On the diameter of total domination vertex critical graphs

Tao Wang [a, b, *]
[a]Institute of Applied Mathematics
Henan University, Kaifeng, 475004, P. R. China
[b]College of Mathematics and Information Science
Henan University, Kaifeng, 475004, P. R. China

December 2, 2013

Abstract

In this paper, we consider various types of domination vertex critical graphs, including total domination vertex critical graphs and independent domination vertex critical graphs and connected domination vertex critical graphs. We provide upper bounds on the diameter of them, two of which are sharp.

MSC (2010): 05C12, 05C69

Keywords: total domination vertex critical graphs; independent domination vertex critical graphs; connected domination vertex critical graphs

1 Introduction

All graphs considered here are finite, undirected and simple. Let G be a graph with vertex set V and edge set E. The *neighborhood* of a vertex v in a graph G, denoted by $N_G(v)$, is the set of all the vertices adjacent to the vertex v, i.e., $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* of a vertex v in G, denoted by $N_G[v]$, is defined by $N_G[v] = N_G(v) \cup \{v\}$. A vertex of degree one is called a *leaf vertex*, the edge connected to that vertex is called a *pendant edge* and the only neighbor of a leaf vertex is called a *support vertex*. We denote the distance between u and v in G by $\mathrm{dist}_G(u, v)$, and denote the diameter of G by $\mathrm{diam}(G)$. The *degree* of a vertex v in G, denoted by $\deg(v)$, is the number of incident edges of G. A vertex of degree k is called a k-vertex, and a vertex of degree at most or at least k is call a k^-- or k^+-vertex, respectively.

A vertex subset $S \subseteq V$ is called a *dominating set* of a graph G if every vertex in V is an element of S or is adjacent to a vertex in S. The *domination number* of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A graph is *domination vertex critical* if the removal of any vertex decreases its domination number. If G is domination vertex critical and $\gamma(G) = k$, we say that G is a k-γ-vertex-critical graph.

A vertex subset $S \subseteq V$ is a *total dominating set* of a graph G if every vertex in V is adjacent to a vertex in S. Every graph without isolated vertices has a total dominating set, since V is such a set. The *total domination number* of a graph G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A graph is *total domination vertex critical* if the removal of any vertex that is not adjacent to a vertex of degree one decreases its total domination number. If G is total domination vertex critical and $\gamma_t(G) = k$, we say that G is a k-γ_t-vertex-critical graph.

A vertex subset $S \subseteq V$ is an *independent dominating set* of a graph G if it is a dominating set and it is also an independent set in G. Equivalently, an independent dominating set is a maximal independent set. The *independent domination number* of a graph G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set of G. A graph is *independent domination vertex critical* if the removal of any vertex decreases its independent domination number. If G is independent domination vertex critical and $i(G) = k$, we say that G is a k-i-vertex-critical graph.

A vertex subset $S \subseteq V$ is a *connected dominating set* of a graph G if it is a dominating set of G and the subgraph induced by S is connected. Every connected graph has a connected dominating set, since V is such a

*Corresponding author: wangtao@henu.edu.cn

set. The *connected domination number* of a graph G, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set of G. A graph is *connected domination vertex critical* if the removal of any vertex decreases its connected domination number. If G is connected domination vertex critical and $\gamma_c(G) = k$, we say that G is a k-γ_c-vertex-critical graph. A necessary condition for a graph to be k-γ_c-vertex-critical is 2-connected. For more details on connected domination vertex critical graphs, see [6].

The total domination vertex critical graphs were first investigated by Goddard et al. [4] and the independent domination vertex critical graphs were studied by Ao [1].

Goddard et al. [4] characterized the class of k-γ_t-vertex-critical graphs with leaf vertices.

Theorem 1.1. *Let G be a connected graph of order at least three with at least one leaf vertex. Then G is k-γ_t-vertex-critical if and only if $G = \mathrm{cor}(H)$ for some connected graph H of order k with $\delta(H) \geq 2$.*

For the connected k-γ_t-vertex-critical graph without leaf vertices, they gave an upper bound on the diameter.

Theorem 1.2 (Goddard et al. [4])**.** *If G is a connected k-γ_t-vertex-critical graph without leaf vertices, then* $\mathrm{diam}(G) \leq 2k - 3$.

The following observation is used frequently, we present it here.

Observation 1. *If D is a total dominating set of a graph G, then for every vertex v in G, the set D contains a neighbor of v.*

Lemma 1. *If G is a k-γ_t-vertex-critical graph without leaf vertices, then for any vertex w, there exists a minimum total dominating set of G containing w, and $\gamma_t(G - w) = \gamma_t(G) - 1$.*

Proof. Let v be a neighbor of w in G, and let D be a minimum total dominating set of $G - v$. It follows that $w \notin D$ and $D \cap N_G(w) \neq \emptyset$, thus $D \cup \{w\}$ is a total dominating set of G. Furthermore, we have that $|D \cup \{w\}| = |D| + 1 \leq k$, and then $D \cup \{w\}$ is a minimum total dominating set of G containing w and $\gamma_t(G - w) = |D| = \gamma_t(G) - 1$. □

Lemma 2. *If G is a k-i-vertex-critical graph, then for any vertex v, there exists a minimum independent dominating set of G containing v, and $i(G - v) = i(G) - 1$.*

The method developed in [3] is a powerful technique to obtain sharp upper bounds on various types of domination vertex critical graphs, it has been used on the k-γ-vertex-critical graphs [3] and paired domination vertex critical graphs [5].

Edwards and MacGillivray [2] presented better upper bounds on the diameter of total domination and independent domination vertex critical graphs, but the proofs have big gaps (the gaps have been confirmed by Edwards through personal email). In this paper, we also adopt the same technique in [2, 3, 5] to obtain sharp upper bounds on the diameter, one of which is a slightly improvement on a result in [2].

2 Upper bounds on the diameter

Theorem 2.1. *If G is a connected k-γ_t-vertex-critical graph without leaf vertices and $k \geq 4$, then* $\mathrm{diam}(G) \leq \frac{5k-7}{3}$.

