Orthogonal polynomials approach to the Hankel transform of sequences based on Motzkin numbers

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Abstract

In this paper we use a method based on orthogonal polynomials to give closed-form evaluations of the Hankel transform of sequences based on the Motzkin numbers. It includes linear combinations of consecutive two, three and four Motzkin numbers. In some cases, we were able to derive the closed-form evaluation of the Hankel transform, while in the others we showed that the Hankel transform satisfies a particular difference equation. As a corollary, we reobtain known results and show some new results regarding the Hankel transform of Motzkin and shifted Motzkin numbers. Those evaluations also give an idea on how to apply the method based on orthogonal polynomials on the sequences having zero entries in their Hankel transform.

Key words: Hankel transform, Orthogonal polynomials, Motzkin numbers.

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1 Introduction

The Motzkin number m_n is the number of different ways of drawing non-intersecting chords on a circle between n points. It is denoted by A001006 in the On-Line Encyclopedia of Integer Sequences [22] and the first few members are given by $1, 1, 2, 4, 9, 21, 51, 127, \ldots$. The sequence of Motzkin numbers has very diverse applications in geometry, combinatorics and number theory [1]. It satisfies the following recurrence relation:

$$m_{n+1} = m_n + \sum_{i=0}^{n-1} m_i \cdot m_{n-1-i} = \frac{2n+3}{n+3} \cdot m_n + \frac{3n}{n+3} \cdot m_{n-1}$$

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It is also known that

$$m_n = \sum_{k=0}^{[n/2]} \frac{n!}{k!(k+1)!(n-2k)!}.$$

The generating function $M(x) = \sum_{k=0}^{\infty} m_k x^k$, is given by

$$M(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2} \tag{1}$$

and satisfies $M(x) = 1 + xM(x) + x^2 M^2(x)$.

Motzkin numbers represent the number of planar paths which do not descend below the x-axis, starting at $(0,0)$ and ending at $(n,0)$, where allowed steps are $(1,0)$, $(1,1)$ and $(1,-1)$. Adding a weight t to $(1,0)$ steps and weight 1 to $(1,1)$ and $(1,-1)$ steps, one obtains a weighted version of Motzkin numbers, called *t-Motzkin numbers* and denoted by m_n^t. If we avoid the condition that the path do not descend below the x-axis, then such paths are counted by the sequence of *central trinomial coefficients* c_n. Recall that c_n is the coefficient of x^n in the expansion $(1+x+x^2)^n$. In the literature, there are a lot of papers dealing with the (generalized) central trinomial coefficients and their Hankel transform [12, 21, 19]. There is a similar situation with the Motzkin and weighted Motzkin numbers [8, 4, 14, 15].

In a recent paper [4], Cameron and Yip evaluated the Hankel transform of the sequences $m_n^t + m_{n+1}^t$ and $m_{n+1}^t + m_{n+2}^t$ using the combinatorial Gessel-Viennot-Lindstrom (GVL) metod [12]. On the other hand, method based on orthogonal polynomials is successfully applied on the similar sequences involving (generalized) Catalan numbers [6, 20] and (generalized) central trinomial coefficients [19]. The aim of this paper is to consider the Hankel transform evaluation of some linear combinations of two, three and four consecutive Motzkin numbers. Using these results, we can reobtain known Hankel transform evaluations of the Motzkin and shifted Motzkin numbers, and also show some new interesting evaluations involving concrete linear combinations of Motzkin numbers. This paper also gives an idea of how to apply the method based on orthogonal polynomials on the sequences which have zero entries in their Hankel transforms.

2 Hankel transform of the moment sequences

The Hankel transform is an important not invertible transform on integer sequences that has been studied much recently [6, 17, 19].

Definition 2.1. *The* **Hankel transform** *of a given sequence $a = (a_n)_{n\in\mathbb{N}_0}$ is the sequence of Hankel determinants $(h_n)_{n\in\mathbb{N}_0}$ where $h_n = \det[a_{i+j-2}]_{i,j=1}^n$, i.e*

$$a = (a_n)_{n\in\mathbb{N}_0} \quad \stackrel{\mathcal{H}}{\Longrightarrow} \quad h = (h_n)_{n\in\mathbb{N}_0}: \quad h_n = \det \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{bmatrix} \tag{2}$$

We denote Hankel transform by $\mathcal{H}$ and hence we write $h = \mathcal{H}(a)$.

Hankel determinants are sometimes also called *persymmetric* or *Turanian* determinants. Although the determinants of Hankel matrices had been defined and explored before, the term Hankel transform was introduced in 2001 by Layman [17]. Many different evaluations of the

Hankel transform are known in the literature. We particularly denote method based on *continued fractions* [3], method based on the *exponential generating function* [13], method based on differential-convolution equations [9, 10] and method based on the orthogonal polynomials [6, 19, 20]. A concise review of different methods for determinant evaluations, including Hankel determinants, is given in the papers of Krattenthaler [14, 15].

In this paper, we use the method based on the orthogonal polynomials for the Hankel transform evaluation. This method was developed in [6] and later in [19, 20].

