SOME RESULTS ON NEARLY PAIRWISE COMPACT SPACES

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Abstract. The object of this paper is to obtain some results on a weaker separation axiom along with preservation theorems under some weaker form of continuity and openness of a function on spaces weaker than pairwise compactness. We also obtain some results concerning subspaces of a bitopological space.

2010 Mathematics Subject Classifications: 54D30, 54E55.
Keywords and phrases: $(\mathscr{P}_i, \mathscr{P}_j)$regularly open; $(\mathscr{P}_i, \mathscr{P}_j)$regularly closed; nearly pairwise compact; pairwise almost continuous; pairwise almost open.

1 Introduction

Kelly [5] introduced the notion of a bitopological space: the triple $(X, \mathscr{P}_1, \mathscr{P}_2)$ where $\mathscr{P}_1$ and $\mathscr{P}_2$ are two topologies on X, is called a bitopological space. Singal and Mathur [11] generalized the concept of compactness of a topological space, and introduced the notion of near compactness. The study of a product space is one among some interesting fields of study in general topology. Datta [3] (see also Swart [12]) introduced the notion of product bitopological spaces. Kiliçman and Salleh [6] studied the product properties for pairwise Lindelöf bitopological spaces [7]. Nandi [9] introduced a notion of near compactness in bitopological settings and he called it pairwise nearly compact. So it appears that the generalizations of topological concepts in a bitopological setting are among core areas of study by topologists. The notion of pairwise near compactness [9] is not a generalization of the notion of pairwise compactness. This motivates us to introduce a bitopological version

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of near compactness in [8] as a generalization of pairwise compactness, and we termed it as 'nearly pairwise compact' (Definition 2.6). We showed by examples in [8] (Example 2.1 and 2.2) that the notions of pairwise near compactness and near pairwise compactness are independent. Bose and Sinha introduced a weaker notion of pairwise continuity, namely pairwise almost continutity in [1], and a weaker notion of pairwise openness, namely pairwise almost oppeness in [2]. Here some characterizations of near pairwise compactness are discussed based on the definitions of pairwise almost continuity and pairwise almost openness of a function. We also obtain some results on near pairwise compactness considering a subspace of a bitopological space.

2 Preliminaries

Unless otherwise mentioned, X stands for the bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ and Y stands for the bitopological space $(Y, \mathscr{Q}_1, \mathscr{Q}_2)$. $(\mathscr{T})\text{int}A$ (resp. $(\mathscr{T})\text{cl}A$) denotes the interior (resp. closure) of a set A in a topological space $(X, \mathscr{T})$. For a topological space $(X, \mathscr{T})$ and $A \subset X$, we write $(A, \mathscr{T}_A)$ to denote the subspace on A of $(X, \mathscr{T})$. So the relative bitopological space for $(X, \mathscr{P}_1, \mathscr{P}_2)$ corresponding to $A \subset X$ is $(X, \mathscr{P}_{1A}, \mathscr{P}_{2A})$. Always $i, j \in \{1, 2\}$ and whenever i, j appear together, $j \neq i$. Throughout the paper, N denotes the set of natural numbers and R, the set of real numbers.

To make the article self-sufficient, we recall the following definitions.

Definition 2.1 (Kelly [5]). *X is said to be pairwise Hausdorff if for each pair of distinct points x and y of X, there exist $U \in \mathscr{P}_1$ and $V \in \mathscr{P}_2$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

Definition 2.2 (Kelly [5]). *X is said to be pairwise normal if for any pair of a $(\mathscr{P}_i)$closed set A and a $(\mathscr{P}_j)$closed set B with $A \cap B = \emptyset, i \neq j$, there exist $U \in \mathscr{P}_j$ and $V \in \mathscr{P}_i$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.*

Definition 2.3 (Fletcher, Hoyle III and Patty [4]). *A cover $\mathscr{U}$ of X is a pairwise open cover if $\mathscr{U} \subset \mathscr{P}_1 \cup \mathscr{P}_2$ and for each $i \in \{1, 2\}$, $\mathscr{U} \cap \mathscr{P}_i$ contains a nonempty set.*

Definition 2.4 (Fletcher, Hoyle III and Patty [4]). *A bitopological space X is pairwise compact if each pairwise open cover of X has a finite subcover.*

