Characterizations of finite groups with X-s-semipermutable subgroups

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Abstract

Let A be a subgroup of a group G and X a non-empty subset of G. A is said to be X-s-semipermutable in G if A has a supplement T in G such that A is X-permutable with every Sylow subgroup of T. In this paper, some new criteria for a finite group G to be p-nilpotent or supersoluble in terms of X-s-semipermutable subgroups are obtained. In particular, a characterization of finite groups all of whose subgroups are G-s-semipermutable are presented.
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1 Introduction

In [9, 10, 11, 12, 14], Guo, Shum and Skiba introduced the following new concepts of generalized permutable subgroups. Let A and B be subgroups of a group G and X a nonempty subset of G. Then A is said to be X-permutable with B if there exists some element x in X such that $AB^x = B^xA$ (in particular, if $X = G$, then, in [10], A is said to be conditionally permutable with B); A is said to be X-semipermutable in G if A is X-permutable with all subgroups of some supplement T of A in G. Based on these generalized permutable subgroups, one has given a series of new and interesting characterizations of the structure of finite groups (see [2, 6, 9, 10, 11, 12, 13, 14, 15, 16, 24]).

Later on, as a generalization of X-semipermutability, L. P. Hao et al introduced the concept of X-s-semipermutability in [19]. Let A be a subgroup of a group G and X a non-empty subset of G. Then A is said to be X-s-semipermutable in G if A is X-permutable with every Sylow subgroup of some supplement T of A in G. Obviously, the X-semipermutability and S-permutability imply the X-s-semipermutability. However, the converse does not hold. For example, let $G = [\langle a, b\rangle]\langle \alpha\rangle$, where $a^4 = 1$, $a^2 = b^2 = [a, b]$ and $a^\alpha = b$, $b^\alpha = ab$. Let $A = \langle \alpha\rangle$ and $X = 1$. Clearly, A is X-s-semipermutable in G. But A is not X-semipermutable in G. On the other hand, let $G = [C_5]C_4$,

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where C_5 is a group of order 5 and C_4 is the automorphism group of C_5 of order 4. Let H be a subgroup of C_4 of order 2. Then H is G-s-semipermutable in G but not S-permutable in G.

Note that in [28], Li et al introduced the concept of SS-quasinormality. A subgroup H of a group G is said to be SS-quasinormal in G if H has a supplement T in G such that H is permutable with every Sylow subgroup of T. Clearly, SS-quasinormality implies that X-s-semipermutability, where $X = 1$. But the converse does not hold in general. The group $G = [C_5]C_4$ mentioned in the foregoing paragraph is a counterexample. Let H be a subgroup of C_4 of order 2. Then H is G-s-semipermutable in G, but not SS-quasinormal in G.

In [19, 20], Hao investigated the influence of X-s-semipermutable subgroups on the supersolubility and p-nilpotency of finite groups. Our object in this paper is to study further this kind of generalized permutable subgroups. Moreover, we will present some new characterizations of p-nilpotency and supersolubility of finite groups under the assumption that some subgroups are X-s-semipermutable. One of our results obtained in this paper characterizes the structure of groups G all of whose subgroups are all G-s-semipermutable.

All groups considered in this paper are finite. For notation and terminology not given in this paper, the reader is referred to [18, 8, 22] if necessary. For some related topics, the reader is also referred to [1, 5, 21, 25, 26, 27, 29, 33, 35, 36].

2 Preliminaries

We begin by stating some elementary facts about the classes of finite groups.

Let $\mathfrak{F}$ be a class of groups. $\mathfrak{F}$ is said to be a formation if $\mathfrak{F}$ is a homomorph and every group G has a smallest normal subgroup (denoted by $G^{\mathfrak{F}}$) whose quotient is still in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if $G/\Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$. A chief factor H/K of a group G is said to $\mathfrak{F}$-central (or $\mathfrak{F}$-eccentric) in G if $[H/K](G/C_G(H/K)) \in \mathfrak{F}$ (or $[H/K](G/C_G(H/K)) \notin \mathfrak{F}$ respectively). In this paper, $Z_\infty^{\mathfrak{F}}(G)$ denotes the $\mathfrak{F}$-hypercenter of a group G, that is, the product of all such normal subgroups H of G whose G-chief factors are $\mathfrak{F}$-central. We use $\mathfrak{N}$ and $\mathfrak{U}$ to denote the class of all nilpotent groups and the class of all supersoluble groups, respectively.