Let $(a_n)_{n\in\mathbb{N}_0}$ be the moment sequence with respect to some measure $d\lambda(x)$. In other words, let

$$a_n = \int_{\mathbb{R}} x^n d\lambda(x) \quad (n = 0, 1, 2, \ldots) . \tag{3}$$

Then the Hankel transform $h = \mathcal{H}(a)$ of the sequence $a = (a_n)_{n\in\mathbb{N}_0}$ can be expressed by the following relation known as the Heilermann formula (for example, see Krattenthaler [15])

$$h_n = a_0^{n+1}\beta_1^n\beta_2^{n-1}\cdots\beta_{n-1}^2\beta_n. \tag{4}$$

The sequence $(\beta_n)_{n\in\mathbb{N}_0}$ appears as a sequence of coefficients in the three-term recurrence relation

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \beta_n P_{n-1}(x), \tag{5}$$

satisfied by the monic orthogonal polynomials $(P_n(x))_{n\in\mathbb{N}_0}$ with respect to the measure $d\lambda(x)$. Weight function (measure) transformations are often used to derive the closed-form expression for the coefficient β_n.

3 Moment representation, orthogonal polynomials and Hankel transform of the Motzkin numbers

In this section we introduce the moment representation of the Motzkin numbers and evaluate its Hankel transform.

In the rest of the paper, we also deal with the *shifted Motzkin numbers* $(m_n^*)_{n\in\mathbb{N}_0}$, $(m_n^{**})_{n\in\mathbb{N}_0}$ and $(m_n^{***})_{n\in\mathbb{N}_0}$ defined by $m_n^* = m_{n+1}$, $m_n^{**} = m_{n+2}$ and $m_n^{***} = m_{n+3}$. Furthermore, denote by h_n, h_n^*, h_n^{**} and h_n^{***} the Hankel transforms of m_n, m_n^*, m_n^{**} and m_n^{***} respectively.

The following theorem gives an explicit expression of the weight function which moment sequence is $(m_n)_{n\in\mathbb{N}_0}$. Its formulation can be found for example in [22] or in the paper [2] where the proof based on the Stieltjes-Perron inversion formula (see for example [5, 16]) is shown.

Theorem 3.1. [2] *Motzkin numbers $(m_n)_{n\in\mathbb{N}_0}$ are moments of the weight function*

$$w(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4-(x-1)^2}, & x \in [-1, 3] \\ 0, & \text{otherwise} \end{cases}. \tag{6}$$

To compute the Hankel transform h_n using the Heilermann formula (4), we need the coefficients α_n and β_n of the three-term recurrence relation, corresponding to the weight function $w(x)$. These coefficients will be obtained by applying weight function transformations. Lemma 3.2 and Lemma 3.3 provide relations between the coefficients α_n and β_n of the original and transformed weight function.

Lemma 3.2. *Let $w(x)$ and $\tilde{w}(x)$ be the weight functions and denote by $(\pi_n(x))_{n\in\mathbb{N}_0}$ and $(\tilde{\pi}_n(x))_{n\in\mathbb{N}_0}$ the corresponding orthogonal polynomials. Also denote by $(\alpha_n)_{n\in\mathbb{N}_0}, (\beta_n)_{n\in\mathbb{N}_0}$ and $(\tilde{\alpha}_n)_{n\in\mathbb{N}_0}, \left(\tilde{\beta}_n\right)_{n\in\mathbb{N}_0}$ the three-term relation coefficients corresponding to $w(x)$ and $\tilde{w}(x)$ respectively. The following transformation formulas are valid:*

(1) *If $\tilde{w}(x) = Cw(x)$ where $C > 0$ then we have $\tilde{\alpha}_n = \alpha_n$ for $n \in \mathbb{N}_0$ and $\tilde{\beta}_0 = C\beta_0$, $\tilde{\beta}_n = \beta_n$ for $n \in \mathbb{N}$. Additionally holds $\tilde{\pi}_n(x) = \pi_n(x)$ for all $n \in \mathbb{N}_0$.*

(2) *If $\tilde{w}(x) = w(ax + b)$ where $a, b \in \mathbb{R}$ and $a \neq 0$ there holds $\tilde{\alpha}_n = \frac{\alpha_n - b}{a}$ for $n \in \mathbb{N}_0$ and $\tilde{\beta}_0 = \frac{\beta_0}{|a|}$ and $\tilde{\beta}_n = \frac{\beta_n}{a^2}$ for $n \in \mathbb{N}$. Additionally holds $\tilde{\pi}_n(x) = \frac{1}{a^n}\pi_n(ax + b)$.*

Proof. In both cases, we directly check the orthogonality of $\bar{\pi}_n(x)$ and obtain the coefficients $\bar{\alpha}_n$ and $\bar{\beta}_n$ by putting $\bar{\pi}_n(x)$ in the three-term recurrence relation for $\pi_n(x)$. □

Lemma 3.3. (Linear multiplier transformation) [11] *Consider the same notation as in Lemma 3.2. Let the sequence $(r_n)_{n\in\mathbb{N}_0}$ be defined by*

$$r_0 = c - \alpha_0, \qquad r_n = c - \alpha_n - \frac{\beta_n}{r_{n-1}} \qquad (n \in \mathbb{N}_0). \tag{7}$$

If $\tilde{w}(x) = (x - c)w(x)$ where $c < \inf\sup(w)$, there holds

$$\begin{aligned} \tilde{\beta}_0 &= \int_{\mathbb{R}} \tilde{w}(x)\, dx, \quad \tilde{\beta}_n = \beta_n \frac{r_n}{r_{n-1}}, \qquad (n \in \mathbb{N}), \\ \tilde{\alpha}_n &= \alpha_{n+1} + r_{n+1} - r_n, \quad (n \in \mathbb{N}_0). \end{aligned} \tag{8}$$

In the following theorem, we give a new proof of the well-known result about the Hankel transform of the Motzkin numbers (see for example [1, 4]). The proof is based on the weight function transformation shown in Lemma 3.2. Derived expressions for the coefficients α_n and β_n will be used for the further evaluations shown in the latter sections.

