On normalizers of the nilpotent residuals of subgroups of a finite group *

Lü Gong[1] and Xiuyun Guo[2†]

1. School of Sciences, Nantong University, Jiangsu 226007, P. R. China

2. Department of Mathematics, Shanghai University, Shanghai 200444, P. R. China

E-mail: lieningzai1917@126.com, xyguo@staff.shu.edu.cn,

Abstract: The aim of this paper is to study the structure of finite groups whose nilpotent residuals of non-normal subgroups are normal.

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1 Introduction

All groups considered in this paper are finite.

A Dedekind group is a group in which every subgroup is normal, which has been generalised in a number of ways. Romalis and Sesekin investigated metahamiltonian groups, in which every subgroup is normal or abelian [23, 24]. Russo and Vincenzi considered groups in which every subgroup is normal or a T-group [25]. Kemhadze investigated the structure of groups in which every subgroup is either subnormal or nilpotent [14], as did Phillips and Wilson, who gave necessary conditions [21, Lemma 7] and a detailed description of such groups with trivial centre [21, Proposition 2]. Recently, Ballester-Bolinches and Cossey have obtained the equivalent condition of non-nilpotent groups in which every subgroup is either subnormal or nilpotent [5].

Recall that the norm $N(G)$ of a group G is the intersection of the normalizes of all subgroups of G, which was first introduced by Baer in 1934 [1] and many useful properties and results on norm have been given [1, 2, 4, 7, 16, 17, 26]. It is clear that $G = N(G)$ if and only if G is a Dedekind group.

In addition, we have studied a characteristic subgroup $N^{\mathcal{N}}(G)$ of a group G, which is the intersection of the normalizers of the nilpotent residuals of all subgroups of G [9], also it is called $S(G)$ in [27], that is,

$$N^{\mathcal{N}}(G) = \bigcap_{H \leq G} N_G(H^{\mathcal{N}}),$$

where $H^{\mathcal{N}}$ is the nilpotent residual of subgroup H of G.

There exists a series of normal subgroups:

$$1 = N^{\mathcal{N}}(G)_0 \leq N^{\mathcal{N}}(G)_1 \leq N^{\mathcal{N}}(G)_2 \leq \cdots \leq N^{\mathcal{N}}(G)_n \leq \cdots,$$

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†Corresponding author.

in which $N^{\mathcal{N}}(G)_{i+1}/N^{\mathcal{N}}(G)_i = N^{\mathcal{N}}(G/N^{\mathcal{N}}(G)_i)$ for $i = 0, 1, 2, \cdots$, and let $N^{\mathcal{N}}(G)_\infty$ be the terminal term of the ascending series.

Our aim in this paper is to consider the structure of groups whose nilpotent residuals of non-normal subgroups are normal, which is a generalization of the class of groups in which every subgroup is nilpotent or normal.

The terminology and notation employed agree with standard usage [6, 8, 22]. Let $\mathcal{N}^2$ denote the class of metanilpotent groups and $Int_{\mathcal{N}^2}(G)$ denote the intersection of all maximal metanilpotent subgroups of a group G. The metanilpotent hypercenter, nilpotent residual, metanilpotent residual, Fitting and Frattini subgroup of a group G will be denoted by $Z_{\mathcal{N}^2}(G)$, $G^{\mathcal{N}}$, $G^{\mathcal{N}^2}$, $F(G)$, $\Phi(G)$ respectively. $K_i(G)$ and $\pi(G)$ denote the ith term of the lower central series of a group G and the set of primes divide the order of G respectively.

2 The structure of $N^{\mathcal{N}}(G)$

It is easy to see that the intersection of all maximal abelian subgroups of a group G is its center $Z(G)$, it is also proved that the intersection of all $\mathcal{F}$-maximal subgroups of a group G is its $\mathcal{F}$-hypercenter $Z_{\mathcal{F}}(G)$ for certain non-empty hereditary saturated formations $\mathcal{F}$ in [10, 12, 19, 28, 29, 31].

In addition, it is well-known that the intersection of all maximal nilpotent subgroups of a group G is its hypercenter $Z_\infty(G)$ in [3, Corollary 4], and it is also proved that G equals $N^{\mathcal{N}}(G)_\infty$ if G is metanilpotent in [9, Theorem 4.4] or [27, Theorem 3.3].

A natural question arises:

Does the intersection of all maximal metanilpotent subgroups of the group G coincide with the hypercenter of $\mathcal{N}^2$?

In this section, we obtain a positive answer to this question. Firstly, we will list here some lemmas on $N^{\mathcal{N}}(G)$ of a group G, which will be useful in the sequel.

Lemma 2.1 *Let G be a group. Then $C_G(G^{\mathcal{N}})$ is nilpotent.*

Proof. Let $C_G(G^{\mathcal{N}}) = C$. Then $CG^{\mathcal{N}}/G^{\mathcal{N}} \cong C/(G^{\mathcal{N}} \cap C)$ is nilpotent. By $[G^{\mathcal{N}} \cap C, C] = 1$, $G^{\mathcal{N}} \cap C \leq Z(C)$. Thus $C/Z(C)$ is nilpotent, so is C.

Lemma 2.2 [9, Proposition 2.2, Theorem 2.12 and Proposition 4.3] or [27, Theorem 2.5 and Proposition 2.3] *Let G be a group. Then*

(1) $Z_\infty(G) \leq C_G(G^{\mathcal{N}}) \leq N^{\mathcal{N}}(G)$.

(2) If $Z(G^{\mathcal{N}}) = 1$, then $N^{\mathcal{N}}(G) = C_G(G^{\mathcal{N}})$.

(3) If $N \trianglelefteq G$ and $N \leq N^{\mathcal{N}}(G)_\infty$, then $N^{\mathcal{N}}(G/N)_\infty = N^{\mathcal{N}}(G)_\infty/N$.

Lemma 2.3 [9, Theorem 4.7] *Let G be a group. Then*

(1) $Z_\infty(G^{\mathcal{N}}) \leq N^{\mathcal{N}}(G)_\infty$.

(2) $N^{\mathcal{N}}(G)_\infty/Z_\infty(G^{\mathcal{N}}) = N^{\mathcal{N}}(G/Z_\infty(G^{\mathcal{N}})) = C_{G/Z_\infty(G^{\mathcal{N}})}(G^{\mathcal{N}}/Z_\infty(G^{\mathcal{N}}))$.