Lemma 2.1. [19, Lemma 2.1] *Let A and X be subgroups of a group G and let N be a normal subgroup of G.*

(1) *If A is X-s-semipermutable in G, then AN/N is XN/N-s-semipermutable in G/N.*

(2) *If A is X-s-semipermutable in G, $A \le D \le G$ and $X \le D$, then A is X-s-semipermutable in D.*

(3) *If A is X-s-semipermutable in G and $X \le D$, then A is D-s-semipermutable in G.*

Lemma 2.2. [23, Lemma 3.3] *Let G be a group and X a normal p-soluble subgroup of G. Then G is p-soluble if and only if a Sylow p-subgroup P of G is X-permutable with all Sylow q-subgroups of*

G, where $q \neq p$.

Lemma 2.3. [32, Lemma 2.10] *Let G be a group. Suppose that p is the smallest prime dividing the order of G and P is a non-cyclic Sylow p-subgroup of G. If every maximal subgroup of P has a p-nilpotent supplement in G, then G is p-nilpotent.*

Lemma 2.4. [31, Corollary 1] *Let A be an S-permutable subgroup of a group G. Then A is subnormal in G.*

Lemma 2.5. [6, Lemma 2.8] *Let G be a group, p a prime and $(|G|, p-1) = 1$. If M is a subgroup of G with index p, then M is normal in G.*

Lemma 2.6. [17, Lemma 2.6] *Let H be a nilpotent normal subgroup of a group G. If $H \neq 1$ and $H \cap \Phi(G) = 1$, then H has a complement in G and H is a direct product of some minimal normal subgroups of G.*

Lemma 2.7. [29, Theorem 1.3] *Let p be a prime dividing the order of a group G and P a Sylow p-subgroup of G. If every maximal subgroup of P has a p-nilpotent supplement in G, then G is p-nilpotent.*

3 Main results

Theorem 3.1. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if G has a normal soluble subgroup E such that $G/E \in \mathfrak{F}$ and for every non-cyclic Sylow subgroup P of $F(E)$, every cyclic subgroup of P of order prime or order 4 (if P is a non-abelian 2-group and $H \nsubseteq Z_\infty(G)$) not having a supersoluble supplement in G is G-s-semipermutable in G.*

Proof. The necessity is clear and we need only to prove the sufficiency.

