reduced quadratic modules by RQM.

5.1 From Crossed Modules to Reduced Quadratic Modules of Algebras

Muro, [20], gave suspension functor by using central push-out from crossed modules to reduced quadratic modules over groups, and showed that this functor preserves the free crossed modules of groups.

Let I have exactly three elements, $I = \{1, 2, 3\}$, with quasi-order $1 < 2$ and $1 < 3$. A direct system with index set I can be given by the diagram

$$\begin{array}{ccc} F_1 & \longrightarrow & F_3 \\ \big\downarrow & & \\ F_2 & & \end{array}$$

Then for this direct system, the following diagram is a push-out:

$$\begin{array}{ccc} F_1 & \xrightarrow{\varphi_3^1} & F_3 \\ {\scriptstyle \varphi_2^1}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ F_2 & \xrightarrow[g]{} & D \end{array}$$

where $D = (F_2 \oplus F_3)/W$, $W = \{(\varphi_2^1(a), -\varphi_3^1(a)) : a \in F_1\}$, $f : b \longmapsto (0, b) + W$ and $g : c \longmapsto (c, 0) + W$.

Let $\partial : L \to M$ be a crossed module of algebras. We take $F_1 = L \otimes M$, $F_2 = L$ and $F_3 = \left(M/M^2 \otimes M/M^2\right)/K$. We can define

$$\begin{array}{rccc} \varphi_2^1 : & F_1 & \longrightarrow & F_2 \\ & (l \otimes m) & \longmapsto & l \cdot m. \end{array}$$

This morphism satisfies the following; $\varphi_2^1(id \otimes \partial)(l \otimes l') = \varphi_2^1(l \otimes \partial l') = l \cdot \partial l' = ll' = w'([l] \otimes [l'])$ and $\partial\varphi_2^1(l \otimes m) = \partial(l \cdot m) = (\partial l)m = w'(\partial \otimes id)(l \otimes m)$. Thus $\partial\varphi_2^1 = w'(\partial \otimes id) : L \otimes M \to M$. We can define the morphism

$$\varphi_3^1 : F_1 = L \otimes M \to (M/M^2 \otimes M/M^2)/K = F_3$$

by composition of the following maps

$$L \otimes M \xrightarrow{q \otimes q} L/L^2 \otimes M/M^2 \xrightarrow{\partial^2 \otimes id} (M/M^2 \otimes M/M^2)/K,$$

where $q : M \to M/M^2$ is the quotient map and K is the image of

$$\partial^2 \otimes id + id \otimes \partial^2 : L/L^2 \otimes M/M^2 \longrightarrow M/M^2 \otimes M/M^2.$$

That is, $\varphi_3^1(l \otimes m) = \partial^2 q(l) \otimes q(m) + K$. Therefore, we have the following push-out diagram

$$\begin{array}{ccc} L \otimes M & \xrightarrow{\varphi_3^1} & (M/M^2 \otimes M/M^2)/K \\ {\scriptstyle \varphi_2^1}\downarrow & & \downarrow{\scriptstyle \omega} \\ L & \xrightarrow[r]{} & L^{\Sigma} \end{array}$$

in which

$$L^{\Sigma} = \frac{L \times (M/M^2 \otimes M/M^2)/K}{W},$$

and

$$W = \{(l \cdot m, \partial^2 q(l) \otimes qm + K) : l \in L, m \in M\}.$$

There is a morphism w given by

$$\begin{array}{rcl} w : (M/M^2 \otimes M/M^2)/K & \longrightarrow & M^{nil} \\ (qm \otimes qm') + K & \longmapsto & mm'. \end{array}$$