Lemma 2.4 [29, Theorem A] *Let $\mathcal{F}$ be a hereditary saturated formation with $\pi(\mathcal{F}) \neq \emptyset$, for any $p \in \pi(\mathcal{F})$, $\mathcal{F}(p)$ denote the intersection of all formations containing the set $\{G/O_{p',p}(G)|G \in \mathcal{F}\}$, and let $F(p)$ denote the class of all groups G such that $G^{\mathcal{F}(p)}$ is a p-group.*

Then $Z_{\mathcal{F}}(G) = Int_{\mathcal{F}}(G)$ hold for each group G if and only if $\mathcal{F}$ satisfies the boundary condition, where we say that $\mathcal{F}$ satisfies the boundary condition if for any $p \in \pi(\mathcal{F})$, $G \in \mathcal{F}$ whenever G is an $F(p)$-critical group.

Lemma 2.5 [9, Corollary 4.5] or [27, Theorem 3.4] *If H is an $\mathcal{N}^2$-subgroup of a group G, then $N^{\mathcal{N}}(G)_\infty H$ is also an $\mathcal{N}^2$-group.*

From Lemma 2.5, the every maximal metanilpotent subgroup of a group G contains the hypercenter of $N^{\mathcal{N}}(G)$. We shall see in the sequel that in fact the latter subgroup is the intersection of all maximal metanilpotent subgroup of a group G. We need the following lemma.

Lemma 2.6 *Let G be a group. Then*

(1) The nilpotent residual of $N^{\mathcal{N}}(G)_\infty$ is nilpotent.

(2) $N^{\mathcal{N}}(G)_\infty \cap G^{\mathcal{N}} = Z_\infty(G^{\mathcal{N}})$.

Proof. (1) If $Z(G^{\mathcal{N}}) = 1$, then, by Lemma 2.2 (2), $N^{\mathcal{N}}(G) = C_G(G^{\mathcal{N}})$. So $N^{\mathcal{N}}(G)$ is nilpotent by Lemma 2.1. Further more, we can prove $N^{\mathcal{N}}(G)_\infty = N^{\mathcal{N}}(G)$.

It is easy to see that $Z(G^{\mathcal{N}}) = 1$ if and only if $N^{\mathcal{N}}(G) \cap G^{\mathcal{N}} = 1$ by Lemma 2.2 (1) and (2). Since $Z(G^{\mathcal{N}}) = 1$, we have $Z((G/N^{\mathcal{N}}(G))^{\mathcal{N}}) = 1$. Then $N^{\mathcal{N}}(G/N^{\mathcal{N}}(G)) \cap (G/N^{\mathcal{N}}(G))^{\mathcal{N}} = 1$, that is,

$$N^{\mathcal{N}}(G)_2 \cap G^{\mathcal{N}} \leq N^{\mathcal{N}}(G) \cap G^{\mathcal{N}} = 1.$$

By the same way, we have $N^{\mathcal{N}}(G)_\infty \cap G^{\mathcal{N}} = 1$. Hence, by Lemma 2.2 (1) and the definition of series, $N^{\mathcal{N}}(G)_\infty = N^{\mathcal{N}}(G)$.

If $Z(G^{\mathcal{N}}) \neq 1$, then we consider $G/Z_\infty(G^{\mathcal{N}})$. By Lemma 2.3 (2), we see

$$N^{\mathcal{N}}(G)_\infty/Z_\infty(G^{\mathcal{N}}) = N^{\mathcal{N}}(G/Z_\infty(G^{\mathcal{N}})) = C_{G/Z_\infty(G^{\mathcal{N}})}(G^{\mathcal{N}}/Z_\infty(G^{\mathcal{N}})).$$

Then $N^{\mathcal{N}}(G)_\infty/Z_\infty(G^{\mathcal{N}})$ is nilpotent by Lemma 2.1, thus $N^{\mathcal{N}}(G)_\infty$ is metanilpotent.

(2) If $Z(G^{\mathcal{N}}) = 1$, then, by the same argument with (1), we have $N^{\mathcal{N}}(G)_\infty \cap G^{\mathcal{N}} = 1 = Z_\infty(G^{\mathcal{N}})$.

If $Z(G^{\mathcal{N}}) \neq 1$, then we consider $G/Z_\infty(G^{\mathcal{N}})$. Then

$$N^{\mathcal{N}}(G/Z_\infty(G^{\mathcal{N}})) \cap (G/Z_\infty(G^{\mathcal{N}}))^{\mathcal{N}} = 1.$$

It follows from Lemma 2.3 that $N^{\mathcal{N}}(G)_\infty \cap G^{\mathcal{N}} = Z_\infty(G^{\mathcal{N}})$.

Now, we are ready to answer the question.

Theorem 2.1 *Let G be a group. Then $Z_{\mathcal{N}^2}(G) = N^{\mathcal{N}}(G)_\infty$.*

Proof. We proceed by induction on the order of G.

(1) $Z_{\mathcal{N}^2}(G) \leq N^{\mathcal{N}}(G)_\infty$.

Let N be a minimal normal subgroup of G and $N \leq Z_{\mathcal{N}^2}(G)$. Since N is $\mathcal{N}^2$-central in G,

$$[N](G/C_G(N)) \in \mathcal{N}^2. \qquad (*)$$

Let $X = [N](G/C_G(N))$. Then X is a primitive group and $Soc(X) = N = F(X)$. It follows from $(*)$ that $X/F(X) \cong G/C_G(N) \in \mathcal{N}$. Hence $N \leq C_G(G^{\mathcal{N}}) \leq N^{\mathcal{N}}(G) \leq N^{\mathcal{N}}(G)_\infty$. By induction and Lemma 2.2 (3),

$$Z_{\mathcal{N}^2}(G)/N = Z_{\mathcal{N}^2}(G/N) \leq N^{\mathcal{N}}(G/N)_\infty = N^{\mathcal{N}}(G)_\infty/N,$$

thus $Z_{\mathcal{N}^2}(G) \leq N^{\mathcal{N}}(G)_\infty$.

(2) $N^{\mathcal{N}}(G)_\infty \leq Z_{\mathcal{N}^2}(G)$.