Proof. Let x and x_n be vertices such that $\mathrm{dist}(x, x_n) = \mathrm{diam}(G) = n$. If $n \leq 4$, then we are done. So we may assume that $n \geq 5$. Let $xx_1 \dots x_{n-1}x_n$ be a shortest path between x and x_n. Define $L_0, L_1, \dots, L_n$ by $L_i = \{v \in V(G) \mid \mathrm{dist}_G(x, v) = i\}$ for $0 \leq i \leq n$. In particular, $L_0 = \{x\}$ and $L_1 = N_G(x)$. Let $R_i = L_0 \cup L_1 \cup \cdots \cup L_i$ for $0 \leq i \leq n$. Let D be a minimum total dominating set in G. If $|D \cap R_j| \geq \frac{3j+10}{5}$, then we say that R_j is *sufficient with respect to D*.

Let m be the maximum integer j such that $|D \cap R_j| \geq \frac{3j+10}{5}$. Notice that the value of m depends on the minimum total dominating set D, we may assume that D is chosen such that m is maximum among all the minimum total dominating set.

Firstly, we must show the existence of m. Let D_1 be a minimum total dominating set of $G - x_1$. It is obvious that $x \notin D_1$ and $D_1 \cap L_1 \neq \emptyset$ and $|D_1 \cap (L_1 \cup L_2)| \geq 2$. Suppose that the value of m does not exist, it follows that $1 + |D_1 \cap R_j| < \frac{3j+10}{5}$, otherwise R_j is sufficient with respect to $D_1 \cup \{x\}$. Hence, we have that $|D_1 \cap L_1| = 1$ and $|D_1 \cap (L_1 \cup L_2)| < 2.2$. In fact $|D_1 \cap L_2| = 1$. From the fact that $|D_1 \cap R_3| < 2.8$, we have that $D_1 \cap L_3 = \emptyset$. If $D_1 \cap L_4 \neq \emptyset$, then we can conclude that $|D_1 \cap (L_4 \cup L_5)| \geq 2$ from the fact that D_1 is a total dominating set of

$G - x_1$, and then R_5 is sufficient with respect to $D_1 \cup \{x\}$, a contradiction. So we may assume that $D_1 \cap L_4 = \emptyset$. Let D_0 be a minimum total dominating set of $G - x_4$. If $|D_0 \cap R_3| \geq 3$, then R_3 is sufficient with respect to $D_0 \cup \{x_3\}$, a contradiction. Thus, we have $|D_0 \cap R_3| = 2$ and $D_0 \cap L_3 = \emptyset$. If $D_0 \cap L_4 = \emptyset$, then $D_0 \cap R_3$ totally dominates R_3, and then $(D_0 \cap R_3) \cup (D_1 \setminus R_3)$ is a smaller total dominating set of G, a contradiction. Hence we have that $D_0 \cap L_4 \neq \emptyset$ and $|D_0 \cap (L_4 \cup L_5)| \geq 2$. Therefore, we have that $|D_0 \cap R_5| \geq 4$ and the set R_5 is sufficient with respect to $D_0 \cup \{x_3\}$, which leads to a contradiction.

Now, we know that the value of m must exist. If $m = n$, then $n = m \leq \frac{5k-10}{3} \leq \frac{5k-7}{3}$, we are done. So we may assume that $m < n$.

If $m = 5t+2$, then $|D \cap R_m| \geq 3t + 3.2$ and $|D \cap R_{m+1}| < 3t + 3.8$, which is a contradiction. If $m = 5t+4$, then $|D \cap R_m| \geq 3t + 4.4$ and $|D \cap R_{m+1}| < 3t + 5$, which is also a contradiction. So we have that $m = 5t, 5t+1$ or $5t+3$. We further assume that $m + 2 \leq n$.

If $m = 5t$, then $|D \cap R_m| \geq 3t+2$ and $|D \cap R_{m+1}| < 3t+2.6$, which implies that $|D \cap R_m| = 3t+2$ and $D \cap L_{m+1} = \emptyset$. From the fact that $|D \cap R_{m+2}| < 3t + 3.2$ and D is a total dominating set and $|D \cap R_{m+3}| < 3t + 3.8$ (if L_{m+3} exists), we can conclude that $D \cap L_{m+2} = \emptyset$ and $|D \cap L_{m+3}| = 1$. Consequently, the set L_{m+4} exists and $D \cap L_{m+3}$ dominates L_{m+2}. But $|D \cap R_{m+4}| < 3t + 4.4$, so we have that $|D \cap L_{m+4}| = 1$.

If $m = 5t + 1$, then $|D \cap R_m| \geq 3t + 2.6$, $|D \cap R_{m+1}| < 3t + 3.2$ and $|D \cap R_{m+2}| < 3t + 3.8$, which implies that $|D \cap R_m| = 3t + 3$ and $D \cap L_{m+1} = D \cap L_{m+2} = \emptyset$. In order to dominate L_{m+2}, the set L_{m+3} exists and $D \cap L_{m+3}$ dominates L_{m+2}. But $|D \cap R_{m+3}| < 3t + 4.4$, so we have that $|D \cap L_{m+3}| = 1$. The set D totally dominates G, it follows that L_{m+4} exists and $D \cap L_{m+4} \neq \emptyset$. Hence $|D \cap R_{m+4}| \geq 3t + 5$ and R_{m+4} is sufficient with respect to D, a contradiction to the maximality of m.

If $m = 5t + 3$, then $|D \cap R_m| \geq 3t + 3.8$, $|D \cap R_{m+1}| < 3t + 4.4$ and $|D \cap R_{m+2}| < 3t + 5$, which implies that $|D \cap R_m| = 3t + 4$ and $D \cap L_{m+1} = D \cap L_{m+2} = \emptyset$. In order to dominate L_{m+2}, the set L_{m+3} exists and $D \cap L_{m+3}$ dominates L_{m+2}. But $|D \cap R_{m+3}| < 3t + 5.6$, so we have that $|D \cap R_{m+3}| = 1$. Since D is a total dominating set in G, it follows that L_{m+4} exists and $D \cap L_{m+4} \neq \emptyset$, but with $|D \cap R_{m+4}| < 3t + 6.2$, we have that $|D \cap L_{m+4}| = 1$.

