COEFFICIENT ESTIMATES FOR HARMONIC v-BLOCH MAPPINGS AND HARMONIC K-QUASICONFORMAL MAPPINGS

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ABSTRACT. Let $f(z) = h(z) + \overline{g(z)}$ be a harmonic v-Bloch mapping defined in the unit disk $\mathbb{D}$ with $\|f\|_{B_v} \leq M$, where $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic in $\mathbb{D}$. In this paper, we obtain the coefficient estimates for f as follows: $|a_n|^2 + |b_n|^2 \leq A_n(v, M)$, where $A_n(v, M)$ is given in Theorem 1. Furthermore we prove that for $v < 1$, $\lim_{n\to\infty} A_n(v, M) = 0$ and for $v \geq 1$, $A_n(v, M) \leq O(n^{2v-2})$. Moreover if f is a harmonic K-quasiconformal self-mapping of $\mathbb{D}$, then $|a_n| + |b_n| \leq B_n(K)$, where $B_n(K)$ is given in Theorem 3 such that $\lim_{n\to\infty} B_n(K) = 0$ and $B_n(1) = \frac{4}{n\pi}$.

1. INTRODUCTION

A complex-valued function $f(z)$ of class C^2 is said to be a harmonic mapping, if it satisfies $f_{z\bar{z}} = 0$. Assume that $f(z)$ is a harmonic mapping defined in a simply connected domain $\Omega \subseteq \mathbb{C}$. Then $f(z)$ has the canonical decomposition $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in Ω. Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk, throughout this paper we consider harmonic mappings $f(z)$ in $\mathbb{D}$.

For $z \in \mathbb{D}$, let

$$\Lambda_f(z) = \max_{0\leq\theta\leq2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = |f_z(z)| + |f_{\bar{z}}(z)|$$

and

$$\lambda_f(z) = \min_{0\leq\theta\leq2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = ||f_z(z)| - |f_{\bar{z}}(z)||.$$

It is well known that f is locally univalent and sense-preserving in $\mathbb{D}$ if and only if its Jacobian satisfies

$$J_f(z) = \lambda_f(z)\Lambda_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 > 0 \quad \text{for} \quad z \in \mathbb{D}.$$

Let

$$\beta_h = \sup_{z,w\in\mathbb{D},\ z\neq w} \frac{|f(z) - f(w)|}{\rho(z, w)}$$

2000 *Mathematics Subject Classification.* Primary: 30C62; Secondary: 30C20, 30F15.

Key words and phrases. Harmonic quasiconformal mappings; Coefficient estimates; S_H^0 Class; harmonic Bloch mappings.

File: Zhu(2013).tex, printed: 28-11-2013, 15.28.

The author of this work was supported by the National Natural Science Foundation of China under Grant 11101165 and Promotion Program for Young and Middle-aged Teacher in Science and Technology Research of Huaqiao University (ZQN-YX110).

be the Bloch constant of f, where ρ denotes the hyperbolic distance in $\mathbb{D}$, and $\rho(z,w)=\frac{1}{2}\ln\left(\frac{1+r}{1-r}\right)$ where r is the modulus of $\frac{z-w}{1-\bar{z}w}$. In [7], we see that the Bloch constant of $f=h+\bar{g}$ can be expressed in terms of the modulus of the derivatives of h and g as follows:

$$\beta_h=\sup_{z\in\mathbb{D}}(1-|z|^2)(|h'(z)|+|g'(z)|).$$

For the extensive discussions on harmonic Bloch mappings, see [3], [4], [5] and [13].

For $v\in(0,\infty)$, a harmonic mapping f is called a harmonic v-Bloch mapping if and only if

$$\|f\|_{B_v}:=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^v\Lambda_f(z)<\infty. \tag{1}$$

Harmonic mappings are nature generalizations of analytic functions. Many classical results of analytic functions under some suitable restrictions can be extended to harmonic mappings. One of the well-known results is the Landau type theorems for harmonic mappings. Many authors have considered such an active topic.