Let N be a minimal normal subgroup of G and $N \leq N^{\mathcal{N}}(G)_\infty$. If $N \leq Z(G)$, then $N \leq C_G(G^{\mathcal{N}^2})$. If $N \leq G^{\mathcal{N}}$, then $N \leq N^{\mathcal{N}}(G)_\infty \cap G^{\mathcal{N}}$. By Lemma 2.6 (2) and [8, Theorem 6.14],

$$N \leq Z_\infty(G^{\mathcal{N}}) = C_U((G^{\mathcal{N}})^{\mathcal{N}}) \leq C_G(G^{\mathcal{N}^2}),$$

where U is maximal nilpotent subgroup of $G^{\mathcal{N}}$.

In addition, $N \leq M$ hold for any maximal metanilpotent subgroups M by Lemma 2.5. Hence, by [8, Theorem 6.14], $N \leq C_M(G^{\mathcal{N}^2}) = Z_{\mathcal{N}^2}(G)$. By induction and Lemma 2.2 (3),

$$Z_{\mathcal{N}^2}(G)/N = Z_{\mathcal{N}^2}(G/N) \geq N^{\mathcal{N}}(G/N)_\infty = N^{\mathcal{N}}(G)_\infty/N,$$

thus $Z_{\mathcal{N}^2}(G) \geq N^{\mathcal{N}}(G)_\infty$.

Theorem 2.2 *Let G be a group. Then $N^{\mathcal{N}}(G)_\infty = Z_{\mathcal{N}^2}(G) = Int_{\mathcal{N}^2}(G)$.*

Proof. Let $F(p) = \mathcal{S}_p\mathcal{N}$ for all p. Then F is the canonical local definition of $\mathcal{N}^2$. If G is $F(p)$-critical for all p, then every maximal subgroup of G is nilpotent since they are in $\mathcal{S}_p\mathcal{N}$ for all p. Then G is either nilpotent or a Schmidt group. In any case, G is metanilpotent, that is, $\mathcal{N}^2$ satisfies the boundary condition. Thus the equality is clear from Lemma 2.4 and Theorem 2.1.

Following [15], we denote the intersection of the normalizers of the derived subgroups of all subgroups in a group G by $D(G)$, $D_\infty(G)$ is the terminal term of the ascending series $1 = D_0(G) \leq D_1(G) \leq D_2(G) \leq \cdots \leq D_n(G) \leq \cdots$, where $D_{i+1}(G)/D_i(G) = D(G/D_i(G))$ for $i = 0, 1, 2, \cdots$.

By the similar way, we can prove following theorem with the help of [15, Lemma 2.3, Theorem 2.6, Problem 5.1 and 5.2], [8, Theorem 6.14] and Lemma 2.4.

Theorem 2.3 *Let G be a group and $\mathcal{F}$ denote the class of group G that G' is nilpotent. Then*

(1) The derived subgroup of $D_\infty(G)$ is nilpotent and $D_\infty(G) \cap G' = Z_\infty(G')$.

(2) $D_\infty(G) = Z_{\mathcal{F}}(G) = Int_{\mathcal{F}}(G)$.

3 $N^{\mathcal{N}}$-groups

In this section, we begin to discuss groups in which the nilpotent residual of every subgroup is normal.

Definition 3.1 *A group G is said to be $N^{\mathcal{N}}$-group ([9]) or S-group ([27]) if $G = N^{\mathcal{N}}(G)$. A group G is called D-group if $G = D(G)$.*

Clearly, we see that the nilpotent residual of every subgroup in a $N^{\mathcal{N}}$-group is normal. So it is easy to see that the class of groups in which every subgroup is nilpotent or normal in properly contained in the class of $N^{\mathcal{N}}$-groups. For the sake of completeness, we list here some basic results on $N^{\mathcal{N}}$-groups which have been proved in [9].

Lemma 3.1 [9, Proposition 4.9] or [27, Theorem 4.2] *Let G be a $N^{\mathcal{N}}$-group.*

(1) If $H \leq G$, then H is a $N^{\mathcal{N}}$-group.

(2) If $K \trianglelefteq G$, then G/K is a $N^{\mathcal{N}}$-group.

Lemma 3.2 [9, Proposition 2.7] or [27, Theorem 1.4] *If $G = A \times B$ is the direct product of a group A and a group B with $(|A|, |B|) = 1$, then $N^{\mathcal{N}}(G) = N^{\mathcal{N}}(A) \times N^{\mathcal{N}}(B)$.*

The following example illustrates that the condition $(|A|, |B|) = 1$ could not be removed in Lemma 3.2. It also shows that the direct product of two $N^{\mathcal{N}}$-groups may be not a $N^{\mathcal{N}}$-group.

Example 3.1 *Let $G = S_3 \times S_3 = \langle a, b, c, d | a^3 = b^2 = c^3 = d^2 = 1, a^b = a^{-1}, c^d = c^{-1}, [a, c] = [a, d] = [b, c] = [c, d] = 1\rangle$, $H = \langle ac, bd\rangle$ and $K = \langle ac\rangle$. It is clear that H is isomorphic to S_3 and K is the nilpotent residual of H. It follows from $K \ntrianglelefteq G$ that $N^{\mathcal{N}}(G) < N^{\mathcal{N}}(S_3) \times N^{\mathcal{N}}(S_3) = G$. Then G is a supersoluble non-$N^{\mathcal{N}}$-group in which all proper subgroups are $N^{\mathcal{N}}$-groups.*

Example 3.2 [9, Proposition 4.10] *The following groups are $N^{\mathcal{N}}$-groups:*

(1) Groups all of whose non-nilpotent subgroups are normal.

(2) Groups with the cyclic nilpotent residual.

(3) Groups with an abelian normal subgroup of index a prime.

Lemma 3.3 [9, Proposition 2.3, Theorem 3.2, 3.5, 4.4, 4.11] *If G is a $N^{\mathcal{N}}$-group, then*

(1) G is a meta-nilpotent group.

(2) $l_p(G) \leq 1$ for a prime $p \in \pi(G)$ and the Fitting length of G is bounded by 2.

(3) $F(G/\Phi(G)) = C_{G/\Phi(G)}((G/\Phi(G))^{\mathcal{N}}) = (G/\Phi(G))^{\mathcal{N}} \times Z(G/\Phi(G))$.