Definition 2.5 (Singal and Singal [10]). *A subset A of X is said to be $(\mathscr{P}_i, \mathscr{P}_j)$regularly open if $A = (\mathscr{P}_i)int((\mathscr{P}_j)clA)$.*

A subset of X is called $(\mathscr{P}_i, \mathscr{P}_j)$regularly closed if its complement is $(\mathscr{P}_i, \mathscr{P}_j)$regularly open. In other words, a set $A \subset X$ is $(\mathscr{P}_i, \mathscr{P}_j)$regularly closed iff $A = (\mathscr{P}_i)cl((\mathscr{P}_j)intA)$.

Definition 2.6 (Mukharjee and Bose [8]). *A bitopological space X is said to be nearly pairwise compact if for each pairwise open cover $\mathscr{U}$ of X, there exists a finite subcollection $\mathscr{V} \subset \mathscr{U}$ such that $\{(\mathscr{P}_i)int((\mathscr{P}_j)clV) \mid V \in \mathscr{V} \cap \mathscr{P}_i, i \in \{1,2\}\}$ covers X.*

Definition 2.7 (Mukharjee and Bose [8]). *A bitopological space X is said to be almost pairwise compact if for each pairwise open cover $\mathscr{U}$ of X, there exists a finite subcollection $\mathscr{V} \subset \mathscr{U}$ such that $\{(\mathscr{P}_j)clV \mid V \in \mathscr{V} \cap \mathscr{P}_i, i \in \{1,2\}\}$ covers X.*

Definition 2.8 (Mukharjee and Bose [8]). *A collection $\mathscr{U}$ of subsets of X is said to be pairwise regularly open if each member of $\mathscr{U}$ is $(\mathscr{P}_i, \mathscr{P}_j)$regularly open for some $i \in \{1,2\}$ and contains at least one $(\mathscr{P}_i, \mathscr{P}_j)$regularly open set for each $i \in \{1,2\}$. $\mathscr{U}$ is said to be a pairwise regularly open cover if it covers X.*

Definition 2.9 (Bose and Sinha [2]). *A mapping $f : X \to Y$ is said to be pairwise almost open if the image of each $(\mathscr{P}_i, \mathscr{P}_j)$regularly open set in X is $(\mathscr{Q}_i)$open in Y.*

Definition 2.10 (Bose and Sinha [1]). *A mapping $f : X \to Y$ is said to be pairwise almost continuous if inverse image of each $(\mathscr{Q}_i, \mathscr{Q}_j)$regularly open set in Y is $(\mathscr{P}_i)$open in X.*

In the sequel, we use the following theorems.

Theorem 2.11 (Mukharjee and Bose [8]). *A bitopological space X is nearly pairwise compact iff each pairwise regularly open cover of X has a finite subcover.*

Theorem 2.12 (Bose and Sinha [1]). *For a map $f : X \to Y$, the following statements are equivalent:*

(*i*) *f is pairwise almost continuous.*
(*ii*) *$f^{-1}(A) \subset (\mathscr{P}_i)int(f^{-1}((\mathscr{Q}_i)int((\mathscr{Q}_j)clA)))$ for each $(\mathscr{Q}_i)$open subset A of Y.*
(*iii*) *$(\mathscr{P}_i)cl(f^{-1}((\mathscr{Q}_i)cl((\mathscr{Q}_j)intB))) \subset f^{-1}(B)$ for each $(\mathscr{Q}_i)$closed subset B of Y.*

3 Near Pairwise Compactness

We begin with a note that a pairwise compact space is a nearly pairwise compact space. Here we give some examples to show that there are results which are not true (resp. true) for a pairwise compact space but true (resp. not true) for a nearly pairwise compact space.

Fletcher et al. [4] showed that if X is pairwise compact and $C \subset X$ is $(\mathscr{P}_1)$closed then C is $(\mathscr{P}_2)$compact. But this is not true in a nearly pairwise compact space.