First, we claim that any chief factor of G below $F(E)$ is of prime order. Assume that the assertion is not true and let L/K be a counterexample with $|K|$ minimal, that is, L/K is not of prime order but for every chief factor U/V of G below $F(E)$ with $|V| < |K|$, U/V is of prime order. Since E is soluble, we see that L/K is a p-chief factor for some prime p. Noticing that $L/K \simeq L \cap O_p(E)/K \cap O_p(E)$, we obtain by the choice of L/K that $L/K = L \cap O_p(E)/K \cap O_p(E)$ and so $L \subseteq O_p(E)$. Let P be the Sylow p-subgroup of $F(E)$. If P is cyclic, then L/K is cyclic of order p, a contradiction. Hence we can assume that P is non-cyclic. Let R/K be a chief factor of G_p/K, where G_p is a Sylow p-subgroup of G and $R \subseteq L$. Then $R = \langle x \rangle K$ for any $x \in R \setminus K$. Now we assume that there is some element $x \in R \setminus K$ of order p or 4 (if P is non-abelian 2-group and $\langle x \rangle \nsubseteq Z_\infty(G)$) not having a supersoluble supplement in G is G-s-semipermutable in G and prove that L/K is of order p, reaching a contradiction. If $x \in Z_\infty(G)$, then $xK/K \in L/K \cap Z_\infty(G/K)$ and so $L/K \subseteq Z_\infty(G/K)$, which implies that L/K is of order p, a contradiction. If $\langle x \rangle$ has a supersoluble supplement T in G, then $L/K \cap TK/K = 1$ or L/K. If $L/K \cap TK/K = L/K$, then L/K is a chief factor of $G/K = TK/K$, which is supersoluble. Therefore L/K is cyclic of order p, a contradiction. If $L/K \cap TK/K = 1$, then $L/K = L/K \cap (\langle x\rangle K/K)(TK/K) = \langle x\rangle K/K(L/K \cap TK/K) = \langle x\rangle K/K$, a contradiction again. These contradictions together with our hypothesis show that $\langle x\rangle$ is G-s-semipermutable in G. Therefore G has a subgroup T such that $\langle x\rangle$ is G-permutable with every Sylow subgroup of T. Let T_q be a Sylow q-subgroup of T, where $q \neq p$. Then $\langle x\rangle(T_q)^g = (T_q)^g\langle x\rangle$ for some $g \in G$. Since $R/K = \langle x\rangle K/K$ is subnormal in G/K, $\langle x\rangle K/K$ is subnormal in $(\langle x\rangle K/K)((T_q)^g K/K)$ and so $\langle x\rangle K/K$ is normalized by $(T_q)^g K/K$. Now one can see that $R/K = \langle x\rangle K/K$ is normal in G/K and therefore $L/K = R/K$ is cyclic. This contradiction means that all elements of $R \setminus K$ of order p or order 4 (if P is a nonabelian 2-group) are contained in K. Since $L/K = (R/K)^{G/K} = R^G/K$, we have that all elements of L of order p or 4 (if P is a non-abelian 2-group) are contained in K.

Let U/V be any chief factor of G below K. Then, by the choice of L/K, U/V is of order p and so $G/C_G(U/V)$ is abelian of exponent dividing $p-1$. Put $X = \bigcap_{U\subseteq K} C_G(U/V)$. Then X is normal in G and G/X is abelian of exponent dividing $p-1$. Let Q be any Sylow q-subgroup of X, where $q \neq p$. Then Q acts trivially on K by [18, Lemma 3.2.3]. Moreover, since all elements of L of order p or 4 (if P is a non-abelian 2-group) are contained in K, Q acts trivially on L/K by the well known Blackburn's theorem, from which we conclude that $X/C_X(L/K)$ is a p-group. It follows that $X \subseteq C_G(L/K)$ as $O_p(G/C_G(L/K)) = 1$ by [18, Lemma 1.7.11] and thereby $G/C_G(L/K)$ is abelian of exponent dividing $p-1$. Now, by [34, I, Lemma 1.3], we have that L/K is of order p, which contradicts our assumption for L/K. Hence our claim holds. Thus $F(E) \subseteq Z_\infty^{\mathfrak{U}}(G)$ and thereby $F(E) \subseteq Z_\infty^{\mathfrak{F}}(G)$ (see [18, Theorem 3.1.6]).

Let M/N be any chief factor of G below $F(E)$ and put $C = \bigcap C_E(M/N)$. Then $F(E) \subseteq C$ since $F(G) \subseteq C_G(M/N)$. We assert that $F(E) = C$. Suppose that it is not true and let $R/F(E)$ be a minimal normal subgroup of $G/F(E)$ with $F(E) < R \leq C$. Then $R \subseteq Z_\infty(R)$ and $R/F(E)$ is an elementary group as E is soluble. It follows that R is nilpotent and consequently $R \subseteq F(E)$, a contradiction. Hence $F(E) = C$. Since $G/C_G(M/N)$ is abelian by the preceding argument and $\mathfrak{F}$ is a saturated formation, $G/F(E) = G/C \in \mathfrak{F}$. Since $F(E) \subseteq Z_\infty^{\mathfrak{F}}(G)$, we obtain that $G \in \mathfrak{F}$. Thus the proof is complete. □

By Theorem 3.1, we have the following corollary.