In [14], Liu proved the following theorems.

Theorem A. *Suppose that f is a harmonic mapping of $\mathbb{D}$ with $f(0)=\lambda_f(0)-1=0$. If $\Lambda_f(z)\le\Lambda$ for all $z\in\mathbb{D}$, then*

$$|a_n|+|b_n|\le\frac{\Lambda^2-1}{n\Lambda}\quad n=2,3,\ldots. \tag{2}$$

The above estimates are sharp for all $n\ge 2$ with extremal functions $f_n(z)=\Lambda^2 z-\int_0^z\frac{(\Lambda^3-\Lambda)dz}{\Lambda+z^{n-1}}$.

Theorem B. *Let f be a harmonic mapping of $\mathbb{D}$ with $f(0)=\lambda_f(0)-1=0$, and $\Lambda_f(z)\le\Lambda$ for all $z\in\mathbb{D}$. Then f is univalent in the disk D_{r_1} with $r_1=\frac{1}{1+\Lambda-\frac{1}{\Lambda}}$ and $f(D_{r_1})$ contains a schlicht disk D_{σ_1} with*

$$\sigma_1=\begin{cases}1+(\Lambda-\frac{1}{\Lambda})\ln\frac{\Lambda-\frac{1}{\Lambda}}{1+\Lambda-\frac{1}{\Lambda}} & \Lambda>1\\ 1 & \Lambda=1.\end{cases}$$

The result is sharp when $\Lambda=1$.

Subsequently, in 2011 SH.Chen et al. [4] proved the following theorems.

Theorem C. *Let $f=h+\bar{g}$ be a harmonic v-Bloch mapping, where h and g are analytic in $\mathbb{D}$ with the expansions*

$$h(z)=\sum_{n=1}^{\infty}a_nz^n\quad\textit{and}\quad g(z)=\sum_{n=1}^{\infty}b_nz^n. \tag{3}$$

If $\lambda_f(0)=\alpha$ for some $\alpha\in(0,1)$ and $\|f\|_{B_v}\le M$ for $M>0$. Then for $n\ge 2$,

$$|a_n|+|b_n|\le A_n(\alpha,v,M)=\inf_{0<r<1}\mu(r)$$

where

$$\mu(r) = \frac{M^2 - \alpha^2(1-r^2)^{2v}}{nr^{n-1}(1-r^2)^v M}.$$

Particularly, if $v = M = \alpha = 1$*, then* $A_2(1,1,1) = 0$*,* $A_3(1,1,1) = \frac{1}{3}$ *and for* $n \geq 4$*,* $A_n(1,1,1) < \frac{(n+1)eM}{2n}$*. The above results are sharp for* $n = 2$ *and* $n = 3$*.*

Theorem D. *Let* f *be a harmonic mapping with* $f(0) = \lambda_f(0) - \alpha = 0$ *and* $\|f\|_{B_v} \leq M$*, where* M *and* $\alpha \in (0,1]$ *are constants. Then* f *is univalent in* $\mathbb{D}_{\rho_0}$*, where*

$$\rho_0 = \psi(r_0) = \max_{0<r<1} \psi(r), \quad \psi(r) = \frac{\alpha r(1-r^2)M}{\alpha M(1-r^2)^v - \alpha^2(1-r^2)^{2v} + M^2}.$$

Moreover, $f(\mathbb{D}_{\rho_0})$ *contains a univalent disk* $\mathbb{D}_{R_0}$ *with*

$$R_0 = r_0 \left[\alpha + \frac{M^2 - \alpha^2(1-r^2)^{2v}}{M(1-r_0^2)^v} \log \frac{M^2 - \alpha^2(1-r^2)^{2v}}{\alpha M(1-r_0^2)^v - \alpha^2(1-r_0^2)^{2v} + M^2}\right].$$

The coefficient estimates is crucial in obtaining Landau type theorems. In the second part of this paper by using Parseval equation we first obtain the coefficient estimates for harmonic v-Bloch mappings and then for $0 < v < \frac{1}{2}$ we obtain its Landau type theorems.