Example 3.1 (Mukharjee and Bose [8]). *Let b be a fixed real number. We define*

$$\begin{aligned}\mathscr{P}_1 &= \{\emptyset, R\} \bigcup \{(-\infty, b], (b, \infty)\},\\ \mathscr{P}_2 &= \{\emptyset, R\} \cup \{(b, \infty)\} \bigcup \left\{ \left(b + \frac{1}{n}, \infty \right) \mid n \in N \right\}.\end{aligned}$$

$(R, \mathscr{P}_1, \mathscr{P}_2)$ is nearly pairwise compact but it is not pairwise compact. Also (b, ∞) is $(\mathscr{P}_1)$closed but it is not $(\mathscr{P}_2)$compact.

Since in a nearly pairwise compact space, a $(\mathscr{P}_1)$closed set may not be $(\mathscr{P}_2)$compact, a $(\mathscr{P}_1)$closed set may not be $(\mathscr{P}_2)$closed in presence of Hausdorffness of $(X, \mathscr{P}_2)$. So the following result due to Fletcher et al. [4] is not true in general in a nearly pairwise compact space: if $(X, \mathscr{P}_1)$ and $(X, \mathscr{P}_2)$ are Hausdorff spaces and $(X, \mathscr{P}_1, \mathscr{P}_2)$ is pairwise compact, then $\mathscr{P}_1 = \mathscr{P}_2$.

Now we consider Example 4.2 (Swart [12]): the bitopological space is $(X, \mathscr{P}, \mathscr{Q})$ where $X = [0, 1]$, $\mathscr{P} = \{\emptyset, X\} \cup \{[0, b) \mid b \in [0, 1]\}$ and $\mathscr{Q} = \{\emptyset, X, \{1\}\}$. It is not pairwise compact but the spaces $(X, \mathscr{P})$ and $(X, \mathscr{Q})$ are compact. The bitopological space of the example is nearly pairwise compact.

Example 3.2. *For a fixed real number b, we define*

$$\begin{aligned}\mathscr{P}_1 &= \{\emptyset, R\} \bigcup \{(b - \frac{1}{n}, \infty) \mid n \in N\} \cup [b, \infty),\\ \mathscr{P}_2 &= \{\emptyset, R\} \bigcup \{(-\infty, b - \frac{1}{n}) \mid n \in N\} \cup (-\infty, b).\end{aligned}$$

$(R, \mathscr{P}_1, \mathscr{P}_2)$ is not nearly pairwise compact although the spaces $(X, \mathscr{P}_1)$ and $(X, \mathscr{P}_2)$ are compact.

Thus a bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ may not be nearly pairwise compact even if the spaces $(X, \mathscr{P}_1)$ and $(X, \mathscr{P}_2)$ are compact.

The above discussions epitomize that the notion of near pairwise compactness is not merely generalization of pairwise compactness but it is a striking notion within the framework of bitopological settings.

Theorem 3.3. *Let there exist a pairwise almost continuous, pairwise almost open mapping $f : X \to Y$ with $f(X) = Y$. Then Y is nearly pairwise compact if X is so.*

Proof. Suppose X is nearly pairwise compact, and $\mathscr{U}^{(Y)} = \{U_\alpha \mid \alpha \in A\}$ is a pairwise regularly open cover of Y. Then using pairwise almost continuity of f, we see $\mathscr{U}^{(X)} = \{f^{-1}(U_\alpha) \mid \alpha \in A\}$ is a pairwise open cover of X. Since X is nearly pairwise compact, there exists a finite subcollection $\mathscr{V}^{(X)} = \{f^{-1}(U_{\alpha_k}) \mid \alpha_k \in A,\ k = 1, 2, \ldots, n\}$ such that $\{(\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}(f^{-1}(U_{\alpha_k})) \mid U_{\alpha_k} \in \mathscr{U}^{(Y)} \cap \mathscr{Q}_i,\ k = 1, 2, \ldots, n\}$ covers X. Now we have

$$\begin{aligned} Y &= f(X) \\ &= f\left(\bigcup_{k=1}^{n} \left\{(\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}(f^{-1}(U_{\alpha_k}))) \mid f^{-1}(U_{\alpha_k}) \in \mathscr{V}^{(X)} \cap \mathscr{P}_i\right\}\right) \\ &= \bigcup_{k=1}^{n} \{f\left((\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}(f^{-1}(U_{\alpha_k})))\right) \mid \\ &\qquad\qquad f^{-1}(U_{\alpha_k}) \in \mathscr{V}^{(X)} \cap \mathscr{P}_i\Big\}. \end{aligned} \tag{3.1}$$