Lemma 3.4 *If $G = MN$ is the product of a normal subgroup N and a subgroup M, then $G^{\mathcal{N}} \leq M^{\mathcal{N}}N$. In particular, $(M \times N)^{\mathcal{N}} = M^{\mathcal{N}} \times N^{\mathcal{N}}$.*

Proof. By $[G, G] \leq [M, M]N$ and induction, we get $K_i(G) \leq K_i(M)N$ for any integer i, then $G^{\mathcal{N}} \leq M^{\mathcal{N}}N$. If $M \trianglelefteq G$, then $G^{\mathcal{N}} \leq M^{\mathcal{N}}N^{\mathcal{N}}(M \cap N)$. Hence $(M \times N)^{\mathcal{N}} = M^{\mathcal{N}} \times N^{\mathcal{N}}$.

The class of all $N^{\mathcal{N}}$-groups is not a saturated class as the following example shows.

Example 3.3 *Let $G = \langle a, b, c, d | a^2 = 1, b^3 = 1, c^3 = 1, d^3 = 1, c^a = c^{-1}, c^b = cd, d^a = d^{-1}\rangle$. Then it is easy to see that G is not a $N^{\mathcal{N}}$-group but $G/\Phi(G)$ is a $N^{\mathcal{N}}$-group.*

It is clear that the class of nilpotent groups is contained in the class of $N^{\mathcal{N}}$-groups. Now we first begin to consider some special non-nilpotent $N^{\mathcal{N}}$-groups.

Proposition 3.1 *Let G be a non-nilpotent $N^{\mathcal{N}}$-group. If there is a maximal subgroup M of G with $M_G = 1$, then*

(1) $G = G^{\mathcal{N}} \rtimes M$ with $G^{\mathcal{N}}$ a minimal normal subgroup of G and M nilpotent.

(2) Every subgroup of G is either nilpotent or subnormal.

Proof. Since M is a maximal subgroup of G and $M_G = 1$, $G = G^{\mathcal{N}}M$. By $C_G(C_G(G^{\mathcal{N}}) \cap M) \geq G^{\mathcal{N}}$ and $N_G(C_G(G^{\mathcal{N}}) \cap M) \geq M$, we see $G = N_G(C_G(G^{\mathcal{N}}) \cap M)$. It follows that $C_G(G^{\mathcal{N}}) \cap M = 1$. For any non-trivial normal subgroup H of G that contained in $C_G(G^{\mathcal{N}})$, $G = HM$ and $C_G(G^{\mathcal{N}}) = H$, which implies $C_G(G^{\mathcal{N}})$ is a minimal normal subgroup of G.

Noticing Lemma 3.3 (3), we get $C_G(G^{\mathcal{N}}) = G^{\mathcal{N}} \times Z(G)$. Then $C_G(G^{\mathcal{N}}) = G^{\mathcal{N}}$ and $Z(G) = 1$. Therefore $G^{\mathcal{N}} \cap M = 1$ and M is nilpotent.

Since $G^{\mathcal{N}}$ is a minimal normal subgroup of G, $K^{\mathcal{N}} = 1$ or $G^{\mathcal{N}}$ for every subgroup K of G. So every subgroup of G is either nilpotent or subnormal in G.

Proposition 3.2 *If G is a primitive non-nilpotent D-group, then every subgroup of G is either nilpotent or normal.*

In Proposition 3.1 above, $G^{\mathcal{N}} = F(G)$ is a minimal normal subgroup of a $N^{\mathcal{N}}$-group G, which is an elementary p-group for some $p \in \pi(G)$. So it is interesting to consider the non-nilpotent $N^{\mathcal{N}}$-groups such that $F(G)$ is a p-group.

Proposition 3.3 *If G is a non-nilpotent $N^{\mathcal{N}}$-group and $F(G)$ is a p-group for some $p \in \pi(G)$, then*

(1) $G = O_{pp'}(G)$ and $F(G)$ is a Sylow p-subgroup of G.

(2) $|G|$ divides $p^n(p^r-1)(p^{r-1}-1)\cdots(p-1)$ if $|P| = p^n$ and $|P/\Phi(P)| = p^r$.

Proof. (1) By $F(G)$ is a p-group, $F(G) = O_p(G)$, and then $F(G/F(G)) = O_{p'}(G/F(G))$. Apply Lemma 3.3 (2), we see $G/F(G)$ is a p'-group, so $P \leq F(G)$ for $P \in \mathrm{Syl}_p(G)$, it implies $F(G) = P$. Then $G/O_p(G) = O_{p'}(G/O_p(G)) = O_{pp'}(G)/O_p(G)$, so $G = O_{pp'}(G)$.

(2) By Lemma 3.3 (1), $N^{\mathcal{N}}$-groups are solvable. According to a Theorem of Hall-Higman [22, Theorem 9.3.2] and (1), we have $P = C_G(P/\Phi(P))$, and $G/C_G(P/\Phi(P))$ is isomorphic to a subgroup of $\mathrm{Aut}(P/\Phi(P))$. Let $|P| = p^n$ and $|P/\Phi(P)| = p^r$. Then $|G/P|$ divides $(p^r-1)(p^r-p)\cdots(p^r-p^{r-1})$, by G/P is a p'-group, we see $|G|$ divides $p^n(p^r-1)(p^{r-1}-1)\cdots(p-1)$.

Remark 3.1 *According to Proposition 3.3, we can see that $F(G)$ is neither a cyclic group nor a dihedral group if $p = 2$, also we can see $G = Q_8 \rtimes C_3$ if $F(G)$ is a quaternion 2-group.*

Lemma 3.5 *Let p be a prime and P a Sylow p-subgroup of a group G. Then G is p-nilpotent if and only if $N_G(P)$ and $G^{\mathcal{N}}$ are p-nilpotent.*

Proof. It is clear that $N_G(P)$ and $G^{\mathcal{N}}$ are p-nilpotent if G is p-nilpotent. Conversely, let G be a counter example of minimal order and $H = O_{p'}(G^{\mathcal{N}})$.

If $H \neq 1$, then we consider G/H. Since $N_{G/H}(PH/H) = N_G(P)H/H$, $(G/H)^{\mathcal{N}}$ are p-nilpotent and the minimality of G, we see G/H is p-nilpotent, so G is, a contradiction.

If $H = 1$, then, by $G^{\mathcal{N}}$ is p-nilpotent, $G^{\mathcal{N}}$ is a p-group. It follows from $P/G^{\mathcal{N}} \trianglelefteq G/G^{\mathcal{N}}$ that $P \trianglelefteq G$, so G is p-nilpotent, a contradiction. Now we complete the proof.