Assume that

$$f(z) = P[F](z) = \int_0^{2\pi} P(r, x-\varphi)F(x)dx$$

is a sense-preserving univalent harmonic mapping of $\mathbb{D}$ with the boundary function $F(x) = e^{i\gamma(x)}$ where

$$P(r, x-\varphi) = \frac{1-r^2}{2\pi(1 - 2r\cos(x-\varphi) + r^2)}$$

is the Poission kernel and $z = re^{i\varphi} \in \mathbb{D}$. Then $f(z)$ is called a harmonic K-quasiconformal mapping if there exists a constant k such that

$$\sup_{z\in\mathbb{D}} \left|\frac{f_{\bar{z}}(z)}{f_z(z)}\right| \leq k = \frac{K-1}{K+1}.$$

For harmonic K-quasiconformal mappings defined in $\mathbb{D}$, there are many interesting results (see [10], [12], [14], [18], [21], [22], [23], [24]). In [17] D.Partyka and K.Sakan proved the following theorem.

Theorem E. *Given* $K \geq 1$ *and let* $f(z) = P[F](z)$ *be a harmonic* K*-quasiconformal mapping of* $\mathbb{D}$ *onto itself, with the boundary function* $F(t)$*. If* $f(0) = 0$*, then for a.e.* $z = e^{it} \in \partial\mathbb{D}$

$$\frac{2^{5(1-K^2)/2}}{(K^2+K-1)^K} \leq |F'(t)| \leq K^{3K} 2^{5(K-\frac{1}{K})/2}. \tag{4}$$

Using this theorem, we obtain the coefficient estimates for $f = P[F]$ as follows:

$$|a_n| + |b_n| \le B_n(K) = \frac{4}{n\pi} K^{3K} 2^{5(K-\frac{1}{K})/2}, \quad n = 1, 2, \ldots.$$

2. Coefficient estimates for harmonic v-Bloch mappings

Theorem 1. *Assume that $f(z) = h(z) + \overline{g(z)}$ is a harmonic v-Bloch mapping such that $f(0) = 0$ and $\|f\|_{B_v} \le M$ for some constants $M > 0$, where $h(z)$ and $g(z)$ are given by* (3)*. Then the following inequality*

$$|a_n|^2 + |b_n|^2 \le A_n(v, M) \tag{5}$$

holds for all $n = 1, 2, 3, \ldots$, where

$$A_n(v, M) = \begin{cases} \frac{M^2}{n} \inf\limits_{0<t<1} \frac{1-(1-t^2)^{1-2v}}{t^{2n}(1-2v)} & v \neq \frac{1}{2} \\ \frac{M^2}{n} \inf\limits_{0<t<1} \frac{-\ln(1-t^2)}{t^{2n}} & v = \frac{1}{2}. \end{cases}$$

Furthermore if $0 < v < 1$, then $\lim\limits_{n\to\infty} A_n(v, M) = 0$. If $v \ge 1$, then $A_n(v, M) \le \frac{M^2}{2v-1} \frac{(n+1)^{2v-1}-1}{n} (1 + \frac{1}{n})^n = O(n^{2v-2})$.

Proof. Using the assumption that $f(0) = 0$ and $\|f\|_{B_v} \le M$, according to (1) we have

$$\Lambda_f(z) = |h'(z)| + |g'(z)| \le \frac{M}{(1 - |z|^2)^v} := \Lambda_r$$

holds for any $z = re^{i\theta} \in \mathbb{D}$. Using $f_\theta(z) = i\left[zh'(z) - \overline{zg'(z)}\right]$ and applying Parseval equation, then

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |f_\theta(re^{i\theta})|^2 d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=1}^{\infty} n a_n r^n e^{in\theta} - \sum_{n=1}^{\infty} n \overline{b_n} r^n e^{-in\theta} \right|^2 d\theta \\ &= \sum_{n=1}^{\infty} n^2 (|a_n|^2 + |b_n|^2) r^{2n}. \end{aligned}$$