Also, $f\left((\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}(f^{-1}(U_{\alpha_k})))\right) \subset f\left((\mathscr{P}_j)\text{cl}(f^{-1}(U_{\alpha_k}))\right)$ and $f\left((\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}(f^{-1}(U_{\alpha_k})))\right)$ is $(\mathscr{Q}_i)$open in Y. Thus we have,

$$f\left((\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}(f^{-1}(U_{\alpha_k})))\right) \subset (\mathscr{Q}_i)\text{int}\left(f\left((\mathscr{P}_j)\text{cl}(f^{-1}(U_{\alpha_k}))\right)\right). \tag{3.2}$$

Now we put $V_{\alpha_k} = (\mathscr{Q}_j)\text{cl}U_{\alpha_k}$. Hence we have

$$U_{\alpha_k} \subset V_{\alpha_k} \Rightarrow U_{\alpha_k} \subset (\mathscr{Q}_j)\text{cl}((\mathscr{Q}_i)\text{int}V_{\alpha_k}).$$

Therefore from (3.2) we obtain

$$\begin{aligned} & f\left((\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}(f^{-1}(U_{\alpha_k})))\right) && (3.3)\\ \subset\ & (\mathscr{Q}_i)\text{int}\left(f\left((\mathscr{P}_j)\text{cl}(f^{-1}((\mathscr{Q}_j)\text{cl}((\mathscr{Q}_i)\text{int}V_{\alpha_k}))))\right)\right) \\ \subset\ & (\mathscr{Q}_i)\text{int}\left(f(f^{-1}(V_{\alpha_k}))\right) \quad \text{(by Theorem 2.12)} && (3.4)\\ =\ & (\mathscr{Q}_i)\text{int}V_{\alpha_k} \quad \text{(since } f \text{ is onto)} \\ =\ & U_{\alpha_k}. \end{aligned}$$

Hence from (3.1) we obtain,

$$Y \subset \bigcup_{k=1}^{n} U_{\alpha_k}.$$

Therefore Y is nearly pairwise compact. ∎

Lemma 3.4. *A bitopological space X is almost pairwise compact iff each pairwise regularly open cover $\mathscr{U}$ of X has a finite subfamily $\mathscr{V}$ such that $\{(\mathscr{P}_j)clV \mid V \in \mathscr{V} \cap \mathscr{P}_i\}$ covers X.*

Proof. We need to prove only the sufficiency of the lemma. Let $\mathscr{G} = \{G_\alpha \mid \alpha \in A\}$ be any pairwise open cover of X. We write $H_\alpha = (\mathscr{P}_i)\mathrm{int}((\mathscr{P}_j)\mathrm{cl}G_\alpha)$, $\alpha \in A$ whenever $G_\alpha \in \mathscr{G}$ is $(\mathscr{P}_i)$open. Then $\mathscr{H} = \{H_\alpha \mid \alpha \in A\}$ is a pairwise regularly open cover of X. Hence we have a finite subfamily $\mathscr{V} = \{H_{\alpha_k} \mid k = 1, 2, \ldots, n\}$ of $\mathscr{H}$ such that $\{(\mathscr{P}_j)\mathrm{cl}H_{\alpha_k} \mid H_{\alpha_k} \in \mathscr{V} \cap \mathscr{P}_i\}$ covers X. Now $(\mathscr{P}_j)\mathrm{cl}H_{\alpha_k} = (\mathscr{P}_j)\mathrm{cl}((\mathscr{P}_i)\mathrm{int}((\mathscr{P}_j)\mathrm{cl}G_{\alpha_k})) \subset (\mathscr{P}_j)\mathrm{cl}G_{\alpha_k}$. So $\{(\mathscr{P}_j)\mathrm{cl}G_{\alpha_k} \mid k = 1, 2, \ldots, n\}$ covers X. ■

Theorem 3.5. *Let there exist a pairwise almost continuous mapping $f : X \to Y$ with $f(X) = Y$. Then Y is almost pairwise compact if X is almost pairwise compact.*