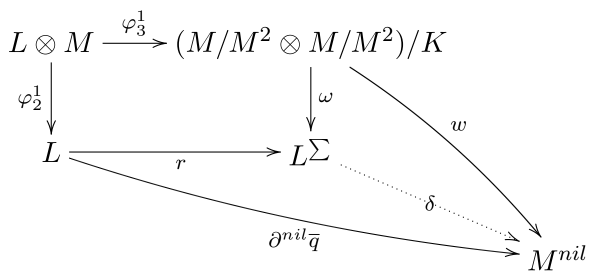
Furthermore, there is also a morphism

$$L \xrightarrow{\partial^{nil}\overline{q}} M^{nil}$$

given by composition of the following maps

$$L \xrightarrow{\overline{q}} L^{nil} \xrightarrow{\partial^{nil}} M^{nil}$$

where, $\overline{q} : L \to L^{nil}$ is the quotient map. Obviously, according to above descriptions, we obtain $w(\partial^2 \otimes id)(q \otimes q) = \partial^{nil}\overline{q}\varphi_2^1$. Thus, we have the following diagram



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There is a unique morphism $\delta : L^{\Sigma} \to M^{nil}$ satisfying $\delta\omega = w$ and $\delta r = \partial^{nil}\overline{q}$.

5.2 From Free Crossed Modules to Free Reduced Quadratic Modules

Definition 5.2 *Let*

$$C \otimes C \xrightarrow{\omega} L \xrightarrow{\delta} M$$

be a reduced quadratic module, let Y be a set and let $\nu : Y \to L$ be a function and M is free nil(2)-algebra, then this reduced quadratic module is called the totally free reduced quadratic *module with basis $\nu : Y \to L$, or alternatively on the function $\delta\nu : Y \to M$, if for any reduced quadratic module $(L', M, \delta', \omega')$ and a function $\nu' : Y \to L'$ such that $\delta'\nu' = \delta\nu$, there is a unique morphism $\Phi : L \to L'$ such that $\Phi\nu = \nu'$.*

Recall from [4] that given a presentation $P = (R; x_1, ..., x_n)$ of an R-algebra B and $\mathsf{E}^{(1)}$ the 1-skeleton of the free simplicial algebra generated by this presentation, then

$$\delta : NE_1^{(1)}/\partial_2(NE_2^{(1)}) \to NE_0^{(1)}$$

is the *free crossed module* on $(x_1, ..., x_n) \to R$.

Now we give a totally free *reduced quadratic module* by using the suspension functor. Let R be a free algebra and let Y be a set and $f : Y \to R$ be a function with codomain R. Let $E = R^+[Y]$, the positively graded part of the polynomial ring on Y so that R acts on E by multiplication. The function f induces a morphism of R-algebras $\theta : R^+[Y] \to R$ given by $\theta(y) = f(y)$. Let P_2 be Peiffer ideal of $R^+[Y]$, then take $C = R^+[Y]/P_2$. Thus from the 1-skeleton we obtain a totally free crossed module $C \to R$ (cf. [4]). Then, we have functions: $\varphi : C \otimes R \to C$ given by $\varphi(y \otimes r) = y \cdot r$ and $\varphi' : C \otimes R \to (R/R^2 \otimes R/R^2)/K$ given by $\varphi'(y \otimes r) = \theta^2 q_1(y) \otimes q_2(r) + K$, where $q_1 : C \to C/C^2$ and $q_2 : R \to R/R^2$ are the quotient maps and K is image of the function

$$\theta^2 \otimes id + id \otimes \theta^2 : C/C^2 \otimes R/R^2 \to R/R^2 \otimes R/R^2.$$

Thus the diagram

$$\begin{array}{ccc} C \otimes R & \xrightarrow{\theta^2(q_1 \otimes q_2)} & (R/R^2 \otimes R/R^2)/K \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\omega} \\ C & \xrightarrow[r]{} & C^{\Sigma} \end{array}$$

is a push-out, where

$$C^{\Sigma} = \frac{R^+[Y]/P_2 \times (R/R^2 \otimes R/R^2)/K}{W}$$

and

$$W = \{(y \cdot r, \theta^2(q_1 \otimes q_2)(y, r)) : y \in R^+[Y]/P_2, r \in R\}.$$

Thus, the diagram

$$\begin{array}{ccc} (R/R^2 \otimes R/R^2)/K & & \\ {\scriptstyle\omega}\downarrow & \searrow^{w} & \\ C^{\Sigma} & \xrightarrow[\delta]{} & R^{nil} \end{array}$$

is the required totally free reduced quadratic module on the function $f^{nil} : Y \to R^{nil}$, where $\delta = \theta^{nil}\overline{q}$ and $\overline{q} : C \to C^{nil}$ is the quotient map. Of course, this follows immediately from the description of the suspension functor.