Corollary 3.2. (Asaad, Csörgö [3].) *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups. Then a group $G \in \mathfrak{F}$ if and only if G has a normal soluble subgroup E such that $G/E \in \mathfrak{F}$ and the subgroups of prime order or order 4 of $F(E)$ are S-permutable in G.*

Theorem 3.3. *Let G be a group and $\mathfrak{F}$ a saturated formation containing all supersoluble groups. Then $G \in \mathfrak{F}$ if and only if G has a normal soluble subgroup E such that $G/E \in \mathfrak{F}$ and every maximal subgroup of each non-cyclic Sylow subgroup of the Fitting subgroup $F(E)$ not having a supersoluble supplement in G is G-s-semipermutable in G.*

Proof. The necessity part is obvious. We only need to prove the sufficiency part. Assume that the assertion is false and let G be a counterexample of minimal order. Then

(1) $\Phi(G) \cap E = 1$.

Suppose that $\Phi(G) \cap E \neq 1$. Let p be a prime divisor of $|\Phi(G) \cap E|$ and P a Sylow p-subgroup of $\Phi(G) \cap E$. Since $\Phi(G) \cap E$ is a nilpotent normal subgroup of G, P is normal in G and so $P \leq F(E)$. Consider the factor group G/P. It is clear that $F(E/P) = F(E)/P$ (see [18, Lemma 1.8.1]) and $(G/P)/(E/P) \simeq G/E$ is contained in $\mathfrak{F}$ by the hypothesis. Then by Lemma 2.1(2), we can see that G/P satisfies the hypothesis. Hence $G/P \in \mathfrak{F}$ by the choice of G. It follows that $G \in \mathfrak{F}$ as $\mathfrak{F}$ is a saturated formation, a contradiction.

(2) $F(E) = N_1 \times N_2 \times \cdots \times N_t$, where N_i is a minimal normal subgroup of G, for $i = 1, 2, ..., t$.

This follows directly from Lemma 2.6 and (1).

(3) N_i is a cyclic group of prime order, for all $i \in \{1, 2, ..., t\}$.

Without loss of generality, we may assume that $P = N_1 \times N_2 \times \cdots \times N_s$ is a Sylow p-subgroup of $F(E)$, where $s \leq t$. Let L_1 be a maximal subgroup of N_1 such that L_1 is normal in some Sylow p-subgroup G_p of G and write $B = N_2 \times \cdots \times N_s$. Then $L = L_1B$ is a maximal subgroup of P. If P is cyclic, then clearly $N_1 = P$ is cyclic of order p. Hence we assume that P is not cyclic. Now, by the hypothesis, L has a supersoluble supplement in G or is G-s-semipermutable in G. Suppose that L has a supersoluble supplement T in G. Then $(N_1 \cap BT)^G = (N_1 \cap BT)^{L_1BT} \subseteq N_1 \cap BT$ and so $N_1 \cap BT = 1$ or N_1. If $N_1 \cap BT = 1$, then $N_1 = N_1 \cap L_1BT = L_1(N_1 \cap BT) = L_1$, a contradiction. If $N_1 \cap BT = N_1$, then $G = BT$ and therefore G/B is supersoluble. Since N_1B/B is a chief factor of G/B, $N_1 \simeq N_1B/B$ is of order p, as desired. Now assume that L is G-s-semipermutable in G. Then G has a subgroup T such that L is G-permutable with every Sylow subgroup of T. Let T_q be a Sylow q-subgroup of T, where $q \neq p$. Then, for some element g of G, $L(T_q)^g = (T_q)^gL$. Since L is subnormal in G, L is subnormal in $L(T_q)^g$ and so L is normalized by $(T_q)^g$. Since L is also normalized by G_p, we conclude that L is normal in G. Consequently $L_1 = L_1(N_1 \cap B) = N_1 \cap L_1B = N_1 \cap L$ is normal in G, which implies that N_1 is cyclic of order p. Similarly we can prove that N_i is a cyclic group of prime order for $i = 2, ..., t$.