Theorem 3.4. *The Hankel transform of the sequence Motzkin numbers $(m_n)_{n\in\mathbb{N}_0}$ is the sequence of all 1's. Coefficients α_n and β_n of the three-term recurrence relation are given by:*

$$\alpha_n = \beta_n = 1, \qquad (n \in \mathbb{N}_0).$$

Proof. The monic Chebyshev polynomials of the second kind

$$Q_n^{(1)}(x) = S_n(x) = \frac{\sin\big((n+1)\arccos x\big)}{2^n \cdot \sqrt{1 - x^2}}$$

are orthogonal with respect to the weight $w^{(0)}(x) = \sqrt{1 - x^2}$. The corresponding coefficients in three-term relation are

$$\beta_0^{(1)} = \frac{\pi}{2}, \quad \beta_n^{(1)} = \frac{1}{4} \quad (n \geq 1), \qquad \alpha_n^{(1)} = 0 \quad (n \geq 0)\ .$$

Let us introduce new weight function $w^{(1)}(x) = \sqrt{1 - \left(\frac{x-1}{2}\right)^2}$. It satisfies $w^{(1)}(x) = w^{(0)}(ax+b)$, where $a = 1/2$ and $b = -1/2$. Hence we get (see Lemma 3.2):

$$\beta_0^{(1)} = \pi, \quad \beta_n^{(1)} = 1 \quad (n \in \mathbb{N}), \qquad \alpha_n^{(1)} = 1 \qquad (n \in \mathbb{N}_0)\ .$$

Since $w(x) = \frac{1}{\pi}w^{(1)}(x)$, Lemma 3.2 implies $\beta_0 = \frac{1}{\pi}\beta_0^{(1)} = 1$, $\beta_n = \beta_n^{(1)} = 1$ for $n \geq 1$ and $\alpha_n = \alpha_n^{(1)} = 1$. The expression for the Hankel transform of the Motzkin numbers now follows directly from the Heilermann formula (4). □

4 Linear combination of two consecutive Motzkin numbers

The advantage of the method based on orthogonal polynomials is the fact that, by knowing the coefficients α_n and β_n corresponding to some sequence, we can effectively obtain the coefficients and the Hankel transform of the linear combination of consecutive members of that sequence. That is demonstrated on the sequence of Motzkin numbers.

Theorem 4.1. *The Hankel transform $\bar{h}_n(c)$ of the sequence $(m_{n+1} - c \cdot m_n)_{n\in\mathbb{N}_0}$, where $(m_n)_{n\in\mathbb{N}_0}$ is the sequence of Motzkin numbers and $c \in \mathbb{R}$, is given by:*

$$\bar{h}_n(c) = \frac{1}{\sqrt{c^2-2c-3}} \cdot \left[\left(\frac{1-c+\sqrt{c^2-2c-3}}{2} \right)^{n+2} - \left(\frac{1-c-\sqrt{c^2-2c-3}}{2} \right)^{n+2} \right] \tag{9}$$

Proof. We start the proof by introducing the following weight function transformation

$$\bar{w}(x) = (x-c)w(x)$$

and by applying Lemma 3.3. The coefficients $\bar{\alpha}_n$ and $\bar{\beta}_n$ are given by

$$\bar{\alpha}_n = \alpha_{n+1} + \bar{r}_{n+1} - \bar{r}_n = 1 + \bar{r}_{n+1} - \bar{r}_n, \qquad n \geq 0, \tag{10}$$

$$\bar{\beta}_0 = \int_{-1}^{3} \bar{w}(x)dx = 1 - c, \qquad \bar{\beta}_n = \beta_n \frac{\bar{r}_n}{\bar{r}_{n-1}} = \frac{\bar{r}_n}{\bar{r}_{n-1}}, \quad n \geq 1, \tag{11}$$

where the sequence $(\bar{r}_n)_{n\in\mathbb{N}_0}$ is determined by the following recurrence relation:

$$\bar{r}_0 = c - 1, \qquad \bar{r}_n = c - \alpha_n - \frac{\beta_n}{\bar{r}_{n-1}} = c - 1 - \frac{1}{\bar{r}_{n-1}}, \quad n \geq 1. \tag{12}$$

Using previous expression we obtain $\bar{r}_1 = c - 1 - \frac{1}{c-1}$, $\bar{r}_2 = c - 1 - \frac{c-1}{c(c-2)} = \frac{(c-1)(c^2-3c+1)}{c(c-2)}$. According to the Heilermann formula (4) there holds

$$\frac{\bar{h}_{n+1}(c)}{\bar{h}_n(c)} = \bar{\beta}_0 \cdot \bar{\beta}_1 \cdot \bar{\beta}_2 \cdots \bar{\beta}_{n+1} = -\bar{r}_{n+1}.$$

Using the recurrence relation (12) along with the previous expression, we obtain the following difference equation

$$\bar{h}_n(c) + (c-1)\bar{h}_{n-1}(c) + \bar{h}_{n-2}(c) = 0, \qquad n \geq 2 \tag{13}$$

with initial values

$$\bar{h}_0(c) = 1 - c, \qquad \bar{h}_1(c) = c^2 - 2c.$$

By solving linear difference equation (13), we get (9). □

As a direct corollary of the previous theorem, we re-obtain the Hankel transform of the shifted sequence $m_n^* = m_{n+1}$. This result appears in [4], and with several other extensions, using the G-V-L method, in [18].