6 Free Quadratic Chain Complexes Over Algebras

In this section we will give a functor from the category of simplicial algebras to that of quadratic chain complexes of algebras. We shall give a totally free quadratic chain complex of algebras from a free simplicial algebra by using this functor. First we give the definition of a quadratic chain complex of algebras analogously to that given by Baues [11] for the group case.

Definition 6.1 *A quadratic chain complex of algebras is a diagram of homomorphisms between algebras*

$$\begin{array}{ccccccccc} & & & & & C \otimes C & & & \\ & & & & \overset{\omega}{\swarrow} & \downarrow w & & & \\ \mathsf{C} : \cdots & \longrightarrow & C_3 & \xrightarrow[\partial_3]{} & C_2 & \xrightarrow[\partial_2]{} & C_1 & \xrightarrow[\partial_1]{} & C_0 \end{array}$$

in which
(i) C_0 acts on C_n for $n \geqslant 1$, the action of $\partial_1(C_1)$ being trivial on C_n for $n \geqslant 3$;
(ii) each ∂_n is a C_0-module homomorphism and $\partial_i \partial_{i+1} = 0$ for all $i \geqslant 1$;
(iii)

$$\begin{array}{ccccc} & & C \otimes C & & \\ & \overset{\omega}{\swarrow} & \downarrow w & & \\ C_2 & \xrightarrow[\partial_2]{} & C_1 & \xrightarrow[\partial_1]{} & C_0 \end{array}$$

is a quadratic module of algebras.

A quadratic chain map $f : \mathsf{C} \to \mathsf{C}'$ between quadratic chain complexes of algebras is a family of homomorphisms between algebras ($n \geqslant 0$), $f_n : C_n \to C'_n$ with $f_n d_{n+1} = d_{n+1} f_{n+1}$ such that (f_2, f_1, f_0) is a map between quadratic modules of algebras. We will denote the category of quadratic chain complexes of algebras by $\mathsf{Quadchain}$.

A quadratic chain complex of algebras C will be said to be *free* if for $n \geqslant 3$, C_n are free algebras and the quadratic module at the base is a free quadratic module of algebras. It will be *totally free* if in addition the base quadratic module is totally free quadratic module of algebras.

A crossed complex of algebras is a chain complex of algebras

$$\rho : \qquad \cdots \longrightarrow \rho_2 \xrightarrow{\partial_2} \rho_1 \xrightarrow{\partial_1} \rho_0$$

in which

(i) $\partial_1 : \rho_1 \to \rho_0$ is a crossed module of algebras, (*ii*) for $n > 1$, ρ_n is a ρ_0-module and $\partial_1(\rho_1)$ acts trivially, and each ∂_n is a ρ_0-module homomorphism,

(*iii*) for $n \geqslant 1$ $\partial_n\partial_{n+1} = 0$.

6.1 From Simplicial Algebras to Quadratic Chain Complexes

For a simplicial algebra E,let

$$\mathsf{C}^{(1)}(\mathsf{E})_n = \rho_n = \frac{NE_n}{(NE_n \cap D_n) + d_{n+1}(NE_{n+1} \cap D_{n+1})}.$$

This gives a crossed complex of algebras $\mathsf{C}^{(1)}(\mathsf{E})$ starting from the Moore complex $(\mathsf{NE}, \partial_\mathsf{n})$ of E. The map $\partial_n : \mathsf{C}^{(1)}(\mathsf{E})_n \to \mathsf{C}^{(1)}(\mathsf{E})_{n-1}$ will be that induced by d_n^n. If E is a free simplicial algebra, then $\mathsf{C}^{(1)}(\mathsf{E})$ is a free crossed complex of algebras.