By the above arguments, we may assume that $D \cap L_{m+1} = D \cap L_{m+2} = \emptyset$ and $|D \cap L_{m+3}| = |D \cap L_{m+4}| = 1$, where $m = 5t$ or $m = 5t + 3$. Without loss of generality, we assume that $D \cap L_{m+3} = \{x_{m+3}\}$ and $D \cap L_{m+4} = \{x_{m+4}\}$. Let D_3 and D_4 be a minimum total dominating set of $G - x_{m+3}$ and $G - x_{m+4}$, respectively.

Recall that the vertex x_{m+3} dominates L_{m+2}, then $D_3 \cap L_{m+2} = \emptyset$ and $D_3 \cap R_{m+1}$ totally dominates R_{m+1}. If $|D_3 \cap R_{m+1}| < |D \cap R_{m+1}|$, then $(D_3 \cap R_{m+1}) \cup (D \setminus R_{m+1})$ is a smaller total dominating set in G, which leads to a contradiction. If $|D_3 \cap R_{m+1}| > |D \cap R_{m+1}|$, then R_{m+1} is sufficient with respect to the minimum total dominating set $D_3 \cup \{x_{m+4}\}$. Hence we have that $|D_3 \cap R_{m+1}| = |D \cap R_{m+1}|$. Notice that maybe L_{m+5} does not exist, if this happens, then we view L_{m+5} as an empty set. If $|D_3 \cap (L_{m+3} \cup L_{m+4} \cup L_{m+5})| \geq 2$, then $|(D_3 \cup \{x_{m+4}\}) \cap R_{m+5}| \geq |D \cap R_{m+1}| + 3$, and then R_{m+5} (or R_{m+4} if L_{m+5} does not exist) is sufficient with respect to $D_3 \cup \{x_{m+4}\}$, which contradicts the maximality of m. Hence, we have that $|D_3 \cap (L_{m+3} \cup L_{m+4} \cup L_{m+5})| \leq 1$, which implies that L_{m+5} exists and $D_3 \cap L_{m+4} = \emptyset$ and $L_{m+3} = \{x_{m+3}\}$.

Notice that $D_4 \cap L_{m+3} = \emptyset$ and $D_4 \cap R_{m+2}$ totally dominates R_{m+2}. If $|D_4 \cap R_{m+2}| < |D \cap R_{m+2}|$, then $(D_4 \cap R_{m+2}) \cup (D \setminus R_{m+2})$ is a smaller total dominating set of G, which leads to a contradiction. If $|D_4 \cap R_{m+2}| > |D \cap R_{m+2}|$, then $|(D_4 \cup \{x_{m+3}\}) \cap R_{m+3}| \geq |D \cap R_m| + 2$, and then R_{m+3} is sufficient with respective to $D_4 \cup \{x_{m+3}\}$, which leads to a contradiction. Hence, we have that $|D_4 \cap R_{m+2}| = |D \cap R_{m+2}|$.

If $D_4 \cap L_{m+2} \neq \emptyset$, then $(D_4 \cap R_{m+2}) \cup (D_3 \setminus R_{m+2})$ is a smaller total dominating set of G, a contradiction. It follows that $D_4 \cap L_{m+2} = \emptyset$. In order to dominate the vertex x_{m+3}, we must have that $D_4 \cap L_{m+4} \neq \emptyset$. Hence, we can conclude that $|D_4 \cap (L_{m+4} \cup L_{m+5})| \geq 2$ and R_{m+5} is sufficient with respect to $D_4 \cup \{x_{m+3}\}$, a contradiction.

Finally, we have to deal with the case that $m = n - 1$. Recall that m is the maximum integer j such that $|D \cap R_j| \geq \frac{3j+10}{5}$, it follows that $D \cap L_{m+1} = D \cap L_n = \emptyset$, and then $|D \cap R_m| = k$ and $n = m + 1 \leq \frac{5k-10}{3} + 1 = \frac{5k-7}{3}$. □

The *coalescence* of two graphs G_1 and G_2 with respect to a vertex x in G_1 and a vertex y in G_2, is the graph $G_1(x * y)G_2$ obtained by identifying x and y; in other words, replacing the vertices x and y by a new vertex w adjacent to the same vertices in G_1 as x and the same vertices in G_2 as y. If there is no confusion, then we write $G_1 * G_2$ instead of $G_1(x * y)G_2$.

Theorem 2.2. *If G is a connected k-i-vertex-critical graph, then* $\operatorname{diam}(G) \leq 2(k-1)$.

Proof. Let x and x_n be vertices such that $\operatorname{dist}(x, x_n) = \operatorname{diam}(G) = n$. Let $xx_1 \ldots x_{n-1}x_n$ be a shortest path between x and x_n. Define $L_0, L_1, \ldots, L_n$ by $L_i = \{v \in V(G) \mid \operatorname{dist}_G(x, v) = i\}$ for $0 \leq i \leq n$. In particular, $L_0 = \{x\}$ and

$L_1 = N_G(x)$. Let $R_i = L_0 \cup L_1 \cup \cdots \cup L_i$ for $0 \le i \le n$. Let D be a minimum independent dominating set in G. If $|D \cap R_j| \ge \frac{j+2}{2}$, then we say that R_j is *sufficient with respect to D*.

Let m be the maximum integer j such that $|D \cap R_j| \ge \frac{j+2}{2}$. The value of m depends on the minimum independent dominating set D, we may assume that D is chosen such that m is maximum among all the minimum independent dominating set. Let D_1 be a minimum independent dominating set of $G - x_1$. It is obvious that $x \notin D_1$ and $D_1 \cap L_1 \ne \emptyset$. Thus $D_1 \cup \{x_1\}$ is a minimum independent dominating set of G with $|(D_1 \cup \{x_1\}) \cap R_1| \ge 2$, and then the value of m exists and $m \ge 1$. If $m = n$, then $n = m \le 2(k-1)$, we are done. So we may assume that $m < n$.