Corollary 4.2. *The Hankel transform of the sequence shifted Motzkin numbers $(m_n^*)_{n\in\mathbb{N}_0}$ is given by:*

$$h_n^* = \begin{cases} 1, & n = 6k \quad \text{or} \quad n = 6k+5 \\ 0, & n = 6k+1 \quad \text{or} \quad n = 6k+4 \\ -1, & n = 6k+2 \quad \text{or} \quad n = 6k+3 \end{cases} \quad (k \in \mathbb{N}_0) \tag{14}$$

Proof. If we put $c = 0$ in the expression (9) we have:

$$h_n^* = \bar{h}_n(0) = \frac{-1+i\sqrt{3}}{2i\sqrt{3}} \left(\frac{1+i\sqrt{3}}{2}\right)^n + \frac{1+i\sqrt{3}}{2i\sqrt{3}} \left(\frac{1-i\sqrt{3}}{2}\right)^n \tag{15}$$

which is equivalent to (14). □

Moreover, the expression (9) contains some other nice evaluations, given in the next example.

Example 4.1. Consider the special cases of (9), providing the following Hankel transform evaluations:

1. $c = 1$. The Hankel transform of $m_{n+1} - m_n$ is $\bar{h}(1) = (0, -1, 0, 1, \ldots)$.
2. $c = 2$. The Hankel transform of $m_{n+1} - 2m_n$ is: $\bar{h}(2) = (-1, 0, 1, -1, 0, 1, \ldots)$.
3. By direct evaluation using (9), it can be shown that $\bar{h}_n(-c) = (-1)^{n+1}\bar{h}_n(2+c)$.
4. Taking a limit of (9) when $c \to -1$ we find that the Hankel transform of $m_{n+1} + m_n$ is given by $\bar{h}_n(-1) = n + 2$. This also follows from [4, Theorem 4.4].
5. Similarly, when $c \to 3$, we find that the Hankel transform of $m_{n+1} - 3m_n$ is given by $\bar{h}_n(-3) = (-1)^{n+1}(n+2)$.

5 Linear combination of three consecutive Motzkin numbers

Let us consider the linear combination of three consecutive Motzkin numbers, i.e., the sequence $m_{n+2} - a \cdot m_{n+1} + b \cdot m_n$, where a and b are arbitrary constants. Denote its Hankel transform by $\hat{h}_n(a, b)$. Theorem 5.1 shows that $\hat{h}_n(a, b)$ satisfies a particular difference equation (as it was the case in the previous section).

Theorem 5.1. *For arbitrary $a, b \in \mathbb{R}$, the Hankel transform $\hat{h}_n(a, b)$ of the sequence*

$$(m_{n+2} - a \cdot m_{n+1} + b \cdot m_n)_{n\in\mathbb{N}_0}$$

satisfies difference equation

$$\begin{aligned} \left(\bar{h}_{n-1}(a,b)\right)^2 \cdot \hat{h}_n(a,b) - \Big[\sqrt{a^2-4b} \cdot \bar{h}_{n-1}(a,b) \cdot \bar{h}_n(a,b) \\ + \left(\bar{h}_{n-1}(a,b)\right)^2 + \left(\bar{h}_n(a,b)\right)^2\Big] \cdot \hat{h}_{n-1}(a,b) + \left(\bar{h}_n(a,b)\right)^2 \cdot \hat{h}_{n-2}(a,b) = 0 \end{aligned} \tag{16}$$

with initial values: $\hat{h}_0(a,b) = 2 - a + b$, $\hat{h}_1(a,b) = 2 - a + 5b - 2ab + b^2$ where $\bar{h}_n(a,b) = \bar{h}_n(c)$ is given by (9) and $c = \frac{a+\sqrt{a^2-4b}}{2}$.

Proof. Given sequence has weight function

$$\hat{w}(x) = \left(x - \frac{a-\sqrt{a^2-4b}}{2}\right) \cdot \left(x - \frac{a+\sqrt{a^2-4b}}{2}\right) \cdot w(x) = \left(x - \frac{a-\sqrt{a^2-4b}}{2}\right) \cdot \bar{w}(x).$$

As in the proof of the previous theorem, we start with the following transformation $\hat{w}(x) = (x-d)\bar{w}(x)$ where $d = \frac{a-\sqrt{a^2-4b}}{2}$, and apply Lemma 3.3. Recall that $\bar{w}(x) = (x-c)w(x)$, as well as $\bar{r}_n$, $\bar{\alpha}_n$ and $\bar{\beta}_n$, are functions of c. By taking $c = \frac{a+\sqrt{a^2-4b}}{2}$, those expressions are now functions of a and b. The coefficients $\hat{\alpha}_n$ and $\hat{\beta}_n$ are given by

$$\hat{\alpha}_n = \bar{\alpha}_{n+1} + \hat{r}_{n+1} - \hat{r}_n = 1 + \bar{r}_{n+2} - \bar{r}_{n+1} + \hat{r}_{n+1} - \hat{r}_n, \qquad n \geq 0, \tag{17}$$