Apply the Lemma 3.5 and Proposition 3.3 (1), we have:

Corollary 3.1 *Let p be a prime and P a Sylow p-subgroup of a $N^{\mathcal{N}}$-group G. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent.*

Lemma 3.6 [11, Lemma 0.5] *If G is a metanilpotent group and $\overline{G} = G/\Phi(G)$, then $F(\overline{G}) = \overline{G}^{\mathcal{N}} \times Z(\overline{G})$.*

Lemma 3.7 [13, Theorem 2.1, Lemma 2.4 and Theorem 2.5]

(1) If N is an abelian normal subgroup of G with $\Phi(G) = 1$, then N has a complement in G.

(2) If $\Phi(G) = 1$ and $Z(G) \neq 1$, then $G = Z(G) \times H$ with $Z(G)$ elementary abelian and $Z(H) = 1$.

Theorem 3.1 *Let G be a non-nilpotent $N^{\mathcal{N}}$-group and $\Phi(G) = 1$. Then the nilpotent residual of every subgroup has a complement and $G = F(G) \rtimes K$ with $F(G)$ abelian and K nilpotent.*

(1) If $Z(G) = 1$, then $F(G) = G^{\mathcal{N}}$.

(2) If $Z(G) \neq 1$, then $G = Z(G) \times H$, and H satisfying (1).

Proof. By the condition of Theorem and Lemma 3.6, the nilpotent residual of every subgroup is normal and $F(G) = G^{\mathcal{N}} \times Z(G)$ is abelian group. Then the nilpotent residual of every subgroup has a complement by Lemma 3.7 (1). Specially, $F(G)$ has a complement K in G, that is, $G = F(G) \rtimes K$, and K is nilpotent.

(1) If $Z(G) = 1$, then $F(G) = G^{\mathcal{N}}$.

(2) If $Z(G) \neq 1$, then, by Lemma 3.7 (2), $G = Z(G) \times H$, where $Z(G)$ is elementary abelian and $Z(H) = 1$. So H satisfying (1).

The following results are consequences of Theorem 3.1.

Theorem 3.2 *Let G be a $N^{\mathcal{N}}$-group. Then*

(a) G is a nilpotent group.

(b) G is a non-nilpotent group. If $\Phi(G) = 1$, then the nilpotent residual of every subgroup has a complement and $G = F(G) \rtimes K$ with $F(G)$ abelian and K nilpotent.

Theorem 3.3 *Let G be a D-group. Then*

(a) G is a abelian group.

(b) G is a non-abelian group. If $\Phi(G) = 1$, then the derived subgroup of every subgroup has a complement and $G = F(G) \rtimes K$ with $F(G)$ and K abelian.

4 Minimal non-$N^{\mathcal{N}}$-groups

Definition 4.1 *A group G is called a minimal non-$N^{\mathcal{N}}$-group if G is not a $N^{\mathcal{N}}$-group, but every proper subgroup of G is a $N^{\mathcal{N}}$-group.*

The semidirect product of Q_8 with S_3 show that the quotient group of a minimal non-$N^{\mathcal{N}}$-group can be not a $N^{\mathcal{N}}$-group. However, $G/\Phi(G)$ is a minimal non-$N^{\mathcal{N}}$-group or $N^{\mathcal{N}}$-group if G is a minimal non-$N^{\mathcal{N}}$-group and $\Phi(G) \neq 1$.

Lemma 4.1 *If G is a minimal non-$N^{\mathcal{N}}$-group and $\Phi(G) \neq 1$, then either $G/\Phi(G)$ is a minimal non-$N^{\mathcal{N}}$-group or $N^{\mathcal{N}}$-group.*

Proof. Let H be a maximal subgroup of G and $K \leq H$. Since G is a minimal non-$N^{\mathcal{N}}$-group, H is a $N^{\mathcal{N}}$-group, then $K^{\mathcal{N}} \trianglelefteq H$. We consider $G/\Phi(G)$ and its maximal subgroup $H/\Phi(G)$. It is clear that $(K\Phi(G)/\Phi(G))^{\mathcal{N}} \trianglelefteq H/\Phi(G)$, so $H/\Phi(G)$ is a $N^{\mathcal{N}}$-group, and every maximal subgroup of $G/\Phi(G)$ is a $N^{\mathcal{N}}$-group. Then $G/\Phi(G)$ is a minimal non-$N^{\mathcal{N}}$-group or $N^{\mathcal{N}}$-group.

Lemma 4.2 *If G is a minimal non-$N^{\mathcal{N}}$-group and $\Phi(G) = 1$, then G is minimal simple or solvable. And if G is a solvable group, then every proper quotient group of G is a $N^{\mathcal{N}}$-group.*

Proof. Since every proper subgroup of G is a $N^{\mathcal{N}}$-group and Lemma 3.3 (1), very proper subgroup of G is solvable. If G is a simple group, then G is a minimal simple group. If G is not a simple group, then there exists a non-trivial proper normal subgroup N of G. By $\Phi(G) = 1$, there exists a maximal subgroup M of G such that $G = MN$. Then G/N is a $N^{\mathcal{N}}$-group since M is a $N^{\mathcal{N}}$-group. It follows from M and N are $N^{\mathcal{N}}$-groups that G is soluble.

In order to classify the simple minimal non-$N^{\mathcal{N}}$-groups, we need some lemmas.

Lemma 4.3 [20, Theorem 4.3] and [18, Lemma 1] *A group G is called a 2-Con-Cos group if the following conditions are satisfied for a proper derived subgroup $G^{'}$ of G,*

(i) $G^{'}x = cl(x)$, for all x in $G - G^{'}$, (ii) $G^{'} = 1 \cup cl(a)$, for some a in G, where $cl(g)$ denotes the conjugacy class of $g \in G$. Then

(1) G is a 2-Con-Cos group with $Z(G) = 1$ if and only if G is a Frobenius group of the type $C_p^r \rtimes C_{p^r-1}$ for some prime p and some $r \geq 1$.

(2) If G is a 2-Con-Cos group, $N \trianglelefteq G$, then $N = 1$ or $N \geq G^{'}$.