It is easy to see that $|f_\theta(z)| \le |z| \Lambda_f(z) \le r\Lambda_r$. Hence

$$\sum_{n=1}^{\infty} n^2 (|a_n|^2 + |b_n|^2) r^{2n} \le r^2 \Lambda_r^2 \le \frac{r^2 M^2}{(1 - r^2)^{2v}}.$$

This implies that

$$\sum_{n=1}^{\infty} n^2 (|a_n|^2 + |b_n|^2) r^{2n-1} \le \frac{rM^2}{(1 - r^2)^{2v}}.$$

For any $0 < t < 1$, integral from both sides gives

$$\sum_{n=1}^{\infty} n(|a_n|^2 + |b_n|^2) \frac{t^{2n}}{2} \le M^2 \int_0^t \frac{r}{(1 - r^2)^{2v}} dr := M^2 \varphi(t). \tag{6}$$

(i). For $v = \frac{1}{2}$. In this case, $\varphi(t) = \frac{-\ln(1-t^2)}{2}$. It follows from (6) that

$$|a_n|^2 + |b_n|^2 \le \frac{M^2}{n}\frac{-\ln(1-t^2)}{t^{2n}}.$$

If $n = 1$, then $\min\limits_{0<t<1} \frac{M^2}{n}\frac{-\ln(1-t^2)}{t^2} = M^2$. For $n > 1$, since $\lim\limits_{t\to 0}\frac{-\ln(1-t^2)}{t^{2n}} = \infty = \lim\limits_{t\to 1}\frac{-\ln(1-t^2)}{t^{2n}}$, we see that $\inf\limits_{0<t<1}\frac{-\ln(1-t^2)}{t^{2n}}$ exists. Hence

$$|a_n|^2 + |b_n|^2 \le A_n\left(\frac{1}{2}, M\right) = \frac{M^2}{n}\inf_{0<t<1}\frac{-\ln(1-t^2)}{t^{2n}}.$$

Let $t_0 = \sqrt{\frac{n}{n+1}}$. Then

$$A_n\left(\frac{1}{2}, M\right) \le \frac{M^2}{n}\frac{-\ln(1-t_0^2)}{t_0^{2n}} = \frac{M^2\ln(n+1)}{n}(1+\frac{1}{n})^n. \tag{7}$$

This implies that $\lim\limits_{n\to\infty} A_n(\frac{1}{2}, M) = 0$.

(ii). For $v \ne \frac{1}{2}$. In this case $\varphi(t) = \frac{1-(1-t^2)^{1-2v}}{2(1-2v)}$. It follows from (6) that

$$|a_n|^2 + |b_n|^2 \le \frac{M^2}{n}\frac{1-(1-t^2)^{1-2v}}{(1-2v)t^{2n}} := \frac{M^2}{n}m(t).$$

If $v < \frac{1}{2}$, then $\inf\limits_{0<t<1} m(t) = \frac{1}{1-2v}$. Hence

$$A_n(v, M) \le \frac{M^2}{n(1-2v)}, \quad (v < \frac{1}{2}). \tag{8}$$

For $v > \frac{1}{2}$, $m(t) = \frac{1-(1-t^2)^{2v-1}}{(1-t^2)^{2v-1}(2v-1)t^{2n}} > 0$. If $n = 1$, then $\inf\limits_{0<t<1} m(t) = 2v-1$. Else if $n > 1$, then since $\lim\limits_{t\to 0} m(t) = \infty = \lim\limits_{t\to 1} m(t)$ we see that $\inf\limits_{0<t<1} m(t)$ exists. Therefore $A_n(v, M) = \frac{M^2}{n}\inf\limits_{0<t<1} m(t)$ and

$$A_n(v, M) \le \frac{M^2}{n}m(t_0) = \frac{M^2}{2v-1}\frac{(n+1)^{2v-1}-1}{n}(1+\frac{1}{n})^n, \quad (v > \frac{1}{2}). \tag{9}$$

It follows from (7), (8) and (9) that if $v < 1$, then $\lim\limits_{n\to\infty} A_n(v, M) = 0$. If $v = 1$, then $A_n(1, M) \le M^2(1+\frac{1}{n})^n$. If $v > 1$, then $A_n(v, M) \le \frac{M^2}{2v-1}\frac{(n+1)^{2v-1}-1}{n}(1+\frac{1}{n})^n = O(n^{2v-2})$.

