[Revision] Submitted to:
Bulletin of the Malaysian Mathematical Sciences Society - November 29, 2013

A BANACH ALGEBRA REPRESENTATION THEOREM

MOHAMMAD MEHRYAR

Abstract. For a space X denote by $\mathrm{C}_0^\sigma(X)$ the set of all bounded continuous $f : X \to \mathbb{R}$ such that $|f|^{-1}([\epsilon, \infty))$ is σ-compact for all $\epsilon > 0$.

We give sufficient (and in certain cases, necessary) conditions for a Banach algebra to be of the form $\mathrm{C}_0^\sigma(X)$ for some locally compact space X.

1. Introduction

By a *space* it is meant a *topological space*; compact spaces and completely regular spaces are assumed to be Hausdorff. The underlying field of scalars will be the real line $\mathbb{R}$.

Consider a completely regular space X. The set of all continuous bounded $f : X \to \mathbb{R}$ is denoted by $\mathrm{C}_\mathrm{b}(X)$. Suppose that $f \in \mathrm{C}_\mathrm{b}(X)$. The *cozero-set* of f is $X \backslash f^{-1}(0)$ and is denoted by $\mathrm{Coz}(f)$, and the *support* of f is $\mathrm{cl}_X \mathrm{Coz}(f)$ and is denoted by $\mathrm{supp}(f)$. Denote

$$\mathrm{Coz}(X) = \big\{\mathrm{Coz}(f) : f \in \mathrm{C}_\mathrm{b}(X)\big\}.$$

The set of all $f \in \mathrm{C}_\mathrm{b}(X)$ such that $|f|^{-1}([\epsilon, \infty))$ is compact for all $\epsilon > 0$ is denoted by $\mathrm{C}_0(X)$ and the set of all $f \in \mathrm{C}_\mathrm{b}(X)$ such that $|f|^{-1}([\epsilon, \infty))$ is σ-compact for all $\epsilon > 0$ is denoted by $\mathrm{C}_0^\sigma(X)$.

In [10] (see also [9], [12] and [13]) it is proved that certain Banach subalgebras H of $\mathrm{C}_\mathrm{b}(X)$, where X is a completely regular space, are of the form $\mathrm{C}_0(Y)$ for some locally compact space Y. The space Y is constructed as a subspace of the Stone–Čech compactification of X. This motivated us to characterize Banach subalgebras H of $\mathrm{C}_\mathrm{b}(X)$ which are of the form $\mathrm{C}_0^\sigma(Y)$ for some locally compact space Y. Spaces similar to $\mathrm{C}_0^\sigma(Y)$ have been defined and studied in [1], [3] and [14], though the approach here is different. The theory of the Stone–Čech compactification will be used here as a primary tool.

The Stone–Čech compactification. Consider a completely regular space X. By a *compactification* γX of X it is meant a compact space γX which contains X densely. The *Stone–Čech compactification* βX of X is the unique compactification of X such that every continuous $f : X \to K$, where K is a compact space, is extendable to a mapping $f_\beta : \beta X \to K$ continuously. It is known that for every

2010 *Mathematics Subject Classification.* Primary 54D35, 54D65, 46J10, 46J25, 46E25, 46E15; Secondary 54C35, 46H05, 16S60.

Key words and phrases. Stone–Čech compactification; Commutative Gelfand–Naimark Theorem; Real Banach algebra.

completely regular space the Stone–Čech compactification exists. For more information and other background material on the subject the reader is referred to the texts [4] and [5].

2. The representation theorem

Definition 2.1. Consider a space X. Define

$$\mathrm{C}_0^\sigma(X)=\big\{f\in \mathrm{C}_\mathrm{b}(X):|f|^{-1}\big([\epsilon,\infty)\big) \text{ is } \sigma\text{-compact for all } \epsilon>0\big\}.$$

The following subspace of βX is defined in [10] and will be used in what follows.