If $m = 2t+1$, then $|D \cap R_m| \ge t + 1.5$ and $|D \cap R_{m+1}| < t+2$, which is a contradiction. So we have that $m = 2t$. We further assume that $m+2 \le n$. It follows that $|D \cap R_m| \ge t+1$ and $|D \cap R_{m+1}| < t+1.5$ and $|D \cap R_{m+2}| < t+2$, and then $|D \cap R_m| = t+1$ and $D \cap L_{m+1} = D \cap L_{m+2} = \emptyset$. In order to dominate L_{m+2}, the set L_{m+3} must exist and $D \cap L_{m+3}$ dominates L_{m+2}. The fact that $|D \cap R_{m+3}| < t + 2.5$ implies that $|D \cap L_{m+3}| = 1$. Let $D \cap L_{m+3} = \{w\}$. Notice that if L_{m+4} exists, we can conclude that $D \cap L_{m+4} = \emptyset$ from the fact that $|D \cap R_{m+4}| < t+3$. Hence, the vertex w dominates $L_{m+2} \cup L_{m+3}$.

Let D_3 be a minimum independent dominating set of $G - w$. Notice that $D_3 \cap (L_{m+2} \cup L_{m+3}) = \emptyset$. If $|D_3 \cap R_{m+1}| > |D \cap R_{m+1}|$, then R_{m+1} is sufficient with respect to $D_3 \cup \{w\}$. If $|D_3 \cap R_{m+1}| < |D \cap R_{m+1}|$, then $(D_3 \cap R_{m+1}) \cup (D \setminus R_{m+1})$ is a smaller independent dominating set of G, which is a contradiction. Hence, we have that $|D_3 \cap R_{m+1}| = |D \cap R_{m+1}|$.

Suppose that the set L_{m+4} does not exist. It implies that $|D \cap R_m| = k-1 = t+1$. Recall that w dominates $L_{m+2} \cup L_{m+3}$, it follows that $D_3 \subseteq R_{m+1}$ and $L_{m+3} = \{w\}$. Let D_2 be a minimum independent dominating set of $G - x_{m+2}$. Therefore, the set $D_2 \cup \{x_{m+2}\}$ is a minimum independent dominating set with $|(D_2 \cup \{x_{m+2}\}) \cap R_{m+2}| = k = t+2$, thus R_{m+2} is sufficient with respect to $D_2 \cup \{x_{m+2}\}$, which is a contradiction. So we may assume that L_{m+4} exists.

If $|D_3 \cap (L_{m+3} \cup L_{m+4})| \ge 1$, then R_{m+4} is sufficient with respect to $D_3 \cup \{w\}$, which leads to a contradiction. So we have that $D_3 \cap (L_{m+3} \cup L_{m+4}) = \emptyset$ and $L_{m+3} = \{w\}$. Let D_4 be a minimum independent dominating set of $G - x_{m+4}$. Notice that $D_4 \cap L_{m+3} = \emptyset$ and $D_4 \cap R_{m+2}$ totally dominates R_{m+2}. If $|D_4 \cap R_{m+2}| > |D \cap R_{m+2}|$, then R_{m+2} is sufficient with respect to $D_4 \cup \{x_{m+4}\}$.

If $|D_4 \cap R_{m+2}| \le |D \cap R_{m+2}|$ and $D_4 \cap L_{m+2} \ne \emptyset$, then $(D_4 \cap R_{m+2}) \cup (D_3 \setminus R_{m+3})$ is a smaller independent dominating set of G, a contradiction.

If $|D_4 \cap R_{m+2}| = |D \cap R_{m+2}|$ and $D_4 \cap L_{m+2} = \emptyset$, then $D_4 \cap L_{m+4} \ne \emptyset$ in order to dominates w, and then R_{m+4} is sufficient with respect to $D_4 \cup \{x_{m+4}\}$.

If $|D_4 \cap R_{m+2}| < |D \cap R_{m+2}|$ and $D_4 \cap L_{m+2} = \emptyset$, then $(D_4 \cap R_{m+2}) \cup (D \setminus R_{m+2})$ is a smaller independent dominating set of G, a contradiction.

By the above arguments, the theorem is true except the case that $m = 2t = n-1$. Notice that $G(x * x)G$ is a $(2k-1)$-i-vertex-critical graph with diameter $2n$. The theorem is true for the graph $G(x * x)G$, it implies that $2n \le 2(2k-1-1)$, thus $n \le 2(k-1)$. □

Theorem 2.3. *If G is a k-γ_c-vertex-critical graph, then* $\mathrm{diam}(G) \le k$.

Proof. Let x and x_n be vertices such that $\mathrm{dist}(x, x_n) = \mathrm{diam}(G) = n$. Let $xx_1 \ldots x_{n-1}x_n$ be a shortest path between x and x_n. Define $L_0, L_1, \ldots, L_n$ by $L_i = \{v \in V(G) \mid \mathrm{dist}_G(x, v) = i\}$ for $0 \le i \le n$. In particular, $L_0 = \{x\}$ and $L_1 = N_G(x)$. Let D_1 be a minimum connected dominating set of $G - x_1$. It is obviously that $x \notin D_1$ and $D_1 \cap L_1 \ne \emptyset$. Since D_1 is a connected dominating set of G, it follows that $D_1 \cap L_i \ne \emptyset$ for every $1 \le i \le n-1$. Hence we have that $|D_1| = k-1 \ge n-1$, which implies that $\mathrm{diam}(G) = n \le k$. □

3 Sharpness of the upper bounds

We characterize when the coalescence of two total domination vertex critical graphs is still a total domination vertex graph.

Theorem 3.1. *Let G_1 and G_2 be k_1-γ_t-vertex-critical and k_2-γ_t-vertex-critical graphs without leaf vertices, respectively. Let x and y be two vertices in G_1 and G_2, respectively. Then $G_1(x * y)G_2$ is $(k_1 + k_2 - 1)$-γ_t-vertex-critical if and only if $\gamma_t(G_2 - N_{G_2}[y]) \ge k_2 - 1$ and $\gamma_t(G_1 - N_{G_1}[x]) \ge k_1 - 1$.*

Proof. Denote the graph $G_1(x * y)G_2$ by G for short. Let D be a minimum total dominating set of G and w be the new created vertex in G. Let D_1 and D_2 be a minimum total dominating set of $G_1 - x$ and $G_2 - y$, respectively. Thus

$|D_1| = k_1 - 1$ and $|D_2| = k_2 - 1$. It is obvious that $\gamma_t(G - w) = k_1 + k_2 - 2$. For any vertex $v \in V(G_1) \setminus \{x\}$, the union of D_2 and a minimum total dominating set of $G_1 - v$ is a total dominating set of $G - v$, thus $\gamma_t(G - v) \leq k_1 + k_2 - 2$. Similarly, for any vertex $v \in V(G_2) \setminus \{y\}$, the union of D_1 and a minimum total dominating set of $G_2 - v$ is a total dominating set of $G - v$, and then $\gamma_t(G - v) \leq k_1 + k_2 - 2$. Hence, for any vertex v in $V(G)$, we have that $\gamma_t(G - v) \leq k_1 + k_2 - 2$.