This completes the proof. □

Remark 1. *We point out that* $|a_n| + |b_n| \le \sqrt{2(|a_n|^2 + |b_n|^2)} \le \sqrt{2A_n(v, M)}$. *This implies that for* $0 < v < 1$*, the coefficients of harmonic* v*-Bloch mappings would close to 0 as* $n \to \infty$. *Furthermore, our results shows that for* $v \ge 1$, $|a_n|+|b_n| \le O(n^{v-1})$. *The following example shows that Theorem 1 is sharp for* $v = 1$.

Example 1. *For* $v = 1$*, we consider harmonic function*

$$f(z) = \sum_{n=1}^{\infty} z^{2^n}.$$

Then

$$\frac{|zf'(z)|}{1-|z|} \leq \sum_{n=1}^{\infty}\left(\sum_{2^k\leq n} 2^k\right)|z|^n \leq \sum_{n=1}^{\infty} 2n|z|^n = \frac{2|z|}{(1-|z|)^2},$$

hence

$$(1-|z|^2)|f'(z)| \leq 4 \quad (|z|<1).$$

It follows from (1) *that* $f(z)$ *is a* 1*-Bloch harmonic function. Moreover, its coefficients do not tend to 0.*

Theorem 2. *Let* $f(z) = h(z) + \overline{g(z)}$ *be a harmonic* v*-Bloch mapping of* $\mathbb{D}$ *satisfying* $f(0) = \lambda_f(0) - 1 = 0$ *and* $0 < v < \frac{1}{2}$*. Then* f *is univalent in the disk* $\mathbb{D}_{r_*} := \{z : |z| < r_*\}$*, where* r_* *is the root of the following equation:*

$$1 - M\sqrt{\frac{2}{1-2v}}\Phi(r) = 0 \tag{10}$$

and $\Phi(r) := \sum\limits_{n=1}^{\infty} \sqrt{n+1}r^n$.

Proof. Let $z_1 = r_1e^{i\theta_1} \in \mathbb{D}_r$ and $z_2 = r_2e^{i\theta_2} \in \mathbb{D}_r$, where $0 < r < r_*$ and $z_1 \neq z_2$. For $0 < v < \frac{1}{2}$, applying Theorem 1 we have

$$|a_n| + |b_n| \leq \sqrt{2(|a_n|^2 + |b_n|^2)} \leq \sqrt{\frac{2}{1-2v}}\frac{M}{\sqrt{n}}.$$

Then

$$\begin{aligned}
|f(z_1) - f(z_2)| &\geq \lambda_f(0)|z_1 - z_2| - |z_1 - z_2|\sum_{n=2}^{\infty}(|a_n| + |b_n|)nr^{n-1} \\
&\geq |z_1 - z_2|\left(1 - M\sqrt{\frac{2}{1-2v}}\sum_{n=2}^{\infty}\sqrt{n}r^{n-1}\right) \\
&= |z_1 - z_2|\left(1 - M\sqrt{\frac{2}{1-2v}}\Phi(r)\right) \\
&:= |z_1 - z_2|\varphi(r).
\end{aligned}$$

Since $\varphi(r)$ is a continuous decreasing function satisfying $\varphi(0) = 1$, $\lim\limits_{r\to 1^-}\varphi(r) = -\infty$, we see that equation $\varphi(r) = 0$ has the root $0 < r_* < 1$. Then for any $0 < r < r_*$, we have $|f(z_1) - f(z_2)| > 0$. This shows that $f(z)$ is univalent in the disk D_{r_*}.