Proof. Suppose $\mathscr{U} = \{U_\alpha \mid \alpha \in A\}$ is a pairwise regularly open cover of Y. Then $\mathscr{G} = \{f^{-1}(U_\alpha) \mid \alpha \in A\}$ is a pairwise open cover of X. Therefore $\mathscr{G}$ has a finite subfamily $\mathscr{H} = \{f^{-1}(U_{\alpha_k}) \mid k = 1, 2, \ldots, n\}$ such that $\{(\mathscr{P}_j)\mathrm{cl}(f^{-1}(U_{\alpha_k})) \mid f^{-1}(U_{\alpha_k}) \in \mathscr{H} \bigcap \mathscr{P}_i\}$ covers X. Since f is pairwise almost continuous, on using the scheme we adopt to establish (3.4) of Theorem 3.3, we obtain $(\mathscr{P}_j)\mathrm{cl}(f^{-1}(U_{\alpha_k})) \subset f^{-1}((\mathscr{Q}_j)\mathrm{cl}U_{\alpha_k})$. Hence the finite subcollection $\{(\mathscr{Q}_j)\mathrm{cl}U_{\alpha_k}) \mid k = 1, 2, \ldots, n\}$ of $\mathscr{U}$ covers Y. So by Lemma 3.4, Y is almost pairwise compact. ■

Theorem 3.6. *Let A be $(\mathscr{P}_i)$open in X for each $i \in \{1, 2\}$. Then A is nearly pairwise compact iff each pairwise open cover $\mathscr{U}$ with respect to $(X, \mathscr{P}_1, \mathscr{P}_2)$ of A has a finite subcollection $\mathscr{V}$ of $\mathscr{U}$ such that $\{(\mathscr{P}_i)int((\mathscr{P}_j)clV) \mid V \in \mathscr{V} \cap \mathscr{P}_i\}$ covers A.*

Proof. Firstly, suppose A is nearly pairwise compact. Let $\mathscr{U} = \{U_\alpha \mid \alpha \in I\}$ be a pairwise open cover of A with respect to $(X, \mathscr{P}_1, \mathscr{P}_2)$. Now $A \cap U_\alpha$ is $(\mathscr{P}_{iA})$open if U_α is $(\mathscr{P}_i)$open. So $\mathscr{U}^{(A)} = \{A \cap U_\alpha \mid \alpha \in I\}$ is a pairwise open cover of A with respect to $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$. Hence we obtain a finite subcollection $\mathscr{V}^{(A)} = \{A \cap U_{\alpha_k} \mid \alpha_k \in I, k = 1, 2, \ldots, n\}$ such that $\{(\mathscr{P}_{iA})\mathrm{int}((\mathscr{P}_{jA})\mathrm{cl}(A \cap U_{\alpha_k})) \mid U_{\alpha_k} \in \mathscr{P}_i, k = 1, 2, \ldots, n\}$ covers A. The 'only if' part now follows from the following relations.

$$\begin{aligned} A &= \bigcup_{k=1}^{n} \{(\mathscr{P}_{iA})\mathrm{int}((\mathscr{P}_{jA})\mathrm{cl}(A \cap U_{\alpha_k}))\} \\ &= \bigcup_{k=1}^{n} \{(\mathscr{P}_{i})\mathrm{int}((\mathscr{P}_{j})\mathrm{cl}(A \cap U_{\alpha_k}))\} \text{ (since a } (\mathscr{P}_{iA})\text{open set in} \\ &\qquad (A, \mathscr{P}_{1A}, \mathscr{P}_{2A}) \text{ is a } (\mathscr{P}_i)\text{open set in } (X, \mathscr{P}_1, \mathscr{P}_2)) \\ &\subset \bigcup_{k=1}^{n} (\mathscr{P}_{i})\mathrm{int}((\mathscr{P}_{j})\mathrm{cl}U_{\alpha_k}). \end{aligned}$$