Lemma 4.4 *Let G be a Frobenius group of the type $C_p^n \rtimes C_{p^n-1}$. Then G is a $N^{\mathcal{N}}$-group precisely when $p^n - 1$ is a prime.*

Proof. By Lemma 4.3, $C_p^n = G^{'} = G^{\mathcal{N}}$ is the minimal normal subgroup of G. If $p^n - 1$ is a prime, then, by Example 3.2 (3), G is a $N^{\mathcal{N}}$-group. Let $C = C_{p^n-1}$ be a cyclic group which is not of prime order and B a non-trivial maximal subgroup of C and $A = C_p^n$. Since G is a Frobenius group, $V \cong C_p^n$ is an irreducible and faithful module for C over the finite field $GF(p)$ of p elements and dim $V = n$. Now, by [8, Theorem 9.16], the order of B is $(p^n - 1)/p$. So the dimension of every irreducible and faithful module for B over $GF(p)$ is less than n. In particular, A is not irreducible for B. Moreover, $C_B(A) = \mathrm{Ker}(B \text{ on } A) = 1$ since A is faithful for C. According to Clifford theorem, A is completely irreducible and so there exists an irreducible B-submodule of A which is not centralized by B, A_1 say. Now A_1 is the nilpotent residual of A_1B and A_1 cannot be normal in G since A is a minimal normal subgroup of G, that is, G is not a $N^{\mathcal{N}}$-group.

Now, we are ready to classify the simple minimal non-$N^{\mathcal{N}}$-groups.

Theorem 4.1 *If G is a simple minimal non-$N^{\mathcal{N}}$-group, then G is isomorphic to one of following groups.*

(a) $PSL(2, p)$, p is a prime, $p > 3$, $p^2 \not\equiv 1(mod5)$, $p^2 \not\equiv 1(mod16)$.

(b) $PSL(2, 2^q)$, q is a prime and $2^q - 1$ also is a prime.

(c) $PSL(2, 3^q)$, q is an odd prime and $\frac{3^q-1}{2}$ also is a prime.

(d) $Sz(2^q)$, q is an odd prime and $2^q - 1$ also is a prime.

Proof. By the classification of minimal simple groups [30], G may have following 5 types: (1) $PSL(2, p)$, p is a prime, $p > 3$, $p^2 \not\equiv 1(\mathrm{mod}5)$. (2) $PSL(2, 2^q)$, q is a prime. (3) $PSL(2, 3^q)$, q is an odd prime. (4) $Sz(2^q)$, q is a prime. (5) $PSL(3, 3)$.

(a) All maximal subgroups of (1) are: (1.1) Dihedral group of order $2\frac{p\pm1}{2}$, (1.2) $C_p \rtimes C_{\frac{p-1}{2}}$, (1.3) A_4, (1.4) S_4 if $p^2 \equiv 1(\mathrm{mod}16)$.

It is clear that dihedral groups, meta-cyclic groups and Schmidt groups are all $N^{\mathcal{N}}$-groups, then (1.1), (1.2), (1.3) are all $N^{\mathcal{N}}$-groups. However $N^{\mathcal{N}}(S_4) = 1$, then (a) is as required.

(b) All maximal subgroups of (2) are: (2.1) Dihedral group of order $2(2^q \pm 1)$, (2.2) $C_2^q \rtimes C_{2^q-1}$, which is order of $2^q(2^q - 1)$, (2.3) A_4 if $q = 2$.

It is easy to see that (2.1) and (2.3) are all $N^{\mathcal{N}}$-groups. Apply Lemma 4.4 to (2.2), we can see $C_2^q \rtimes C_{2^q-1}$ is a $N^{\mathcal{N}}$-group if $2^q - 1$ is a prime, and then $PSL(2, 2^q)$ is a minimal non-$N^{\mathcal{N}}$-group. Otherwise, $PSL(2, 2^q)$ is not a minimal non-$N^{\mathcal{N}}$-group.

(c) All maximal subgroups of (3) are: (3.1) Dihedral group of order $2(\frac{3^q\pm1}{2})$, (3.2) $C_3^q \rtimes C_{\frac{(3^q-1)}{2}}$, which is order of $3^q\frac{(3^q-1)}{2}$, (3.3) A_4.

It is clear by the similar argument with (b).

(d) All maximal subgroups of (4) are: (4.1) Frobenius group $P_2 \rtimes C_{2^q-1}$, where P_2 is non-abelian, $|P_2| = 2^{2q}$, (4.2) Dihedral group of order $2(2^q-1)$, (4.3) $C_{2^q \pm 2^{\frac{q+1}{2}}+1} \rtimes C_4$.

Also it is clear that (4.1) and (4.3) are all $N^{\mathcal{N}}$-groups. If 2^q-1 is a prime, then $P_2 \rtimes C_{2^q-1}$ is a Schmidt group, so $Sz(2^q)$ is a minimal non-$N^{\mathcal{N}}$-group. If 2^q-1 is not a prime, then $P_2 \rtimes C_{2^q-1}/\Phi(P_2) \cong C_2^q \rtimes C_{2^q-1}$, by Lemma 4.4, it is not a $N^{\mathcal{N}}$-group, so $Sz(2^q)$ is not a minimal non-$N^{\mathcal{N}}$-group.

(e) Since S_4 is a maximal subgroup of $PSL(3,3)$ and $N^{\mathcal{N}}(S_4) = 1$, $PSL(3,3)$ is not a minimal non-$N^{\mathcal{N}}$-group. Now the theorem is complete.

After classifying the simple minimal non-$N^{\mathcal{N}}$-groups, we turn to solvable minimal non-$N^{\mathcal{N}}$-groups.

Theorem 4.2 *If G is a solvable minimal non-$N^{\mathcal{N}}$-group and $\Phi(G) = 1$, then*

(1) Assume $G^{\mathcal{N}}$ is not nilpotent, then

(1.a) $G = F(G) \rtimes H$, where $F(G)$, H are unique minimal normal and maximal subgroup of G.

(1.b) H is a schimdt group. Let $F(G)$ be p-group and $H = Q \rtimes R$ or $Q \rtimes P_1$, where Q, R and P_1 are a normal Sylow q-subgroup, cyclic Sylow r and p-subgroup of H. Then $G = F(G) \rtimes (Q \rtimes R)$ or $F(G) \rtimes (Q \rtimes P_1)$ and $G^{\mathcal{N}} = F(G) \rtimes Q$ is a abelian-by-nilpotent group, $G^{\mathcal{N}^2} = F(G)$.