$$\hat{\beta}_0 = \int_{-1}^{3} \hat{w}(x)dx = 2 - a + b, \qquad \hat{\beta}_n = \bar{\beta}_n \cdot \frac{\hat{r}_n}{\hat{r}_{n-1}} = \frac{\bar{r}_n}{\bar{r}_{n-1}} \cdot \frac{\hat{r}_n}{\hat{r}_{n-1}}, \quad n \geq 1, \tag{18}$$

where the sequence $(\hat{r}_n)_{n\in\mathbb{N}_0}$ is determined by the following recurrence relation:

$$\begin{aligned} \hat{r}_0 &= d - \bar{\alpha}_0 = \frac{4-2a+2b}{a-2+\sqrt{a^2-4b}}, \\ \hat{r}_n &= d - \bar{\alpha}_n - \frac{\bar{\beta}_n}{\hat{r}_{n-1}}, \quad n \geq 1. \end{aligned} \tag{19}$$

According to the previous expression and Heilermann formula (4) we have

$$\frac{\hat{h}_{n+1}(a,b)}{\hat{h}_n(a,b)} = \hat{\beta}_0 \cdot \hat{\beta}_1 \cdot \hat{\beta}_2 \cdots \hat{\beta}_{n+1} = (2-a+b) \cdot \frac{\bar{r}_{n+1}}{\bar{r}_0} \cdot \frac{\hat{r}_{n+1}}{\hat{r}_0} = -\frac{\bar{h}_{n+1}(a,b)}{\bar{h}_n(a,b)} \cdot \hat{r}_{n+1}$$

which implies

$$\hat{r}_{n+1} = -\frac{\hat{h}_{n+1}(a,b)}{\hat{h}_n(a,b)} \cdot \frac{\bar{h}_n(a,b)}{\bar{h}_{n+1}(a,b)}.$$

Using the recurrence relation (19), we obtain:

$$\begin{aligned} \hat{r}_n &= d - \bar{\alpha}_n - \frac{\bar{\beta}_n}{\hat{r}_{n-1}} = d - (1 + \bar{r}_{n+1} - \bar{r}_n) - \frac{\frac{\bar{r}_n}{\bar{r}_{n-1}}}{\hat{r}_{n-1}} \\ &= d - (1 + c - \alpha_{n+1} - \frac{\beta_n}{\bar{r}_n} - \bar{r}_n) + \frac{\bar{r}_n}{\bar{r}_{n-1}} \cdot \frac{\bar{h}_{n-1}(a,b) \cdot \hat{h}_{n-2}(a,b)}{\bar{h}_{n-2}(a,b) \cdot \hat{h}_{n-1}(a,b)} \\ &= -\sqrt{a^2-4b} - \frac{\bar{h}_{n-1}(a,b)}{\bar{h}_n(a,b)} - \frac{\bar{h}_n(a,b)}{\bar{h}_{n-1}(a,b)} + \frac{\bar{h}_n(a,b)}{\bar{h}_{n-1}(a,b)} \cdot \frac{\hat{h}_{n-2}(a,b)}{\hat{h}_{n-1}(a,b)} \end{aligned}$$

which implies (16) with the initial values: $\hat{h}_0(a,b) = 2-a+b$ and $\hat{h}_1(a,b) = 2-a+5b-2ab+b^2$. □

Note that in general case, it is difficult to obtain the closed-form solution of the equation (16). Therefore, we consider two special cases.

1. By putting $b=0$ and $a=c$, our sequence reduces to the linear combination of two shifted Motzkin numbers $m_{n+2} - c \cdot m_{n+1} = m^*_{n+1} - c \cdot m^*_n$. Denote the Hankel transform of this sequence by $\bar{h}^*_n(c)$. The difference equation (16) is now reduced to

$$\left[\bar{h}_{n-1}(c)\right]^2 \cdot \bar{h}^*_n(c) - \left[\left(\bar{h}_{n-1}(c)\right)^2 + \left(\bar{h}_n(c)\right)^2 + c \cdot \bar{h}_{n-1}(c) \cdot \bar{h}_n(c)\right] \cdot \bar{h}^*_{n-1}(c) + \left[\bar{h}_n(c)\right]^2 \cdot \bar{h}^*_{n-2}(c) = 0 \tag{20}$$

with initial values: $\bar{h}_0^*(c) = \bar{h}_1^*(c) = 2 - c$. Unfortunately, it is still difficult to find its solution as the closed-form expression.

However, in the special case $c = -1$ it can be proven by mathematical induction that the Hankel transform of $m_{n+2} + m_{n+1}$ is given by

$$\bar{h}_n^*(-1) = \begin{cases} 6k+3, & n = 6k \quad or \quad n = 6k+1 \\ -1, & n = 6k+2 \\ -6(k+1) & n = 6k+3 \quad or \quad n = 6k+4 \\ 1, & n = 6k+5 \end{cases} \tag{21}$$

Recall that we have already shown in the previous section that $\bar{h}_n(-1) = n + 2$.