To prove the 'if' part, let $\mathscr{G} = \{G_\alpha \mid \alpha \in \Gamma\}$ be a pairwise open cover of A with respect to $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$. Then $\mathscr{G}$ is also a pairwise open cover of A with respect to $(X, \mathscr{P}_1, \mathscr{P}_2)$, since A is $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$. Hence we have a finite subcollection $\mathscr{H} = \{G_{\alpha_k} \mid k = 1, 2, \ldots, m\}$ such that $\{(\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}G_{\alpha_k}) \mid G_{\alpha_k} \in \mathscr{H} \cap \mathscr{P}_i\}$ covers A. Now

$$\begin{aligned} A &= \bigcup_{k=1}^{m} \{A \cap (\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}G_{\alpha_k})\} \\ &= \bigcup_{k=1}^{m} \{(\mathscr{P}_{iA})\text{int}(A \cap (\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}G_{\alpha_k}))\} \\ &\subset \bigcup_{k=1}^{m} \{(\mathscr{P}_{iA})\text{int}((\mathscr{P}_{jA})\text{cl}G_{\alpha_k}))\}. \end{aligned}$$

Hence A is nearly pairwise compact. ■

Lemma 3.7. *Each proper subset of a pairwise Hausdorff bitopological space X has a nontrivial $(\mathscr{P}_i)$open cover.*

Proof. Obvious. ■

Definition 3.8. *A bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ is said to be mildly pairwise normal* if for each pair of a $(\mathscr{P}_i, \mathscr{P}_j)$regularly closed set A and a $(\mathscr{P}_j, \mathscr{P}_i)$regularly closed set B with $A \cap B = \emptyset$, there exist a $(\mathscr{P}_j)$open set U and a $(\mathscr{P}_i)$open set V such that $A \subset U$, $B \subset V$, $U \cap V = \emptyset$, where either U is $(\mathscr{P}_j, \mathscr{P}_i)$regularly open or V is $(\mathscr{P}_i, \mathscr{P}_j)$regularly open in X.*

Obviously, a pairwise normal space is mildly pairwise normal but converse is not true.

Example 3.9. *Let R be the set of real numbers and $a, b \in R$ with $b > a$. We define*

$$\begin{aligned} \mathscr{P}_1 &= \{\emptyset, R\} \cup \{(-\infty, a], (-\infty, b)\}, \\ \mathscr{P}_2 &= \{\emptyset, R\} \cup \{[a, \infty), [b, \infty)\}. \end{aligned}$$

Then the bitopological space $(R, \mathscr{P}_1, \mathscr{P}_2)$ is mildly pairwise normal but the space is not pairwise normal.

Theorem 3.10. *A pairwise Hausdorff and nearly pairwise compact space X is mildly pairwise normal.*

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*Probably M. K. Singal and A. R. Singal introduced a notion of pairwise mild normality. But we do not have the access to the paper.

Proof. Let A and B be $(\mathscr{P}_1,\mathscr{P}_2)$regularly closed and $(\mathscr{P}_2,\mathscr{P}_1)$regularly closed subsets of X respectively with $A \cap B = \emptyset$.

Firstly, we fixed a $y \in B$. Then for each $x \in A$, on using pairwise Hausdorffness of X, we obtain a $(\mathscr{P}_2)$open set U_x and a $(\mathscr{P}_1)$open set V_x such that $x \in U_x, y \in V_x$ with $U_x \cap V_x = \emptyset$. So $\mathscr{U} = \{U_x \mid x \in A\}$ is a $(\mathscr{P}_2)$open cover of A which in turn implies $\mathscr{U} \cup \{X - A\}$ is a pairwise open cover of X. So the near pairwise compactness of X ensures the existence of a finite subcollection $\mathscr{G}$ of $\mathscr{U} \cup \{X - A\}$ such that $A \subset \bigcup\{(\mathscr{P}_i)\text{int}((\mathscr{P}_j)\text{cl}V) \mid V \in \mathscr{G} \cap \mathscr{P}_i\}$. But $(\mathscr{P}_1)\text{int}((\mathscr{P}_2)\text{cl}(X - A)) = X - A$. Hence $\mathscr{G}_A = \mathscr{G} - \{X - A\}$ is a finite subcollection of $\mathscr{U}$ such that $A \subset \bigcup\{(\mathscr{P}_2)\text{int}((\mathscr{P}_1)\text{cl}V \mid V \in \mathscr{G}_A\}$. Let $\mathscr{G}_A = \{U_{x_k} \mid k = 1, 2, \ldots, n\}$. Then $y \in \bigcap_{k=1}^n V_{x_k}$ and