(1.c) $B^{\mathcal{N}} = Q$ or $B^{\mathcal{N}} < F(G)$, where B is a subgroup of G such that $B^{\mathcal{N}} \ntrianglelefteq G$.

(2) Assume $G^{\mathcal{N}}$ is nilpotent, then

(2.a) $G = P \rtimes K$, where P, K are Sylow p-group and nilpotent Hall p'-subgroup of G.

(2.b) $P = F(G) = B^{\mathcal{N}}N = G^{\mathcal{N}} = [P,K]$ is an elementary abelian p-group, where $B^{\mathcal{N}} \ntrianglelefteq G$ and N is a minimal normal subgroup in G.

(2.c) Let $|P| = p^n$. Then $|G|$ divides $|P| \cdot |GL(n,p)|$.

(2.d) Let M be a maximal subgroup of G. If $B \leq M$, then $M \trianglelefteq G$, $|G : M| = q \neq p$. If $B \nleq M$, then $M \trianglelefteq G$ or $M = (F(G) \cap M) \rtimes K$.

Proof. Our theorem will be proved by following two cases according to $G^{\mathcal{N}}$ is nilpotent or not.

(1) If $G^{\mathcal{N}}$ is not nilpotent, then the proof is divided into following 5 steps.

(1.1) $F(G)$ is a unique minimal normal subgroup of G.

By $\Phi(G) = 1$, $F(G)$ is a direct product of some minimal normal subgroups of G. If there exists two different minimal normal subgroups N_1, N_2 of G, then G/N_1, G/N_2 are $N^{\mathcal{N}}$-groups. Then $G^{\mathcal{N}}$ is nilpotent by Lemma 3.3 (1) and 3.4, a contradiction. Hence $F(G)$ is the unique minimal normal subgroup of G.

(1.2) $G = F(G) \rtimes H$, where H is a non-nilpotent maximal subgroup of G.

Again since $\Phi(G) = 1$, there exists a maximal subgroup H of G such that $G = F(G)H$. It follows from $F(G) \cap H \trianglelefteq G$ and (1.1) that $F(G) \cap H = 1$. Therefore $F(G) = C_G(F(G))$ and $H = G/F(G)$ is not nilpotent.

(1.3) H is a Schmidt group.

Since G is solvable, H is solvable, then there exists a maximal normal subgroup M of H such that $|H : M| = q$. Let $F(G)$ be an elementary abelian p-group.

By $M \trianglelefteq H$, we get $N_G(F(G)M) \geq H$ and $N_G(F(G)M) \geq F(G)M$, then $F(G)M \trianglelefteq G$. It follows that $F(F(G)M) = F(G)$. Then M is nilpotent since $M^{\mathcal{N}} \cong (F(G)M/F(F(G)M))^{\mathcal{N}} = 1$. Next we prove that M is a p'-group. If not, let $M_p \in \mathrm{Syl}_p(M)$. Then $M_p \trianglelefteq H$ and $F(G)M_p \trianglelefteq G$, so $F(G)M_p = F(G)$ and $M_p \leq F(G)$, a contradiction.

If $q \neq p$, then H is a p'-group and $F(G)$ is a normal Sylow p-subgroup of G. Let H_1 be a proper subgroup of H. Since $F(F(G)H_1) = F(G) \times O_{p'}(F(G)H_1)$, $O_{p'}(F(G)H_1) = 1$ and $F(F(G)H_1) = F(G)$. Then H_1 is nilpotent since $H_1^{\mathcal{N}} \cong (F(G)H_1/F(G))^{\mathcal{N}} = (F(G)H_1/F(F(G)H_1))^{\mathcal{N}} = 1$, so H is a Schmidt p'-group.

If $q = p$, then there exists subgroup A of order p such that $H = MA$. Let T be a maximal subgroup of H.

Case 1. T does not contain a subgroup of order p, then $F(F(G)T) = F(G)$. By the similar argument above, we can get T is nilpotent.

Case 2. T contains a subgroup $A^h (h \in H)$ of order p. If $F(G)A^h \ntrianglelefteq F(G)T$, then $F(F(G)T) = O_p(F(G)T) = F(G)$, so T is nilpotent. If $F(G)A^h \trianglelefteq F(G)T$, then $A^h = F(G)A^h \cap T \trianglelefteq T$, and $T = A^h M_1$, $M_1 < M$, so $[A^h, M_1] \leq A^h \cap M = 1$, that is, T is nilpotent.

Hence, in either cases above, H is a Schmidt group.

(1.4) $G^{\mathcal{N}} = F(G) \rtimes Q$ is a abelian-by-nilpotent group, and $G^{\mathcal{N}^2} = F(G)$.

By (1.3), $H = Q \rtimes R$ or $Q \rtimes P_1$, where Q, R and P_1 are normal Sylow q-subgroup, cyclic Sylow r and p-subgroup of H. Then $G = F(G) \rtimes (Q \rtimes R)$ or $F(G) \rtimes (Q \rtimes P_1)$ and $G^{\mathcal{N}} = F(G) \rtimes Q$ by Lemma 3.4. Therefore $G^{\mathcal{N}}$ is a $N^{\mathcal{N}}$-group. If $q \neq 2$, then $\exp(Q) = q$, and $G^{\mathcal{N}}$ is metabelian group. If $q = 2$, then $\exp(\Phi(Q)) = 2$, and $G^{\mathcal{N}}$ is abelian-by-nilpotent group.

(1.5) $B^{\mathcal{N}} = Q$ or $B^{\mathcal{N}} < F(G)$, where B is a subgroup of G such that $B^{\mathcal{N}} \ntrianglelefteq G$.

Since G is a minimal non-$N^{\mathcal{N}}$-group, there at least exists a subgroup B of G such that $B^{\mathcal{N}} \ntrianglelefteq G$. If $BF(G) = G$, then, by (1.2), $B \cong H$, and therefore $B^{\mathcal{N}} = Q$ by (1.4). If $BF(G) < G$, then $BF(G)/F(G)$ is nilpotent by (1.3), so $B^{\mathcal{N}} \leq (BF(G))^{\mathcal{N}} \leq F(G)$, that is, $B^{\mathcal{N}} < F(G)$.

(2) If $G^{\mathcal{N}}$ is nilpotent, then $N^{\mathcal{N}}(G) > 1$ and $G^{\mathcal{N}} \leq F(G)$. Also it is easy to see $G = N^{\mathcal{N}}(G)_2$ and the Fitting length of G equals 2. Following proof can be divided into 7 steps.