(4) Final contradiction.

By (3), we see that $G/C_G(N_i)$ is abelian, where $i = 1, 2, ..., t$. Hence $G' \leq C_G(N_i)$ and so $G' \leq C_G(F(E))$. It follows that $G' \cap E \leq C_H(F(E)) = F(E)$. Hence by (2) and (3), every G-chief factor below $G' \cap E$ is cyclic, from which we have that every chief factor of G below $G' \cap E$ is $\mathfrak{F}$-central. On the other hand, since $\mathfrak{F}$ is a saturated formation, $G/(G' \cap E) \in \mathfrak{F}$. This induces that $G \in \mathfrak{F}$. The final contradiction completes the proof. $\square$

The corollaries below follow from Theorem 3.3.

Corollary 3.4. (Ramadan [30].) *Assume that G is a soluble group and every maximal subgroup of the Sylow subgroups of $F(G)$ is normal in G. Then G is supersoluble.*

Corollary 3.5. (Asaad, Ramadan, Shaalan [4].) *A soluble group G is supersoluble if and only if G has a normal subgroup E such that G/E is supersoluble and every maximal subgroup of each Sylow subgroup of $F(E)$ is normal in G.*

Corollary 3.6. (Asaad, Ramadan, Shaalan [4].) *Let G be a group with a normal supersoluble subgroup E such that G/E is supersoluble. If all maximal subgroups of any Sylow subgroup of $F(H)$ is S-permutable in G, then G is supersoluble.*

Corollary 3.7. (Chen, Li [6].) *A group G is supersoluble if and only if G has a normal soluble subgroup E such that G/E is supersoluble and every maximal subgroup of each Sylow subgroup of $F(E)$ is $F(E)$-semipermutable in G.*

Now, we can characterize the structure of groups G with all subgroups G-s-semipermutable in the light of the preceding results.

Theorem 3.8. *Let G be a group. Every subgroup of G is G-s-semipermutable in G if and only if*

(1) *$G = [H]K$, where $H = G^{\mathfrak{N}}$ is a nilpotent Hall subgroup of G with odd order, and*

(2) *$G = HN_G(L)$ for every subgroup L of H.*

Proof. We first prove the necessity. Suppose that each subgroup of G is G-s-semipermutable in G. Then G has a Hall $\{p, q\}$-subgroup for different primes p and q dividing the order of G. By the well-known Arad's result, we see that G is soluble. Moreover, by Theorem 3.1, G is supersoluble. It follows that $G^{\mathfrak{N}}$ is nilpotent. We claim that $G^{\mathfrak{N}}$ is of odd order. If not, assume that $2 \in \pi(G^{\mathfrak{N}})$ and let P be a Sylow 2-subgroup of $G^{\mathfrak{N}}$. Then, P is normal in G and every chief factor of G below P is of order 2. Thus, $P \leq Z_\infty(G)$. Let D be a Hall p'-subgroup of $G^{\mathfrak{N}}$. Then $G^{\mathfrak{N}}$ is contained in D, a contradiction. Hence $G^{\mathfrak{N}}$ is of odd order.

Let $H = G^{\mathfrak{N}}$. We prove H is a Hall subgroup of G by induction. It is trivial if $H = 1$ and so we suppose $H > 1$. Let N be a minimal normal subgroup of G contained in H and $|N| = p$, where p is a prime. Assume that G has a minimal normal subgroup R of prime order q with $q \neq p$. Since the hypothesis holds for the factor group G/R, $(G/R)^{\mathfrak{N}}=G^{\mathfrak{N}}R/R=HR/R$ is a Hall subgroup of G/R by induction. Then the Sylow p-subgroup of H is also a Sylow p-subgroup of G. If there exists $r \in \pi(H)$ with $r \neq p$, then, by considering the factor group G/N, we conclude that the Sylow r-subgroup of H is a Sylow r-subgroup of G. Therefore, H is a Hall subgroup of G. Hence we can suppose that every minimal normal subgroup of G is a p-subgroup. Since G is supersoluble, $O_p(G)$ is a Sylow p-subgroup of G and consequently $H \leq O_p(G)$. If $N < H$, then, by induction, we see that H is a Hall subgroup of G. Hence, we now assume that $H = N$ is a minimal normal subgroup of G. If $H = O_p(G)$, then the conclusion is obvious. Thus, we suppose H is a proper subgroup of $O_p(G)$. We assert that $\Phi = \Phi(O_p(G)) = 1$. Assume this is not true. Then $(G/\Phi)^{\mathfrak{N}}=H\Phi/\Phi$ is a Sylow p-subgroup of G. Since the class of all nilpotent groups ia a saturated formation, we have that H is not contained in Φ. Therefore $H\Phi$ is a Sylow p-subgroup of G, which implies that H is a Sylow p-subgroup of