$$\begin{aligned} A &\subset \bigcup\nolimits_{k=1}^n (\mathscr{P}_2)\text{int}((\mathscr{P}_1)\text{cl}U_{x_k}) \\ &\subset (\mathscr{P}_2)\text{int}\left(\bigcup\nolimits_{k=1}^n (\mathscr{P}_1)\text{cl}U_{x_k}\right) \\ &= (\mathscr{P}_2)\text{int}\left((\mathscr{P}_1)\text{cl}\left(\bigcup\nolimits_{k=1}^n U_{x_k}\right)\right). \end{aligned}$$

We write $U_y = (\mathscr{P}_2)\text{int}\left((\mathscr{P}_1)\text{cl}\left(\bigcup_{k=1}^n U_{x_k}\right)\right)$ and $V_y = \bigcap_{k=1}^n V_{x_k}$. Then U_y is $(\mathscr{P}_2,\mathscr{P}_1)$regularly open with $A \subset U_y$ and V_y is $(\mathscr{P}_1)$open with $y \in V_y$. Now we show $U_y \cap V_y = \emptyset$. If possible, let $z \in U_y \cap V_y$. We note that $z \in U_y = (\mathscr{P}_2)\text{int}\left((\mathscr{P}_1)\text{cl}\left(\bigcup_{k=1}^n U_{x_k}\right)\right) \subset (\mathscr{P}_1)\text{cl}\left(\bigcup_{k=1}^n U_{x_k}\right) = \bigcup_{k=1}^n (\mathscr{P}_1)\text{cl}U_{x_k}$. Hence for some $l \in \{1, 2, \ldots, n\}$, $z \in (\mathscr{P}_1)\text{cl}U_{x_l}$. Since $z \in V_{x_k}$ for all $k = 1, 2, \ldots, n$ and $U_{x_l} \cap V_{x_l} = \emptyset$, we have $z \notin U_{x_l}$. Also $z \in V_{x_l}$ and V_{x_l} is $(\mathscr{P}_1)$open. Thus z cannot be a $(\mathscr{P}_1)$limit point of U_{x_l} which contradicts the fact, $z \in (\mathscr{P}_1)\text{cl}U_{x_l}$.

Now we vary $y \in B$ over B. For each $y \in B$, we obtain a $(\mathscr{P}_2,\mathscr{P}_1)$regularly open set U_y and a $(\mathscr{P}_1)$open set V_y such that $A \subset U_y, y \in V_y$ and $U_y \cap V_y = \emptyset$. So using the steps describe above, we obtain a finite family $\{V_{y_k} \mid k = 1, 2, \ldots, m\}$ of $(\mathscr{P}_1)$open sets with

$$\begin{aligned} B &\subset \bigcup\nolimits_{k=1}^m (\mathscr{P}_1)\text{int}((\mathscr{P}_2)\text{cl}V_{y_k}) \\ &\subset (\mathscr{P}_1)\text{int}\left((\mathscr{P}_2)\text{cl}\left(\bigcup\nolimits_{k=1}^m V_{y_k}\right)\right). \end{aligned}$$

We put $U = \bigcap_{k=1}^m U_{y_k}$ and $V = (\mathscr{P}_1)\text{int}\left((\mathscr{P}_2)\text{cl}\left(\bigcup_{k=1}^m V_{y_k}\right)\right)$. Thus U is $(\mathscr{P}_2)$open and V is $(\mathscr{P}_1,\mathscr{P}_2)$regularly open with $A \subset U, B \subset V$ and $U \cap V = \emptyset$. ■

Corollary 3.11. *Let X be a pairwise Hausdorff and nearly pairwise compact space. If $A \subset X$ is $(\mathscr{P}_j,\mathscr{P}_i)$regularly closed then A is contained in a $(\mathscr{P}_i,\mathscr{P}_j)$regularly open set.*

Proof. Follows from Lemma 3.7 and Theorem 3.10. ∎

Acknowledgement. The authors would like to thank the referees for useful comments and valuable suggestions for improvement of the paper.

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