2. By putting $b = c^2$ and $a = 2c$, our sequence reduces to $m_{n+2} - 2c \cdot m_{n+1} + c^2 \cdot m_n$. Denote its Hankel transform by $\hat{h}_n(c)$, which satisfies the following difference equation

$$\left[\bar{h}_{n-1}(c)\right]^2 \cdot \hat{h}_n(c) - \left[\left(\bar{h}_{n-1}(c)\right)^2 + \left(\bar{h}_n(c)\right)^2\right] \cdot \hat{h}_{n-1}(c) + \left[\bar{h}_n(c)\right]^2 \cdot \hat{h}_{n-2}(c) = 0. \tag{22}$$

with the initial values: $\hat{h}_0(c) = c^2 - 2c + 2$ and $\hat{h}_1(c) = c^4 - 4c^3 + 5c^2 - 2c + 2$. This equation can be solved in closed-form, which is proven by the following theorem.

Theorem 5.2. *For arbitrary $c \in \mathbb{R}$, the Hankel transform $\hat{h}_n(c)$ of the sequence*

$$\left(m_{n+2} - 2c \cdot m_{n+1} + c^2 \cdot m_n\right)_{n \in \mathbb{N}_0}$$

is given by

$$\hat{h}_n(c) = \frac{1}{(c^2 - 2c - 3)^{3/2}} \left[H_1^{2n+5} - H_2^{2n+5}\right] - \frac{5 + 2n}{c^2 - 2c - 3} \tag{23}$$

where

$$H_1 = \frac{1 - c + \sqrt{c^2 - 2c - 3}}{2}, \quad H_2 = \frac{1 - c - \sqrt{c^2 - 2c - 3}}{2}.$$

Proof. Consider the difference equation (22). From here, we conclude that is valid:

$$\begin{aligned} \hat{h}_n(c) - \hat{h}_{n-1}(c) &= \left(\hat{h}_{n-1}(c) - \hat{h}_{n-2}(c)\right) \cdot \frac{\left(\bar{h}_n(c)\right)^2}{\left(\bar{h}_{n-1}(c)\right)^2} \\ &= \left(\hat{h}_1(c) - \hat{h}_0(c)\right) \cdot \frac{\left(\bar{h}_2(c)\right)^2}{\left(\bar{h}_1(c)\right)^2} \cdot \frac{\left(\bar{h}_3(c)\right)^2}{\left(\bar{h}_2(c)\right)^2} \cdots \frac{\left(\bar{h}_n(c)\right)^2}{\left(\bar{h}_{n-1}(c)\right)^2} \\ &= \left(\hat{h}_1(c) - \hat{h}_0(c)\right) \cdot \frac{\left(\bar{h}_n(c)\right)^2}{\left(\bar{h}_1(c)\right)^2} = \left(\bar{h}_n(c)\right)^2. \end{aligned} \tag{24}$$

Furthermore:

$$\begin{aligned} \hat{h}_n(c) &= \hat{h}_{n-1}(c) + \left(\bar{h}_n(c)\right)^2 \\ &= \hat{h}_{n-2}(c) + \left(\bar{h}_{n-1}(c)\right)^2 + \left(\bar{h}_n(c)\right)^2 \\ &= \hat{h}_0(c) + \left(\bar{h}_1(c)\right)^2 + \left(\bar{h}_2(c)\right)^2 + \ldots + \left(\bar{h}_{n-1}(c)\right)^2 + \left(\bar{h}_n(c)\right)^2. \end{aligned} \tag{25}$$

Recall that $\bar{h}_n(c) = D^{-1/2}\left[H_1^{n+2} - H_2^{n+2}\right]$, where $D = c^2 - 2c - 3$ (Theorem 4.1). Direct computation yields to $H_1 H_2 = 1$ and $\bar{h}_n(c)^2 = D^{-1}\left[H_1^{2n+4} + H_2^{2n+4} - 2\right]$. Replacing into (25) we obtain:

$$\hat{h}_n(c) = \hat{h}_0(c) + \frac{1}{D}\left[\frac{H_1^6}{1-H_1^2}(1-H_1^{2n}) + \frac{H_2^6}{1-H_2^2}(1-H_2^{2n}) - 2n\right].$$

The following simplifications can be made also by direct computation:

$$P = \frac{H_1^6}{1-H_1^2} + \frac{H_2^6}{1-H_2^2} = 1 - 2c - 3c^2 + 4c^3 - c^4$$

$$\frac{H_1}{1-H_1^2} = -\frac{1}{\sqrt{D}} \qquad \frac{H_2}{1-H_2^2} = \frac{1}{\sqrt{D}}.$$

Using these simplifications and $\hat{h}_0(c) = c^2 - 2c - 2$ we finally get:

$$\begin{aligned}\hat{h}_n(c) &= \hat{h}_0(c) + \frac{1}{D}\left[P + \frac{1}{\sqrt{D}}\left(H_1^{2n+5} - H_2^{2n+5}\right) - 2n\right] \\ &= \frac{1}{D^{3/2}}\left[H_1^{2n+5} - H_2^{2n+5}\right] - \frac{2n+5}{D}\end{aligned}$$

which completes the proof of the theorem. □

Example 5.1. By taking specific values of c in (23), we obtain the following interesting Hankel transform evaluations:

1. $c = 1$. The Hankel transform of $m_{n+2} - 2m_{n+1} + m_n$ is: $(1, 2, 2, 3, 3, 4, 4, \ldots)$.
2. $c = 2$, $c = 0$. The Hankel transform of m_{n+2} and $m_{n+2} - 4m_{n+1} + 4m_n$ is: $(2, 2, 3, 4, 4, 5, \ldots)$.
3. By direct evaluation using (23), it can be shown that $\hat{h}_n(-c) = \hat{h}_n(2 + c)$.
4. By taking a limit of (23) when $c \to -1$ we find that the Hankel transform of $m_{n+2} + 2m_{n+1} + m_n$ is given by $\hat{h}_n(-1) = (30 + 37n + 15n^2 + 2n^3)/6$. The same sequence is obtained in the case $c \to 3$, i.e. $m_{n+2} - 6m_{n+1} + 9m_n$.