G, a contradiction. Hence $\Phi = 1$ and so $O_p(G)$ is elementary abelian. Let L be any subgroup of $O_p(G)$. We show that L is normal in G. By the hypothesis, L has a supplement T in G and L is G-permutable with the Sylow subgroups of T. Let T_q be a Sylow q-subgroup of T, where $q \neq p$. Then, for some $x \in G$, LT_q^x is a subgroup. Since L is subnormal in G, L is normal in LT_q^x, which means that T_q^x normalizes L. In addition, since $O_p(G)$ is an elementary abelian Sylow p-subgroup, L is normal in G, as wanted. Let $O_p(G) = \langle a \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_t \rangle$ and $H = \langle a \rangle$, where $|a| = |a_i| = p$ for all $i = 2, \cdots, t$. Set $a_1 = aa_2 \cdots a_t$. Then we have that $O_p(G) = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_t \rangle$. Since $1 \neq H \leq O_p(G)$, $O_p(G)$ is not contained in $Z(G)$. Hence there exists an index $i \in \{1, 2, \cdots, t\}$ such that a_i is not contained in $Z(G)$. Pick a p'-element $g \in G \backslash C_G(a_i)$. Then $y = [a_i, g] \neq 1$. Since G/H is nilpotent, we know that $y = [a_i, g] \in H$. On the other hand, $y = [a_i, g] \in \langle a_i \rangle$ as $\langle a_i \rangle$ is normal in G. Hence $\langle a_i \rangle = H$, a contradiction. Therefore $H = G^{\mathfrak{N}}$ is a Hall subgroup of G.

By the well known Shur-Zassenhaus theorem, we see that H has a complement K in G. Since G/H is nilpotent, K is a Hall nilpotent subgroup in G and $G = [H]K$, and therefore (1) holds. Finally, let L be any subgroup of H. By the preceding argument, $N_G(L)$ contains a Hall π-subgroup of G, where $\pi = \pi(K)$. It follows that $K^x \leq N_G(L)$ for some element x in G. Thus, $G = HK{=}HK^x{=}HN_G(L)$, completing the proof of (2).

From now on, we prove the sufficiency. Suppose that G is a group satisfying (1) and (2). We will show that every subgroup of G is G-permutable with all Sylow subgroups of G and so is G-s-semipermutable in G. Let $\pi = \pi(H)$ and π' the set of all primes not in π. Let D be an arbitrary subgroup of G. By the hypothesis, G is soluble and so $D = D_1D_2$, where D_1 and D_2 are Hall subgroups of D with $\pi(D_1) \subseteq \pi$ and $\pi(D_2) \subseteq \pi'$. Let P be any Sylow p-subgroup of G.

Suppose $D_2 = 1$. Then $D = D_1$. If $p \in \pi$, then P is normal in G by the hypothesis and therefore $DP = PD$. If $p \in \pi'$, then by condition (2), there exists an element x in G such that $P^x \leq N_G(D)$. It follows that $DP^x = P^xD$. Hence, in this case, D is G-permutable with all Sylow subgroups of G. Similarly, one can show that D is G-permutable with every Sylow subgroup of G provided $D = D_2$.