($\Longleftarrow$) Suppose that $\gamma_t(G_2 - N_{G_2}[y]) \geq k_2 - 1$ and $\gamma_t(G_1 - N_{G_1}[x]) \geq k_1 - 1$. We want to prove $\gamma_t(G) \geq k_1 + k_2 - 1$.

Notice that either $D \cap V(G_1)$ totally dominates G_1 or $D \cap V(G_2)$ totally dominates G_2. By symmetry, we may assume that $D \cap V(G_1)$ totally dominates G_1 and $|D \cap V(G_1)| \geq k_1$. If $w \notin D$, then $D \cap V(G_2)$ totally dominates $G_2 - y$ and $|D \cap V(G_2)| \geq k_2 - 1$, and then $|D| \geq k_1 + k_2 - 1$. So we may assume that $w \in D$. If $D \cap N_{G_2}(y) \neq \emptyset$, then $D \cap V(G_2)$ is a total dominating set of G_2 and $|D \cap V(G_2)| \geq k_2$, and then $|D| \geq k_1 + k_2 - 1$. If $D \cap N_{G_2}(y) = \emptyset$, then $D \setminus V(G_1) \subseteq V(G_2) \setminus N_{G_2}[y]$ and $D \setminus V(G_1)$ totally dominates $G_2 - N_{G_2}[y]$, and then $|D \setminus V(G_1)| \geq k_2 - 1$ and $|D| \geq k_1 + k_2 - 1$.

($\Longrightarrow$) Suppose that $|D| = k_1 + k_2 - 1$. We want to prove that $\gamma_t(G_2 - N_{G_2}[y]) \geq k_2 - 1$ and $\gamma_t(G_1 - N_{G_1}[x]) \geq k_1 - 1$.

By Lemma 1, let D_1^* be a minimum total dominating set of G_1 containing x. It follows that $\gamma_t(G_2 - N_{G_2}[y]) \geq k_2 - 1$; otherwise, the union of D_1^* and a minimum total dominating set of $G_2 - N_{G_2}[y]$ is a smaller total dominating set of G, a contradiction. Similarly, we can prove that $\gamma_t(G_1 - N_{G_1}[x]) \geq k_1 - 1$. □

Remark 1. From the characterization, the graph $C_6 * C_6$ is not a total domination vertex critical graph as mentioned in [2].

A *pointed graph* is a graph with two assigned diametrical vertices called Left and Right. For a pointed graph G, we define $L_k(G)$ and $R_k(G)$ be the set of vertices which are distance k from the Left-vertex and Right-vertex, respectively.

For two pointed graphs G_1 and G_2, we define $G_1 \bullet G_2$ as the pointed graph obtained by identifying and unassigning the Right-vertex from G_1 and the Left-vertex from G_2.

Let $K_{m,m}$ be a complete bipartite graph with bipartition $\{y_1, y_3, \ldots, y_{2m-1}\}$ and $\{y_2, y_4, \ldots, y_{2m}\}$, where $m \geq 2$. Let F be the graph obtained from $K_{m,m}$ by removing one edge $y_1 y_{2m}$, and let $\bar{F}$ be the complement of F with x_i corresponding to y_i. Notice that $\gamma_t(F) = \gamma_t(\bar{F}) = 2$ and $\{x_1, x_{2m}\}$ totally dominates $\bar{F}$ and every pair of adjacent vertices in $K_{m,m}$ totally dominates $K_{m,m}$. Let R be the pointed graph obtained from the disjoint union of $\bar{F}$ and $K_{m,m}$, by joining every vertex of $\bar{F}$ to every vertex of $K_{m,m}$ except edges between the corresponding vertices, and adding five new vertices z_1, z_2, z_3, Left and Right such that Left is adjacent to every vertex in $\bar{F}$, the vertex z_1 is adjacent to $\{x_1, x_2, \ldots, x_{2m-1}\} \cup \{y_2, y_3, \ldots, y_{2m-1}\}$, the vertex z_2 is adjacent to $\{x_2, x_3, \ldots, x_{2m}\} \cup \{y_2, y_3, \ldots, y_{2m-1}\}$, the vertex z_3 is adjacent to every vertex in $K_{m,m}$ and z_1, while Right is adjacent to every vertex in $K_{m,m}$ and z_2.

Theorem 3.2. *The graph R is 3-γ_t-vertex-critical graph with diameter three.*

Let H be a graph with at least four vertices. Let $V(H) = \{x_1, \ldots, x_t\}$ and $V(\bar{H}) = \{y_1, \ldots, y_t\}$ with x_i corresponding to y_i. Let A be the pointed graph obtained by joining every vertex of H to every vertex of $\bar{H}$ except edges between the corresponding vertices, and adding two new vertices Left and Right such that Left is adjacent to every vertex in H and Right is adjacent to every vertex in $\bar{H}$. It can be shown that A is a 3-γ_t-vertex-critical graph if and only if $\gamma_t(H) = \gamma_t(\bar{H}) = 2$. Simply write the Left-vertex as x and Right-vertex as y. Suppose that $\gamma_t(H) = \gamma_t(\bar{H}) = 2$. A minimum total dominating set of H totally dominates $A - y$ and a minimum total dominating set of $\bar{H}$ totally dominates $A - x$. For any vertex x_i, the two vertices y_i and a nonadjacent vertex x_j of x_i totally dominates $A - x_i$; similarly, for any vertex y_i, the two vertices x_i and a nonadjacent vertex y_j of y_i totally dominates $A - y_i$. But $\gamma_t(A) > 2$, thus A is a 3-γ_t-vertex-critical graph. Conversely, if G is a 3-γ_t-vertex-critical graph, then a minimum total dominating set of $A - y$ is also a minimum total dominating set of H and a minimum total dominating set of $A - x$ is also a minimum total dominating set of $\bar{H}$, and then $\gamma_t(H) = \gamma_t(\bar{H}) = 2$. In what follows, we assume that $\gamma_t(H) = \gamma_t(\bar{H}) = 2$. Notice that $\mathrm{diam}(A) = 3$.