Definition 2.2. Consider a completely regular space X. For a subset H of $\mathrm{C}_\mathrm{b}(X)$ define

$$\mathrm{e}_H X=\bigcup\big\{\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}\mathrm{Coz}(h):h\in H\big\}.$$

Remark 2.3. The space $\mathrm{e}_H X$ has been originally defined and studied in [6] (see also [7] and [11]) with H considered to be the set of all elements of $\mathrm{C}_\mathrm{b}(X)$ with support having a given topological property $\mathscr{P}$. For specific choices of the topological property $\mathscr{P}$ the structure of the space $\mathrm{e}_H X$ is known. For example, it is proved in [8] that

$$\mathrm{e}_H X=\mathrm{int}_{\beta X}\upsilon X$$

if $\mathscr{P}$ is pseudocompactness, where υX is the Hewitt realcompactification of X, and it is proved in [11] that

$$\mathrm{e}_H X=\beta X\backslash\mathrm{cl}_{\beta X}(\upsilon X\backslash X)$$

if $\mathscr{P}$ is realcompactness. (A completely regular space X is said to be *pseudocompact* if there is no unbounded continuous $f:X\to\mathbb{R}$. Also, a completely regular space is *realcompact* if it is homeomorphic to a closed subspace of some product of the real line $\mathbb{R}$. The *Hewitt realcompactification* υX of a completely regular space X – which may be assumed to be a subspace of βX – is a realcompact space containing X densely, such that every continuous $f:X\to\mathbb{R}$ is extendible to a mapping $f_\upsilon:\upsilon X\to\mathbb{R}$ continuously. The Hewitt realcompactification of a completely regular space exists.)

Consider a space X and a dense subspace D of X. Then

$$\mathrm{cl}_X U=\mathrm{cl}_X(U\cap D)$$

for any open subspace U of X.

The following simple lemma will be used in the following.

Lemma 2.4. *Consider a completely regular space X. Suppose that H is a subset of $\mathrm{C}_\mathrm{b}(X)$ such that for any $x\in X$ there exists some $h\in H$ with $h(x)\neq 0$. Then*

$$X\subseteq \mathrm{e}_H X.$$

Proof. Suppose that $x\in X$. Suppose that $h\in H$ with $h(x)\neq 0$. Because

$$\mathrm{Coz}(h_\beta)\subseteq \mathrm{cl}_{\beta X}\mathrm{Coz}(h_\beta)=\mathrm{cl}_{\beta X}\big(X\cap\mathrm{Coz}(h_\beta)\big)=\mathrm{cl}_{\beta X}\mathrm{Coz}(h),$$

then

$$x\in\mathrm{Coz}(h_\beta)\subseteq\mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}\mathrm{Coz}(h)\subseteq \mathrm{e}_H X.$$

□

The following gives sufficient (and in some cases, necessary) conditions for a Banach algebra H to be of the form $\mathrm{C}_0^\sigma(Y)$ for some locally compact space Y. The proof for the sufficiency part is a modification of the proof of Theorem 2.3 in [10]; full detail is provided here for completeness.

Theorem 2.5. *Suppose that H is a Banach algebra contained in $\mathrm{C}_\mathrm{b}(X)$ for a completely regular space X such that*

(1) *for any $x \in X$ there exists some $h \in H$ such that $h(x) \neq 0$, and*
(2) *for any $f \in \mathrm{C}_\mathrm{b}(X)$, if $\mathrm{supp}(f) \subseteq \mathrm{supp}(h)$ for some $h \in H$, then $f \in H$.*

Then

$$H = \mathrm{C}_0^\sigma(Y)$$

for a locally compact space Y. The converse holds true if Y is also metrizable.

Proof. Suppose that H is a Banach subalgebra of $\mathrm{C}_\mathrm{b}(X)$ for a completely regular space X such that conditions (1) and (2) hold true. It will be shown that

$$H = \mathrm{C}_0^\sigma(Y)$$

with $Y = \mathrm{e}_H X$.

For any $f \in \mathrm{C}_\mathrm{b}(X)$ set

$$f_H = f_\beta | \mathrm{e}_H X.$$

Because $X \subseteq \mathrm{e}_H X$ by Lemma 2.4, the mapping f_H extends f.

Claim. *Suppose that $f \in \mathrm{C}_\mathrm{b}(X)$. Then the following are equivalent:*

(i) *$f \in H$.*
(ii) *$f_H \in \mathrm{C}_0^\sigma(\mathrm{e}_H X)$.*

Proof of the claim. (i) *implies* (ii). Because

$$\mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} \mathrm{Coz}(f) \subseteq \mathrm{e}_H X$$

and

$$\mathrm{Coz}(f_\beta) \subseteq \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} \mathrm{Coz}(f),$$

since

$$\mathrm{Coz}(f_\beta) \subseteq \mathrm{cl}_{\beta X} \mathrm{Coz}(f_\beta) = \mathrm{cl}_{\beta X} \big(X \cap \mathrm{Coz}(f_\beta)\big) = \mathrm{cl}_{\beta X} \mathrm{Coz}(f),$$

we have $\mathrm{Coz}(f_\beta) \subseteq \mathrm{e}_H X$. Assume that $\epsilon > 0$. Then

$$|f_H|^{-1}\big([\epsilon, \infty)\big) = |f_\beta|^{-1}\big([\epsilon, \infty)\big)$$

is closed in βX and is then compact, and of course is then σ-compact.