Hence, we suppose that D_1 and D_2 are non-trivial. Note that $D_1 \leq H$ by (1). Since D_1 is subnormal in D by condition (1), D_1 is normal in D. This means that $D \leq N_G(D_1)$. Since $G = HN_G(D_1)$ by (2), $N_G(D_1)$ contains a nilpotent Hall π'-subgroup of G by the solubility of G, B say. Without loss of generality, we may suppose that $D_2 \leq B$. If $p \in \pi$, then, clearly, $PD = DP$ as P is normal in G. If $p \in \pi'$, then G has an element x such that $P^x \leq B$. Since B is nilpotent, P^xD_2 is a subgroup of $N_G(D_1)$ and consequently $P^xD_2D_1 = P^xD$ is a subgroup of G. Thus, in this case, D is also G-permutable with all Sylow subgroups of G, completing the proof of the sufficiency. □

Lemma 3.9. *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$, P a Sylow p-subgroup of G and $X = O_{p'p}(G)$. Then G is p-nilpotent if and only if every maximal subgroup of P not having a p-nilpotent supplement in G is X-s-semipermutable in G.*

Proof. The necessity is obvious and we only need to prove the sufficiency. Suppose that the result is false and let G be a counterexample of minimal order. Then

(1) P is not cyclic.

Assume that P is cyclic. Then $N_G(P)/C_G(P)$ is a p'-group. Since $N_G(P)/C_G(P)$ is isomorphic to a subgroup of Aut(P) and $(|G|, p-1) = 1$, we have $N_G(P) = C_G(P)$ and therefore G is p-nilpotent by [22, IV, Theorem 2.6], a contradiction.

(2) $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \neq 1$. Then, by Lemma 2.1, it is easy to see that $G/O_{p'}(G)$ satisfies the hypothesis. The minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent and so G is p-nilpotent, a contradiction.

(3) $O_p(G) \neq 1$.

If not, then $O_p(G) = 1$ and so $X = 1$. First, we assume that every maximal subgroup of P has a p-nilpotent supplement in G. If $p = 2$, then by Lemma 2.3, G is p-nilpotent, a contradiction. Hence p is an odd prime and so G is also p-nilpotent by Lemma 2.7. Therefore, by the hypothesis, some maximal subgroup R of P is X-s-semipermutable in G. Then G has a subgroup T such that $G = RT$ and R is X-permutable with every Sylow subgroup of T. Indeed, one can easily see that R is permutable with every Sylow q-subgroup of G, where $q \neq p$. We claim that $R \cap T$ is an S-permutable subgroup of T. In fact, let Q be a Sylow subgroup of T. Then $RQ = QR$, whence $(R\cap T)Q = Q(R\cap T)$, as claimed. Thus, by Lemma 2.4, $R\cap P$ is subnormal in T and so $R\cap T \leq O_p(T)$ by [7, A, Lemma 8.6]. Since $|T : R\cap T| = |RT : R| = |G : R|$, $|T/O_p(T)| \leq p$. Similar to (1), we have that $T/O_p(T)$ is p-nilpotent. It follows that T is p-soluble. Let K be a Hall p'-subgroup of T. Then $RK = KR$ since R is permutable with every Sylow subgroup of T. The fact that $|G : RK| = p$ and $(|G|, p-1) = 1$ imply that RK is normal in G by Lemma 2.5. Since R is permutable with all Sylow q-subgroups of G, where $q \neq p$, it follows from Lemma 2.2 that RK is p-soluble, which implies that either $O_{p'}(RK) \neq 1$ or $O_p(RK) \neq 1$. Consequently $O_p(G) \neq 1$ by (2), a contradiction. Thus (3) holds.

(4) $O_p(G)$ is a minimal normal subgroup of G.

It is easy to verify that $G/O_p(G)$ satisfies the hypothesis. The minimal choice of G implies that $G/O_p(G)$ is p-nilpotent. It follows that G is p-soluble. Let N be a minimal normal subgroup of G. Then N is an elementary ebelian p-group by (2). Obviously G/N satisfies the hypothesis and so G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$. Now it is easy to see that $O_p(G) = F(G) = C_G(N) = N$. Hence $O_p(G)$ is a minimal normal subgroup of G.