Now, we give a quadratic chain complex of algebras from the Moore complex of a simplicial algebra. Given a simplicial algebra E with Moore complex NE, define $C_n = \mathsf{C}^{(2)}(\mathsf{E})_n$ by $C_0 = NE_0$, $C_1 = NE_1/P_3$, $C_2 = (NE_2/\partial_3(NE_3 \cap D_3))/P_3'$, and for $n \geqslant 3$, $C_n = NE_n/(NE_n \cap D_n) + d_{n+1}(NE_{n+1} \cap D_{n+1})$ with ∂_n induced by the differential of NE and $C = (C_1^{cr})/(C_1^{cr})^2$ and where P_3 is the ideal of NE_1 generated by triple brackets $\langle x, \langle y, z\rangle\rangle$ and $\langle\langle x, y\rangle, z\rangle$ and P_3' is the ideal of $NE_2/\partial_3(NE_3 \cap D_3)$ generated by elements of the form

$$s_1\langle x, y\rangle[s_1 z - s_0 z] + \partial_3(NE_3 \cap D_3)$$

and

$$s_1 x[s_1\langle y, z\rangle - s_0\langle y, z\rangle] + \partial_3(NE_3 \cap D_3)$$

for $x, y, z \in NE_1$.

Note that this structure for $n = 0, 1, 2$, forms a quadratic module and for $n \geqslant 3$ the algebras C_n are the same as the corresponding crossed complex term $\mathsf{C}^{(1)}(\mathsf{E})_n$ as defined above.

We thus have the following result.

Proposition 6.2 *For a simplicial algebra* E *with the above structure,* $\mathsf{C} = \mathsf{C}^{(2)}(\mathsf{E})$ *is a quadratic chain complex of algebras.*

Proof: The structure $(C_2, C_1, C_0, \omega, w)$ forms a quadratic module. The only thing remaining is to check that $\partial_2\partial_3$ is trivial which is straightforward. □

Thus, we can define a functor from the category of simplicial algebras to that of quadratic chain complexes of algebras. We denote it by

$$\mathsf{C}^{(2)} : \mathsf{SimpAlg} \to \mathsf{Quadchain}.$$

6.2 From Quadratic Chain Complexes to Crossed Complexes of Algebras

Baues [11] defined a functor from quadratic chain complexes to crossed complexes denoting it by λ. Denoting it λ we will give its algebra version. That is, we define a functor from the category of quadratic chain complexes to that of crossed chain complexes of algebras.

This functor carries a quadratic chain complex of algebras C to a crossed complex of algebras $\lambda(\mathsf{C}) = (\rho_n, d_n)_{n\geqslant 0}$. The extra structures are: $\rho_0 = \lambda(\mathsf{C})_0 = C_0$, $\rho_1 = \lambda(\mathsf{C})_1 = C_1/w(C \otimes C) = C_1^{cr}$, $\rho_2 = \lambda(\mathsf{C})_2 = C_2/(\omega(C \otimes C))$ and $\rho_n = C_n$ for $n \geqslant 3$. Thus we have the following complex of algebras

$$\lambda(\mathsf{C}) : \cdots \longrightarrow C_4 \longrightarrow C_3 \xrightarrow{d_3} \rho_2 \xrightarrow{d_2} \rho_1 \xrightarrow{d_1} C_0$$

in which $d_1 : \rho_1 \to C_0$ is a crossed module of algebras since the Peiffer elements are zero in the factor algebra ρ_1.

As a corollary, for a simplicial algebra E there is the following isomorphism

$$\lambda\mathsf{C}^{(2)}(\mathsf{E}) \cong \mathsf{C}^{(1)}(\mathsf{E}).$$

Proposition 6.3 *If* E *is a free simplicial algebra, then* $\mathsf{C}^{(2)}(\mathsf{E})$ *is a totally free quadratic chain complex.*

Proof: We have already seen in Theorem 4.2 that the base quadratic module of $\mathsf{C}^{(2)}(\mathsf{E})$ is totally free on 2-dimensional construction data. It remains to show that C_n for $n \geqslant 3$ are free on the corresponding data, but here we can use the case of crossed complexes, and it was proved in [4] that these terms in higher dimension are free algebras on the corresponding data. □

It is clearly seen that for a free simplicial algebra E, from the isomorphism $\lambda\mathsf{C}^{(2)}(\mathsf{E}) \cong \mathsf{C}^{(1)}(\mathsf{E})$, $\lambda\mathsf{C}^{(2)}(\mathsf{E})$ is a totally free crossed complex of algebras.