The special case $c = 0$, i.e. the Hankel transform of the shifted Motzkin numbers $m_n^{**} = m_{n+2}$, also follows from [4, Corollary 4.2]. For the sake of completeness, we state it as a separate corollary.

Corollary 5.3. *The Hankel transform of the sequence of shifted Motzkin numbers $(m_n^{**})_{n\in\mathbb{N}_0}$ is given by:*

$$h_n^{**} = \begin{cases} 4k+2, & n = 6k \quad \textit{or} \quad n = 6k+1 \\ 4k+3, & n = 6k+2 \\ 4k+4, & n = 6k+3 \quad \textit{or} \quad n = 6k+4 \\ 4k+5, & n = 6k+5 \end{cases} \qquad (k \in \mathbb{N}_0) \tag{26}$$

Note that the difference equation (22) is similar to the equation (20). However, due to the lack of one addend in the middle term, (22) can be solved analytically.

The following corollary gives the expressions for the three-term recurrence relation coefficients $\hat{\alpha}_n$ and $\hat{\beta}_n$, corresponding to the sequence $m_{n+2} - 2c \cdot m_{n+1} + c^2 \cdot m_n$. These expressions will be further used in the following section.

Corollary 5.4. *Coefficients $\hat{\alpha}_n$ and $\hat{\beta}_n$ are given by:*

$$\begin{aligned}
\hat{\alpha}_n &= 1 + \frac{(\bar{h}_{n+1}(c))^2 - \bar{h}_n(c)\cdot \bar{h}_{n+2}(c)}{\bar{h}_n(c)\cdot \bar{h}_{n+1}(c)} \\
&\qquad + \frac{(\hat{h}_n(c))^2 \cdot \bar{h}_{n-1}(c)\cdot \bar{h}_{n+1}(c) - (\bar{h}_n(c))^2\cdot \hat{h}_{n-1}(c)\cdot \hat{h}_{n+1}(c)}{\bar{h}_n(c)\cdot \bar{h}_{n+1}(c)\cdot \hat{h}_{n-1}(c)\cdot \hat{h}_n(c)} \qquad (27)\\
\hat{\beta}_n &= \frac{\hat{h}_n(c)\cdot \hat{h}_{n-2}(c)}{\left(\hat{h}_{n-1}(c)\right)^2}. \qquad (28)
\end{aligned}$$

6 Another linear combination and Hankel transform of m_{n+3}

Finally, consider the following linear combination of four consecutive Motzkin numbers

$$\breve{m}_n = m_{n+3} - 3c\cdot m_{n+2} + 3c^2\cdot m_{n+1} - c^3\cdot m_n$$

which is the moment sequence of $\breve{w}(x) = (x-c)^3 w(x) = (x-c)\hat{w}(x)$. Denote its Hankel transform by $\breve{h}_n(c)$. Proceeding similarly as in the previous section, we can show that $\breve{h}_n(c)$ satisfies difference equation. This is demonstrated by the following theorem.

Theorem 6.1. *The Hankel transform $\breve{h}_n(c)$ of the sequence*

$$(\breve{m}_n)_{n\in\mathbb{N}_0} = \left(m_{n+3} - 3c\cdot m_{n+2} + 3c^2\cdot m_{n+1} - c^3\cdot m_n\right)_{n\in\mathbb{N}_0}$$

satisfies difference equation

$$\begin{aligned}
&\left[\left(\hat{h}_{n-1}(c)\right)^2\cdot \bar{h}_n(c)\right]\cdot \breve{h}_n(c) - \left[\bar{h}_{n+1}(c)\cdot\left(\hat{h}_{n-1}(c)\right)^2 + \bar{h}_{n-1}(c)\cdot\left(\hat{h}_n(c)\right)^2\right]\cdot \breve{h}_{n-1}(c) \\
&\qquad + \bar{h}_n(c)\cdot\left(\hat{h}_n(c)\right)^2\cdot \breve{h}_{n-2}(c) = 0. \qquad (29)
\end{aligned}$$

with the initial values: $\breve{h}_0(c) = 4-6c+3c^2-c^3$ and $\breve{h}_1(c) = 3-18c+21c^2-20c^3+15c^4-6c^5+c^6$.