(2.1) $F(G) = B^{\mathcal{N}}N$, where $B^{\mathcal{N}} \ntrianglelefteq G$ and N is a minimal normal subgroup of G.

Since G is a minimal non-$N^{\mathcal{N}}$-group, there at least exists a subgroup B of G such that $B^{\mathcal{N}} \ntrianglelefteq G$. By $(BN/N)^{\mathcal{N}} \trianglelefteq G/N$ hold for any normal subgroup N of G, $B^{\mathcal{N}}N \trianglelefteq G$. Specially, we consider the case N is a minimal normal subgroup of G.

If N is a unique minimal normal subgroup of G, then, by $\Phi(G) = 1$, $F(G) = N = G^{\mathcal{N}}$. If minimal normal subgroup of G is not unique, then, for another minimal normal group T of G, $T \cap B^{\mathcal{N}}N$ is normal. Since $B^{\mathcal{N}}T \cap B^{\mathcal{N}}N = B^{\mathcal{N}}(T \cap B^{\mathcal{N}}N) \trianglelefteq G$, we get $T \leq B^{\mathcal{N}}N$, it follows that every minimal normal subgroup of G is contained in $B^{\mathcal{N}}N$. By $\Phi(G) = 1$, $F(G) = B^{\mathcal{N}}N$.

(2.2) $F(G) = P$, where $P \in \mathrm{Syl}_p(G)$.

Now we claim that $B^{\mathcal{N}}$ does not contain any non-trivial normal subgroup of G. If not, let $T \trianglelefteq G$ and $T \leq B^{\mathcal{N}}$. Then $(B/T)^{\mathcal{N}} = B^{\mathcal{N}}/T \trianglelefteq G/T$, so $B^{\mathcal{N}} \trianglelefteq G$, a contradiction.

Let N be an elementary abelian p-group. Then $F(G) = N$ or $F(G) = B^{\mathcal{N}}N$ by (2.1). It is easy to see $F(G)_{p'} \trianglelefteq G$, therefore $F(G)_{p'} = 1$ by the claim above. Then $F(G) = O_p(G)$. Thus $G/F(G) = O_{p'}(G/F(G))$, and then $F(G) = P$, where $P \in \mathrm{Syl}_p(G)$.

(2.3) $F(G) = G^{\mathcal{N}}$ and $Z(G) = 1$.

By Lemma 3.6, $F(G) = G^{\mathcal{N}} \times Z(G)$. It follows from (2.1) that $B^{\mathcal{N}}N = G^{\mathcal{N}} \times Z(G)$. If $N \leq Z(G)$, then $Z(G) = N$ and $H^{\mathcal{N}} = G^{\mathcal{N}}$, a contradiction. Then $N \leq G^{\mathcal{N}}$ and $Z(G) = 1$. Hence $F(G) = B^{\mathcal{N}}N = G^{\mathcal{N}}$.

(2.4) $G = P \rtimes K$, where K is nilpotent Hall p'-subgroup, $P = [P, K]$.

By (2.2) and (2.3), there exists a Hall p'-subgroup K such that $G = P \rtimes K$ and K is a nilpotent p'-subgroup. Apply Fitting lemma, we get $P = [P, K] \times C_P(K) = [P, K]$ by $Z(G) = 1$.

(2.5) Let $|P| = p^n$. Then $|G|$ divides $|P| \cdot |GL(n, p)|$.

By (2.2) and $\Phi(G) = 1$, P is elementary p-group, and then G/P is isomorphic to a subgroup of $\mathrm{Aut}(P)$. Hence $|G|$ divides $|P| \cdot |GL(n, p)|$ by (2.4).

(2.6) If $B \leq M$ and M is a maximal subgroup of G, then $M \trianglelefteq G$, $|G : M| = q \neq p$.

Also we claim that B is contained in a unique maximal subgroup M of G. If there exists two different maximal subgroups M, M_1 such that $B \leq M$, $B \leq M_1$, then $H^{\mathcal{N}} \trianglelefteq M$ and $H^{\mathcal{N}} \trianglelefteq M_1$, so $H^{\mathcal{N}} \trianglelefteq \langle M, M_1 \rangle = G$, a contradiction.

If $F(G) \nleq M$, then $G = F(G)M$. By $F(G)$ is abelian group, $B^{\mathcal{N}} \trianglelefteq G$, a contradiction. Then $F(G) \leq M$ and $M/F(G)$ is the maximal subgroup of $G/F(G)$, so $M/F(G) \trianglelefteq G/F(G)$ and $|G/F(G) : M/F(G)| = q$, that is, $M \trianglelefteq G$ and $|G : M| = q$.

(2.7) If $B \nleq E$ and E is a maximal subgroup of G, then $E \trianglelefteq G$ or $E = (F(G) \cap E) \rtimes K$.

It is also clear that there at least exists a maximal subgroup E of G such that the nilpotent residual of every subgroup of E is normal. Otherwise, every maximal subgroup contains subgroup that its nilpotent residual is not normal, then, by (2.6), every maximal subgroup of G is normal and G is nilpotent, a contradiction. If $E \geq F(G)$, then $E/F(G)$ is the maximal subgroup of $G/F(G)$, so $E \trianglelefteq G$. If $E \ngeq F(G)$, then $G = F(G)E$ and $E = (F(G) \cap E) \rtimes K$.

Corollary 4.1 *If G is a soluble minimal non-$N^{\mathcal{N}}$-group and $\Phi(G) = 1$, then,*

(1) for a prime $p \in \pi(G)$, $B^{\mathcal{N}}$ is a p-group for any subgroup B of G such that $B^{\mathcal{N}} \ntrianglelefteq G$.

(2) $1 \leq l_p(G) \leq 2$.

(3) $P \geq F(G)$, where $P \in Syl_p(G)$.

Corollary 4.2 *If G is a minimal non-$N^{\mathcal{N}}$-group and $\Phi(G) = 1$, then $Z(G) = 1$.*

Theorem 4.3 *If G is a minimal non-$N^{\mathcal{N}}$-group and $\Phi(G) \neq 1$, then $G/\Phi(G)$ satisfies Theorem 3.2, 4.1 or 4.2.*

Proof. It is clear from Lemma 4.2.

Hence, we have obtained a simple characterization of minimal non-$N^{\mathcal{N}}$-groups by Theorem above.

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