Remark 2. For every $t \geq 4$, we can find at least one graph H on t vertices with $\gamma_t(H) = 2$ and $\gamma_t(\bar{H}) = 2$. For instance, let K_{t-2} be a complete graph on $t - 2$ vertices, and let H be the graph on t vertices obtained from K_{t-2} by attaching a path xx_1x_2. It is easy to check that $\gamma_t(H) = 2$ and $\gamma_t(\bar{H}) = 2$.

Let Q be the pointed graph obtained from two copies of A, called A_1 and A_2, by deleting the Right-vertex y from A_1 and the Left-vertex x from A_2, and joining every neighbor of y in A_1 to every neighbor of x in A_2. Notice that $\mathrm{diam}(Q) = 5$ and $\gamma_t(Q) = 4$. By Theorem 2.1, the graph Q is not a 4-γ_t-vertex-critical graph. Let $Q^{(1)} = Q$ and $Q^{(n)} = Q^{(n-1)} \bullet Q$. We simple denote $R \bullet Q^{(n)}$ by $\mathfrak{C}_n$.

Let J_1 and J_3 be disjoint union of tK_2 and let J_2 be $t\bar{K}_2$, where $t \geq 2$. Let J be the pointed graph obtained from $J_1 \cup J_2 \cup J_3$ by joining every vertex of J_1 to every vertex of J_2 except the edges corresponding vertices in J_1 and J_2, similarly, joining every vertex of J_2 to every vertex of J_3 except the edges corresponding vertices in J_2 and J_3, adding a new Left vertex x adjacent to every vertex of J_1 and adding a new Right vertex y adjacent to every vertex in J_3. It is easy to check that J is a 4-γ_t-vertex-critical graph with diameter 4.

Theorem 3.3. *(a)* $\gamma_t(R \bullet Q^{(n)}) \geq 3n + 3$*; (b)* $\gamma_t(R \bullet Q^{(n)} - y) \geq 3n + 2$*; (c)* $\gamma_t(R \bullet Q^{(n)} - N[y]) \geq 3n + 2$.

Proof. We prove the results by mathematical induction.

Basis step: If $n = 0$, then the results are trivially true.

Inductive step: Suppose that the results are true for all values less than n. Let D, D_1 and D_2 be a minimum total dominating set of $\mathfrak{C}_n$, $\mathfrak{C}_n - y$ and $\mathfrak{C}_n - N[y]$, respectively. Denote the Left vertex of Q_n by x and the Right vertex of Q_n by y. If $D \cap V(\mathfrak{C}_{n-1})$ totally dominates $\mathfrak{C}_{n-1}$, then $|D \cap V(\mathfrak{C}_{n-1})| \geq 3n$, but $|D \setminus V(\mathfrak{C}_{n-1})| \geq 3$, thus $|D| \geq 3n + 3$. So we may assume that $D \cap V(\mathfrak{C}_{n-1})$ does not totally dominates $\mathfrak{C}_{n-1}$. Notice that $D \cap V(Q_n)$ must totally dominate Q_n and $|D \cap V(Q_n)| \geq 4$. If $x \notin D$, then $D \cap R_1(Q_{n-1}) = \emptyset$ and $D \cap V(\mathfrak{C}_{n-1})$ totally dominates $\mathfrak{C}_{n-1} - x$ and $|D \cap V(\mathfrak{C}_{n-1})| \geq 3n - 1$, thus $|D| \geq 3n - 1 + 4 = 3n + 3$. If $x \in D$, then $D \cap R_1(Q_{n-1}) = \emptyset$ and $D \cap V(\mathfrak{C}_{n-1} - x)$ totally dominates $\mathfrak{C}_{n-1} - N[x]$ and $|D \cap V(\mathfrak{C}_{n-1} - x)| \geq 3n - 1$. Since $D \cap V(Q_n)$ totally dominates Q_n and $x \in D$, it follows that $|D \cap V(Q_n)| \geq 5$, and then $|D| \geq 3n - 1 + 5 = 3n + 4$. Hence, we have that $\gamma_t(R \bullet Q^{(n)}) \geq 3n + 3$.

If $D_1 \cap V(\mathfrak{C}_{n-1})$ totally dominates $\mathfrak{C}_{n-1}$, then $|D_1 \cap V(\mathfrak{C}_{n-1})| \geq 3n$, but $|D_1 \setminus V(\mathfrak{C}_{n-1})| \geq 2$, thus $|D_1| \geq 3n + 2$. So we may assume that $D_1 \cap V(\mathfrak{C}_{n-1})$ does not totally dominates $\mathfrak{C}_{n-1}$. If $x \notin D_1$, then $D_1 \cap R_1(Q_{n-1}) = \emptyset$ and $D_1 \cap V(\mathfrak{C}_{n-1})$ totally dominates $\mathfrak{C}_{n-1} - x$ and $|D_1 \cap V(\mathfrak{C}_{n-1})| \geq 3n - 1$. Notice that $D_1 \setminus V(\mathfrak{C}_{n-1})$ totally dominates $Q_n - y$ and $D_1 \cap L_1(Q_n) \neq \emptyset$ and $|D_1 \setminus V(\mathfrak{C}_{n-1})| \geq 4$, thus $|D_1| \geq 3n - 1 + 4 = 3n + 3$. If $x \in D_1$, then $D_1 \cap R_1(Q_{n-1}) = \emptyset$ and $D_1 \cap V(\mathfrak{C}_{n-1} - x)$ totally dominates $\mathfrak{C}_{n-1} - N[x]$ and $|D_1 \cap V(\mathfrak{C}_{n-1} - N[x])| \geq 3n - 1$. Notice that $x \in D_1$ and $D_1 \cap L_1(Q_n) \neq \emptyset$, and then $|D_1 \cap (Q_n - y)| \geq 4$, thus $|D_1| \geq 3n - 1 + 4 = 3n + 3$. Hence, we have that $\gamma_t(R \bullet Q^{(n)} - y) \geq 3n + 2$.