Proof. We start with the transformation

$$\breve{w}(x) = (x-c)^3\cdot w(x) = (x-c)\cdot \hat{w}(x)$$

and apply Lemma 3.3. The coefficients $\breve{\alpha}_n$ and $\breve{\beta}_n$ are equal to

$$\breve{\alpha}_n = \hat{\alpha}_{n+1} + \breve{r}_{n+1} - \breve{r}_n, \qquad n \geq 0, \qquad (30)$$

$$\breve{\beta}_0 = \int_{-1}^{3} \breve{w}(x)dx = 4 - 6c + 3c^2 - c^3, \qquad \breve{\beta}_n = \hat{\beta}_n\cdot\frac{\breve{r}_n}{\breve{r}_{n-1}} = \frac{\bar{r}_n}{\bar{r}_{n-1}}\cdot\frac{\hat{r}_n}{\hat{r}_{n-1}}\cdot\frac{\breve{r}_n}{\breve{r}_{n-1}}, \quad n\geq 1. \qquad (31)$$

The sequence $(\breve{r}_n)_{n\in\mathbb{N}_0}$ is determined by

$$\breve{r}_n = c - \hat{\alpha}_n - \frac{\hat{\beta}_n}{\breve{r}_{n-1}}, \quad n \geq 1, \qquad (32)$$

with the initial value equal to

$$r_0 = c - \hat{\alpha}_0 = \frac{c^2 - 2c + (c^2 - 2c + 2)^2}{(c-1)(c^2 - 2c + 2)}.$$

According to the previous expression and the Heilermann formula (4) we have:

$$\frac{\breve{h}_n(c)}{\breve{h}_{n-1}(c)} = -\frac{\hat{h}_n(c)}{\hat{h}_{n-1}(c)} \cdot \breve{r}_n,$$

which implies

$$\breve{r}_n = -\frac{\hat{h}_{n-1}(c) \cdot \breve{h}_n(c)}{\hat{h}_n(c) \cdot \breve{h}_{n-1}(c)}. \tag{33}$$

Now by replacing (33) and (27)-(28) (Corollary 5.4) into (32), we obtain (29). □

As the special case, we give closed-form evaluation of the Hankel transform of $(m_{n+3})_{n\in\mathbb{N}_0}$. This is done by the following theorem.

Theorem 6.2. *The Hankel transform of the sequence* $(m_{n+3})_{n\in\mathbb{N}_0}$ *is given by:*

$$h_n^{***} = \begin{cases} 4(2k+1)^2, & n = 6k \\ (2k+1)(4k+3), & n = 6k+1 \\ -2(k+1)(4k+3), & n = 6k+2 \\ -16(k+1)^2, & n = 6k+3 \\ -2(k+1)(4k+5), & n = 6k+4 \\ (4k+5)(2k+3), & n = 6k+5 \end{cases} \quad (k \in \mathbb{N}_0). \tag{34}$$

Proof. By putting $c = 0$ in (29), we get:

$$\left[\left(h_{n-1}^{**}\right)^2 \cdot h_n^*\right] \cdot h_n^{***} - \left[h_{n+1}^* \cdot \left(h_{n-1}^{**}\right)^2 + h_{n-1}^* \cdot (h_n^{**})^2\right] \cdot h_{n-1}^{***} + h_n^* \cdot (h_n^{**})^2 \cdot h_{n-2}^{***} = 0. \tag{35}$$

Denote by

$$h_{i,k}^{***} = h_{6k+i}^{***}, \qquad i \in \{0, 1, \ldots, 5\}, \quad k \in \mathbb{N}_0. \tag{36}$$

Equation (35) now reduces to:

$$\begin{cases} (4k+2)^2 \cdot h_{2,k}^{***} - (4k+2)^2 \cdot h_{1,k}^{***} + (4k+3)^2 \cdot h_{0,k}^{***} = 0 \\ (4k+3)^2 \cdot h_{3,k}^{***} - (4k+4)^2 \cdot h_{2,k}^{***} + (4k+4)^2 \cdot h_{1,k}^{***} = 0 \\ h_{4,k}^{***} - h_{3,k}^{***} + h_{2,k}^{***} = 0 \\ (4k+4)^2 \cdot h_{5,k}^{***} - (4k+4)^2 \cdot h_{4,k}^{***} + (4k+45)^2 \cdot h_{3,k}^{***} = 0 \\ (4k+5)^2 \cdot h_{0,k+1}^{***} - (4k+6)^2 \cdot h_{5,k}^{***} + (4k+6)^2 \cdot h_{4,k}^{***} = 0 \\ h_{1,k+1}^{***} - h_{0,k+1}^{***} + h_{5,k}^{***} = 0 \end{cases} \tag{37}$$

Recall that expressions for h_n^* and h_n^{**} are given by (14) and (26) respectively. The solution of the previous system is given by

$$h_{0,k}^{***} = 4(2k+1)^2, \qquad h_{1,k}^{***} = (2k+1)(4k+3), \qquad h_{2,k}^{***} = -2(k+1)(4k+3),$$
$$h_{3,k}^{***} = -16(k+1)^2, \qquad h_{4,k}^{***} = -2(k+1)(4k+5), \qquad h_{5,k}^{***} = (2k+3)(4k+5).$$

which can be proved by mathematical induction. □

7 Summary

At the end of the paper, we summarize the new Hankel transform evaluations in the following table.

Sequence	Dif. eq.	Closed-form expr.
$m_{n+1} - c \cdot m_n$	(13)	(9)
$m_{n+2} - a \cdot m_{n+1} + b \cdot m_n$	(16)	(spec. cases)
$m_{n+2} - c \cdot m_{n+1}$	(20)	(21), for $c = -1$
$m_{n+2} - 2c \cdot m_{n+1} + c^2 \cdot m_n$	(22)	(23)
$m_{n+3} - 3c \cdot m_{n+2} + 3c^2 \cdot m_{n+1} - c^3 \cdot m_n$	(29)	(spec. cases)
m_{n+3}	(35)	(34)

Hankel transform evaluation of the general form of the second, third and fifth sequence are left as the open problems.

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