(ii) *implies* (i). Fix a natural number n. Then $|f_H|^{-1}([1/n, \infty))$ is a σ-compact subspace of $\mathrm{e}_H X$. Hence

$$|f_H|^{-1}\big([1/n, \infty)\big) \subseteq \bigcup_{i=1}^\infty \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} \mathrm{Coz}(h_i)$$

with $h_1, h_2, \ldots \in H$. (It may be assumed that $h_i \neq \mathbf{0}$ for each natural number i.) We have

$$\begin{aligned} \bigcup_{i=1}^\infty \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} \mathrm{Coz}(h_i) &\subseteq \bigcup_{i=1}^\infty \mathrm{cl}_{\beta X} \mathrm{Coz}(h_i) \\ &\subseteq \mathrm{cl}_{\beta X} \bigg(\bigcup_{i=1}^\infty \mathrm{Coz}(h_i)\bigg) = \mathrm{cl}_{\beta X} \mathrm{Coz}\bigg(\sum_{i=1}^\infty 2^{-i} \frac{h_i^2}{\|h_i^2\|}\bigg). \end{aligned}$$

Set

$$g_n=\sum_{i=1}^\infty 2^{-i}\frac{h_i^2}{\|h_i^2\|}.$$

We have $g_n\in H$. Considering the above relations it follows that

$$|f_H|^{-1}\big([1/n,\infty)\big)\subseteq \mathrm{cl}_{\beta X}\mathrm{Coz}(g_n).$$

Hence

$$\begin{array}{rcl}|f|^{-1}\big([1/n,\infty)\big) & = & X\cap|f_H|^{-1}\big([1/n,\infty)\big)\\ & \subseteq & X\cap\mathrm{cl}_{\beta X}\mathrm{Coz}(g_n)=\mathrm{cl}_X\mathrm{Coz}(g_n)=\mathrm{supp}(g_n).\end{array}$$

Set

$$g=\sum_{n=1}^\infty 2^{-n}\frac{g_n}{\|g_n\|}.$$

(It is clear that $g_n\neq\mathbf{0}$ for each natural number n.) Then $g\in H$. Because

$$\mathrm{Coz}(f)=\bigcup_{n=1}^\infty|f|^{-1}\big([1/n,\infty)\big)\subseteq\bigcup_{n=1}^\infty\mathrm{supp}(g_n)\subseteq\mathrm{supp}(g)$$

we have $\mathrm{supp}(f)\subseteq\mathrm{supp}(g)$. Now (2) implies that $f\in H$. This concludes the proof of the claim.

Claim. *Suppose that $\psi:H\to\mathrm{C}_0^\sigma(\mathrm{e}_HX)$ is defined by $\psi(h)=h_H$ for any $h\in H$. Then ψ is an isometric isomorphism.*

Proof of the claim. The mapping ψ is well-defined; this follows from the first claim. It is also clear that ψ is a homomorphism and it is injective. (Use the fact that any two scalar-valued continuous mappings on e_HX are identical, provided that they are so on the dense subspace X of e_HX.) It will be shown that ψ is surjective. Suppose that $g\in\mathrm{C}_0^\sigma(\mathrm{e}_HX)$. Then $(g|X)_H=g$ (because $(g|X)_H$ and g are identical on X) and consequently $g|X\in H$ by the first claim. We have $\psi(g|X)=g$. To show that ψ is an isometry, take some $h\in H$. We have

$$|h_H|(\mathrm{e}_HX)=|h_H|(\mathrm{cl}_{\mathrm{e}_HX}X)\subseteq\overline{|h_H|(X)}=\overline{|h|(X)}\subseteq\big[0,\|h\|\big]$$

where the bar notation denotes the closure in $\mathbb{R}$. Hence $\|h_H\|\leq\|h\|$. Because h_H extends h, it is clear that $\|h\|\leq\|h_H\|$. This concludes the proof of the claim.

It is clear that e_HX is open in βX by its definition. Because βX is compact, it then follows that e_HX is locally compact. This concludes the proof of the first part of the theorem.