If $D_2 \cap V(\mathfrak{C}_{n-1})$ totally dominates $\mathfrak{C}_{n-1}$, then $|D_2 \cap V(\mathfrak{C}_{n-1})| \geq 3n$, but $|D_2 \setminus V(\mathfrak{C}_{n-1})| \geq 2$, thus $|D_2| \geq 3n + 2$. So we may assume that $D_2 \cap V(\mathfrak{C}_{n-1})$ does not totally dominates $\mathfrak{C}_{n-1}$. If $x \notin D_2$, then $D_2 \cap R_1(Q_{n-1}) = \emptyset$ and $D_2 \cap V(\mathfrak{C}_{n-1})$ totally dominates $\mathfrak{C}_{n-1} - x$ and $|D_2 \cap V(\mathfrak{C}_{n-1})| \geq 3n - 1$. Notice that $D_2 \setminus V(\mathfrak{C}_{n-1})$ totally dominates $Q_n - N[y]$ and $D_2 \cap L_1(Q_n) \neq \emptyset$ and $|D_2 \setminus V(\mathfrak{C}_{n-1})| \geq 3$, thus $|D_2| \geq 3n - 1 + 3 = 3n + 2$. If $x \in D_2$, then $D_2 \cap R_1(Q_{n-1}) = \emptyset$ and $D_2 \cap V(\mathfrak{C}_{n-1} - x)$ totally dominates $\mathfrak{C}_{n-1} - N[x]$ and $|D_2 \cap V(\mathfrak{C}_{n-1} - N[x])| \geq 3n - 1$. Notice that $x \in D_2$ and $D_2 \cap L_1(Q_n) \neq \emptyset$, and then $|D_2 \cap (Q_n - N[y])| \geq 3$, thus $|D_1| \geq 3n - 1 + 3 = 3n + 2$. Hence, we have that $\gamma_t(R \bullet Q^{(n)} - N[y]) \geq 3n + 2$. □

Corollary 1. *(a)* $\gamma_t(R \bullet Q^{(n)}) = 3n + 3$*; (b)* $\gamma_t(R \bullet Q^{(n)} - y) = 3n + 2$*; (c)* $\gamma_t(R \bullet Q^{(n)} - N[y]) = 3n + 2$.

Theorem 3.4. *The graph* $R \bullet Q^{(n)} \bullet J$ *is* $(3n + 6)$-γ_t*-vertex-critical graph with diameter* $5n + 7$.

Proof. If $n = 0$, then the statement follows by Theorem 3.1. So we may assume that $n \geq 1$. Denote the graph $R \bullet Q^{(n)} \bullet J$ by G and denote the i-th copy of Q by Q_i with Left x_i and Right y_i. Denote the Left vertex of J by x and the Right vertex by y. Let D be a minimum total dominating set of G. Notice that there exists a minimum total dominating set $D_{i,l}$ of $Q_i - N[y_i]$ containing x_i, that is, a vertex from each of $L_0(Q_i), L_1(Q_i)$ and $L_2(Q_i)$ totally dominates $L_0(Q_i) \cup L_1(Q_i) \cup L_2(Q_i) \cup L_3(Q_i)$; by symmetry, there exists a minimum total dominating set $D_{i,r}$ of $Q_i - N[x_i]$ containing y_i, that is, a vertex from each of $R_0(Q_i), R_1(Q_i)$ and $R_2(Q_i)$ totally dominates $R_0(Q_i) \cup R_1(Q_i) \cup R_2(Q_i) \cup R_3(Q_i)$. For the graph R, there exists a minimum total dominating set $D_{0,l}$ of $R -$ Right and a minimum total dominating set $D_{0,r}$ of R containing the Right vertex. For the graph J, there exists a minimum total dominating set $D_{n+1,l}$ containing the Left vertex and a minimum total dominating set $D_{n+1,r}$ of $J -$ Left.

If $D \cap V(\mathfrak{C}_n)$ totally dominates $\mathfrak{C}_n$, then $|D \cap V(\mathfrak{C}_n)| \geq 3n + 3$ and $|D| \geq (3n + 3) + 3 = 3n + 6$. So we may assume that $D \cap V(\mathfrak{C}_n)$ does not totally dominates $\mathfrak{C}_n$. If $x \notin D$, then $D \cap R_1(Q_n) = \emptyset$ and $D \cap V(\mathfrak{C}_n)$ totally dominates $\mathfrak{C}_n - y_n$ and $|D \cap V(\mathfrak{C}_n)| \geq 3n + 2$, thus $|D| \geq 3n + 2 + 4 = 3n + 6$. If $x \in D$, then $D \cap R_1(Q_n) = \emptyset$ and $D \cap (\mathfrak{C}_n - y_n)$ totally dominates $\mathfrak{C}_n - N[y_n]$ and $|D| \geq 3n + 2 + 4 = 3n + 6$. There exists a total dominating set with $3n + 6$ vertices, such as $D_{0,r} \cup D_{1,r} \cup D_{2,r} \cup \ldots D_{n,r} \cup D_{n+1,r}$. Hence, we have that $\gamma_t(R \bullet Q^{(n)} \bullet J) = 3n + 6$.

Let v be an arbitrary vertex. If $v \in R$, then a minimum total dominating set of $R - v$ and $D_{1,r} \cup D_{2,r} \cup \ldots D_{n,r} \cup D_{n+1,r}$ form a total dominating set of $G - v$ with $3n + 5$ vertices.

If $v \in J$, then $D_{0,r} \cup D_{1,r} \cup \ldots D_{n-1,r}$ and a minimum total dominating set of $R_2(Q_n)$ and a minimum total dominating set of $J - v$ form a total dominating set of $G - v$ with $3n + 5$ vertices.