Final contradiction.

Since $G/O_p(G)$ satisfies the hypothesis, $G/O_p(G)$ is p-nilpotent and so G is p-soluble. By (3) and (4), we have that $G = [O_p(G)]M$ for some maximal subgroup of G. In view of Lemma 2.3 and Lemma 2.7, P has a maximal subgroup R not having a p-nilpotent supplement in G. By the hypothesis, R is $O_p(G)$-s-semipermutable in G since $X = O_p(G)$ by (1). Hence G has a subgroup T such that R is $O_p(G)$-permutable with every Sylow subgroup of T. Since R is normalized by $O_p(G)$, we can see that R is permutable with every Sylow subgroup of T. Let K be a Hall p'-subgroup of T. Then RK is a subgroup of G of index p by the above arguments and so RK is normal in G by Lemma 2.5. Consequently $RK \cap O_p(G) = 1$ or $O_p(G)$. Note that $O_p(G)$ is not contained in R (if not, R has a p-nilpotent supplement M in G, a contradiction) and so $P = O_p(G)R$. Now, if $O_p(G) \cap RK = O_p(G)$, then $O_p(G)$ is contained in R, a contradiction. Therefore $O_p(G) \cap RK = 1$ and so $O_p(G)$ is of order p. Thus $O_p(G)$ is contained in $Z(G)$ as $G/C_G(O_p(G))$ is isomorphic to a subgroup of $Aut(O_p(G))$ and $(|G|, p-1) = 1$. Since $G/O_p(G)$ is p-nilpotent, it follows that G is p-nilpotent, a final contradiction. □

Theorem 3.10. *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and $\mathfrak{F}$ a saturated formation containing all p-nilpotent groups. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup E such that $G/E \in \mathfrak{F}$ and E has a Sylow p-subgroup P with the property that every maximal subgroup of P not having a p-nilpotent supplement in G is X-s-semipermutable in G, where $X = O_{p'p}(E)$.*

Proof. The necessity is clear and it needs only to prove the sufficiency. By Lemma 3.9, we have that E is p-nilpotent. Let K be a normal p-complement of E. If $K \neq 1$, then G/K satisfies the hypothesis by Lemma 2.1 and so belongs to $\mathfrak{F}$ by induction. Let A/B be a chief factor of G below K. Since K is a p'-group, $G/C_G(A/B)$ is $\mathfrak{F}$-central by [18, §3.1, Example 2] and [18, Corollary 3.1.16]. It follows that $G \in \mathfrak{F}$. Now assume that $K = 1$. Then $E = P$ is a normal p-subgroup of G. Let Q be a Sylow q-subgroup of G, where $q \neq p$. Then PQ is p-nilpotent by the hypothesis and Lemma 3.9. Therefore $Q \leq C_G(N)$. Let L/M be a chief factor of G with $L \leq E$. Then $QM/M \leq C_{G/M}(L/M)$ by above argument. Let G_p be a Sylow p-subgroup of G. Then $L/M \cap Z(G_p/M) \neq 1$ (see [8, II, Theorem 6.4]). Let L_1/M be a subgroup of $L/M \cap Z(G_p/M)$ of order p. Then $G/M \leq C_{G/M}(L_1/M)$ and so $L_1/M \leq Z(G/M)$. Consequently, $L/M = L_1/M \leq Z(G/M)$ as L/M is a chief factor of G, which implies that $E \subseteq Z_\infty(G)$. Since $G/E \in \mathfrak{F}$ by the hypothesis, we have that $G \in \mathfrak{F}$ by [18, Theorem 3.1.6] and so the theorem follows. □

From Theorem 3.10, we have

Corollary 3.11. (Chen, Li [6].) *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$, P a Sylow p-subgroup of G and $X = O_{p'p}(G)$. Then G is p-nilpotent if and only if every maximal subgroup of P not having a p-nilpotent supplement in G is X-semipermutable in G.*

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