References

[1] M. André, Homologie des algèbres commutatives. *Die Grundlehren der Mathematischen Wissenchaften*, **206** *Springer-Verlag* (1970).

[2] M. Artin and B. Mazur, On the van Kampen theorem, *Topology*, **5**, 179-189, (1966).

[3] Z. Arvasi, Crossed squares and 2-crossed modules of commutative algebras, *Theory and Applications of Categories*, Vol. 3, No. **7**, pp 160-181, (1997).

[4] Z. Arvasi and T. Porter, Simplicial and crossed resolution of commutative algebras, *Journal of Algebra*, **181**, 426-448, (1996).

[5] Z. Arvasi and T. Porter, Higher dimensional Peiffer elements in simplicial commutative algebras, *Theory and Applications of Categories*, Vol. 3, No. **1**, pp 1-23, (1997).

[6] Z. Arvasi and T. Porter, Freeness conditions for 2-crossed module of commutative algebras, *Applied Categorical Structures*, **6**, pp 455-477, (1998).

[7] Z. Arvasi and E. Ulualan, On algebraic models for homotopy 3-types, *Journal of Homotopy and Related Structures* Vol.1, No **1**, pp.1-27, (2006).

[8] Z. Arvasi and E. Ulualan, Quadratic and 2-crossed modules of algebras, *Algebra Colloquium* 14 : 4 669-686 (2007).

[9] Z. Arvasi and E. Ulualan, Freeness conditions for quadratic modules and quadratic chain complexes, *Georgian Math. J.*, **18**, 615-637, (2011).

[10] Z. Arvasi and E. Ulualan, Homotopical aspects of commutative algebras I: freeness conditions for crossed squares, *Journal of Homotopy and Related Structures* (In print) Online first articles, (2014).

[11] H.J. Baues, Combinatorial homotopy and 4-dimensional complexes, *Walter de Gruyter*, **15**, 380 pages, (1991).

[12] D. Conduché, Modules croisés généralisés de longueur 2, *Journal of Pure and Applied Algebra*, **34**, pp 155-178, (1984).

[13] G.J. Ellis, Higher dimensional crossed modules of algebras, *Journal of Pure and Applied Algebra*, **52**, 277-282, (1988).

[14] G.J. Ellis, Crossed squares and combinatorial homotopy , *Math.Z.* , **214**, 93-110, (1993).

[15] A.R. Grandjeán and M.J. Vale, 2-modulos cruzados an la cohomologia de André-Quillen, *Memorias de la Reai Academia de Ciencias*, **22**, (1986).

[16] D. Guin-Waléry and J-L. Loday, Obsructioná l'excision en K-theories algébrique, *In: Friedlander, E.M.,Stein, M.R.(eds.) Evanston conf. on algebraic K-Theory 1980*, (Lect. Notes Math., vol.854, pp 179-216), Berlin Heidelberg New York: Springer (1981).

[17] L.Illusie, Complex cotangent et deformations I, II, *Springer Lecture Notes in Math.,* **239** (1971) II **283** (1972).

[18] J-L. Loday, Spaces with finitely many non-trivial homotopy groups, *Journal of Pure and Applied Algebra*, **24**, pp 179-202, (1982).

[19] C. Morgenegg, Sur les invariants d'un anneau local á corps résiduel de caractéristique 2, *Ph.D. thesis,* Lausanne EPFL (1979).

[20] F. Muro, Suspensions of crossed and quadratic complexes, Co-H-stuctures and applications, *Trans. Amer. Math. Soc.*, **357**, 3623-3653, (2005).

[21] T. Porter, Homology of commutative algebras and an invariant of Simis and Vasconceles, *Journal of Algebra* , **99**, pp 458-465, (1986).

[22] J.H.C. Whitehead, Combinatorial homotopy II, *Bull. Amer. Math. Soc.*, **55**, pp 453-496, (1949).

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