The proof is completed. □

For $M = 1$ and some constants $v \in \left(0, \frac{1}{2}\right)$, calculate by computer we obtain some r_* which were shown by the following table.

M	v	r_*
1	1/5	0.264534
1	1/4	0.248227
1	1/3	0.214222
1	49/100	0.0650995

3. Coefficient estimates for harmonic K-quasiconformal mappings

Theorem 3. *Given $K \geq 1$, let $f(z) = P[F](z) = h(z) + \overline{g(z)}$ be a harmonic K-quasiconformal self-mapping of $\mathbb{D}$ satisfying $f(0) = 0$ with the boundary function F, where*

$$h(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic in $\mathbb{D}$. Then

$$|a_n| + |b_n| \leq B_n(K) := \frac{4}{n\pi} K^{3K} 2^{5(K-1/K)/2} \quad n = 1, 2, \ldots. \tag{11}$$

In particular, if $K = 1$ then $|a_n| + |b_n| \leq B_n(1) = \frac{4}{n\pi}$.

Proof. For every $z = re^{i\theta} \in \mathbb{D}$,

$$f(re^{i\theta}) = \sum_{n=1}^{\infty} a_n r^n e^{in\theta} + \sum_{n=1}^{\infty} \overline{b_n} r^n e^{-in\theta}.$$

We find that

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta, \ n = 1, 2, \ldots,$$

$$\overline{b_n} r^n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{in\theta} d\theta, \ n = 1, 2, \ldots.$$

For every n (see [15] and [20]) we set $a_n = |a_n| e^{i\alpha_n}$, $b_n = |b_n| e^{i\beta_n}$ and $\theta_n = \frac{\alpha_n + \beta_n}{2n}$. Then

$$\begin{aligned} (|a_n| + |b_n|) r^n &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) [e^{-i\alpha_n} e^{-in\theta} + e^{i\beta_n} e^{in\theta}] d\theta \right| \\ &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) [e^{-in(\theta+\theta_n)} + e^{in(\theta+\theta_n))}] d\theta \right| \\ &= \left| \frac{1}{\pi} \int_0^{2\pi} f(re^{i\theta}) \cos n(\theta + \theta_n) d\theta \right|. \end{aligned}$$

Integrating by parts we have

$$(12) \qquad (|a_n| + |b_n|)r^n = \left| \frac{1}{n\pi} \int_0^{2\pi} f_\theta(re^{i\theta}) \sin n(\theta + \theta_n) d\theta \right|.$$

In [11, Theorem 2.8], D.Kalaj proved that the radial limits of f_θ and f_r exist almost everywhere and

$$\lim_{r\to 1^-} f_\theta(re^{i\theta}) = F'(\theta),$$

for almost every $z = re^{i\theta} \in \mathbb{D}$. Here F is the boundary function of f. Hence, tending $r \to 1^-$ in (12) and also using (4) we obtain:

$$|a_n| + |b_n| \leq \frac{1}{n\pi} \int_0^{2\pi} |F'(\theta)| |\sin n(\theta + \theta_n)| d\theta \leq \frac{4K^{3K} 2^{5(K-1/K)/2}}{n\pi}.$$

This completes the proof. □

Remark 2. *Given the boundary function $F(t) = \rho(t)e^{i\gamma(t)}$ of $\mathbb{R}$ onto a convex Jordan curve $\gamma \in C^{1,\mu}(0 < \mu \leq 1)$, suppose that $f(z) = P[F](z)$ is a harmonic K-quasiconformal mapping of $\mathbb{D}$ onto the convex domain bounded by γ. According to* [11, Theorem 3.1] *we know that $\|F'(t)\|_\infty < \infty$. Using* (12) *we can see that $|a_n| + |b_n| \leq \frac{4\|F'\|_\infty}{n\pi} \to 0$, as $n \to \infty$.*

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