If $v \in L_1(Q_1) \cup L_2(Q_1)$, then there exists two adjacent vertices in $L_1(Q_1) \cup L_2(Q_1)$ which totally dominates $L_0(Q_1) \cup L_1(Q_1) \cup L_2(Q_1) \cup L_3(Q_1) - v$, denote this two adjacent vertices by D^*. Thus $D_{0,l} \cup D^* \cup D_{2,l} \cup \cdots \cup D_{n,l} \cup D_{n+1,l}$ is a total dominating set of $G - v$ with $3n + 5$ vertices.

If $v \in L_3(Q_n) \cup L_4(Q_n)$, then there exists two adjacent vertices in $L_3(Q_n) \cup L_4(Q_n)$ which totally dominates $L_2(Q_n) \cup L_3(Q_n) \cup L_4(Q_n) \cup L_5(Q_n) - v$, denote this two adjacent vertices by S^*. Thus $D_{0,r} \cup D_{1,r} \cup \cdots \cup D_{n-1,r} \cup S^* \cup D_{n+1,r}$ is a total dominating set of $G - v$ with $3n + 5$ vertices.

Suppose that $v \in L_0(Q_i) \cup L_1(Q_i) \cup L_2(Q_i)$ with $i \geq 2$. Thus $D_{0,r} \cup \cdots \cup D_{i-2,r}$ and two adjacent vertices in $R_2(Q_{i-1})$ and two adjacent vertices in $L_1(Q_i) \cup L_2(Q_i)$ which totally dominates $L_0(Q_i) \cup L_1(Q_i) \cup L_2(Q_i) \cup L_3(Q_i) - v$ and $D_{i+1,l} \cup \cdots \cup D_{n,l} \cup D_{n+1,l}$ form a total dominating set of $G - v$ with $3n + 5$ vertices.

Suppose that $v \in L_3(Q_i) \cup L_4(Q_i) \cup L_5(Q_i)$ with $i \leq n - 1$. Thus $D_{0,r} \cup D_{1,r} \cup \ldots D_{i-1,r}$ and two adjacent vertices in $L_3(Q_i) \cup L_4(Q_i)$ which totally dominates $L_2(Q_i) \cup L_3(Q_i) \cup L_4(Q_i) \cup L_5(Q_i) - v$ and two adjacent vertices in $L_2(Q_{i+1})$ and $D_{i+2,l} \cup \cdots \cup D_{n+1,l}$ form a total dominating set of $G - v$ with $3n + 5$ vertices.

Hence, for any vertex v in V, we have that $\gamma_t(G - v) \leq 3n + 5$, and then G is a $(3n + 6)$-γ_t-vertex-critical graph. □

We can adapt the similar technique to prove that $R \bullet Q^{(n)} \bullet R \bullet R$ is $(3n + 7)$-γ_t-vertex-critical, so we omit the details of the proof.

Theorem 3.5. *The graph $R \bullet Q^{(n)} \bullet R \bullet R$ is a $(3n + 7)$-γ_t-vertex-critical graph with diameter $5n + 9$.*

Theorem 3.6. *For every integer $k \geq 4$, there are infinitely many graphs that are k-γ_t-vertex-critical with diameter $\left\lfloor \frac{5k-7}{3} \right\rfloor$.*

Proof. We divide the graphs into four classes according to the value of k.

(1) Suppose that $k \equiv 2 \pmod 3$ and $k = 3n + 5$. Notice that the graph $A \bullet Q^{(n)} \bullet A$ is a $(3n + 5)$-γ_t-vertex-critical graph and $\mathrm{diam}(A \bullet Q^{(n)} \bullet A) = 5n + 6 = \frac{5k-7}{3}$, which has been proved in [4, Theorem 13].

(2) Suppose that $k \equiv 0 \pmod 3$ and $k = 3n + 6$. Notice that the graph $R \bullet Q^{(n)} \bullet J$ is a $(3n + 6)$-γ_t-vertex-critical graph and $\mathrm{diam}(R \bullet Q^{(n)} \bullet J) = 5n + 7 = \left\lfloor \frac{5k-7}{3} \right\rfloor$.

(3) Suppose that $k \equiv 1 \pmod 3$ and $k = 3n+7$. Notice that the graph $R \bullet Q^{(n)} \bullet R \bullet R$ is a $(3n+7)$-γ_t-vertex-critical graph and $\mathrm{diam}(R \bullet Q^{(n)} \bullet R \bullet R) = 5n + 9 = \left\lfloor \frac{5k-7}{3} \right\rfloor$.

(4) If $k = 4$, then the graph J meet the requirement by Theorem 3.1. □

Remark 3. As in [2], the upper bound in Theorem 2.2 is sharp. We provide infinitely many k-i-vertex-critical graphs with diameter $2(k - 1)$ for each $k \geq 2$. For instance, let B be the complete graph on $2t$ vertices with a perfect matching removed, and let G be the graph whose block graph is a path on $k - 1$ vertices and every block is a copy of B; notice that $i(G) = k$ and $\mathrm{diam}(G) = 2(k - 1)$.

Remark 4. So far, we don't know if the given upper bound on the k-γ_c-vertex-critical graphs is the best possible.

Acknowledgments. This project was supported by NSFC (11026078). The authors would like to express heartfelt thanks to Edwards and MacGillivray for providing us a pdf-file of the paper [2]. In addition, the authors would like to thank the anonymous reviewers for their valuable comments.

References

[1] S. Ao, Independent domination critical graphs, Master's thesis, University of Victoria (1994).

[2] M. Edwards and G. MacGillivray, The diameter of total domination and independent domination vertex-critical graphs, Australas. J. Combin. 52 (2012) 33–39.

[3] J. Fulman, D. Hanson and G. Macgillivray, Vertex domination-critical graphs, Networks 25 (1995) (2) 41–43.

[4] W. Goddard, T. W. Haynes, M. A. Henning and L. C. van der Merwe, The diameter of total domination vertex critical graphs, Discrete Math. 286 (2004) (3) 255–261.

[5] X. Hou and M. Edwards, Paired domination vertex critical graphs, Graphs Combin. 24 (2008) (5) 453–459.

[6] M. D. Plummer, On the structure of graphs vertex critical with respect to connected domination, in Research trends in combinatorial optimization, Springer, Berlin, 2009, pp. 303–315.