For the converse, suppose that

$$H=\mathrm{C}_0^\sigma(Y)$$

where Y is a locally compact metrizable space. It will be shown that H is a Banach subalgebra of $\mathrm{C}_\mathrm{b}(Y)$ satisfying conditions (1) and (2) (with $X=Y$ therein).

It is clear that H is non-empty, because $\mathbf{0}\in H$ trivially. To show that H is closed under addition, suppose that $f,g\in H$. Assume that $\epsilon>0$. Because

$$|f+g|^{-1}\big([\epsilon,\infty)\big)\subseteq|f|^{-1}\big([\epsilon/2,\infty)\big)\cup|g|^{-1}\big([\epsilon/2,\infty)\big)$$

and the latter is σ-compact, since it is the union of two σ-compact spaces, its closed subspace $|f+g|^{-1}([\epsilon,\infty))$ is σ-compact. Consequently $f+g\in H$. Next, it will be shown that H is closed under multiplication. Suppose that $f, g \in H$. Assume that $\epsilon > 0$. Suppose that $M > 0$ such that $\|g\| \leq M$. Because

$$|fg|^{-1}\big([\epsilon, \infty)\big) \subseteq |f|^{-1}\big([\epsilon/M, \infty)\big),$$

and $|f|^{-1}([\epsilon/M, \infty))$ is σ-compact, its closed subspace $|fg|^{-1}([\epsilon, \infty))$ is σ-compact. Consequently $fg \in H$. The proof that H is closed under scalar multiplication is similar. Consequently H is a subalgebra of $\mathrm{C_b}(Y)$. To show that H is a Banach space it clearly suffices to show that H is closed in $\mathrm{C_b}(Y)$. Suppose that f is in the closure in $\mathrm{C_b}(Y)$ of H; we check that $f \in H$. Assume that $\epsilon > 0$. Then $\|f - g\| < \epsilon/2$ for some $g \in H$. Suppose that $t \in |f|^{-1}([\epsilon, \infty))$. Then

$$\epsilon \leq \big|f(t)\big| \leq \big|f(t) - g(t)\big| + \big|g(t)\big| \leq \|f - g\| + \big|g(t)\big| \leq \epsilon/2 + \big|g(t)\big|$$

and consequently $|g(t)| \geq \epsilon/2$. That is $t \in |g|^{-1}([\epsilon/2, \infty))$. Hence

$$|f|^{-1}\big([\epsilon, \infty)\big) \subseteq |g|^{-1}\big([\epsilon/2, \infty)\big).$$

Because the latter is σ-compact, its closed subspace $|f|^{-1}([\epsilon, \infty))$ is σ-compact. Hence $f \in H$.

It will be now verified that H satisfies conditions (1) and (2). To show that H satisfies condition (1), suppose that $y \in Y$. Because Y is locally compact, there exists an open neighborhood U of y in Y with compact closure $\mathrm{cl}_Y U$. There exists a continuous $g : Y \to [0, 1]$ with $g(y) = 1$ and $g|(Y \backslash U) = \mathbf{0}$. Consequently, $|g|^{-1}([\epsilon, \infty)) \subseteq U$ if $\epsilon > 0$, and then $|g|^{-1}([\epsilon, \infty))$ is compact, because it is closed in $\mathrm{cl}_Y U$. Hence $g \in H$. To show that H satisfies condition (2), suppose that $f \in \mathrm{C_b}(X)$ such that $\mathrm{supp}(f) \subseteq \mathrm{supp}(h)$ for some $h \in H$. Because

$$\mathrm{Coz}(h) = \bigcup_{n=1}^{\infty} |h|^{-1}\big([1/n, \infty)\big)$$

is a countable union of σ-compact subspaces, it is consequently σ-compact. But in any metrizable space σ-compactness is identical to separability. Hence, in particular, $\mathrm{Coz}(h)$ is separable, and consequently its closure $\mathrm{supp}(h)$ in Y is separable. Hence $\mathrm{supp}(h)$ is σ-compact and then its closed subspace $\mathrm{supp}(f)$ is also σ-compact. Now, if $\epsilon > 0$, then $|f|^{-1}([\epsilon, \infty))$ is closed in $\mathrm{supp}(f)$, and is consequently σ-compact. Hence $f \in H$. $\square$

Remark 2.6. Consider a space X. By an *open covering* of X it is meant a collection of open subspaces of X whose union of its elements is the whole X. Suppose that $\mathscr{U}$ and $\mathscr{V}$ are open coverings of X. It is said that $\mathscr{U}$ *refine* $\mathscr{V}$ if every element of $\mathscr{U}$ is contained in an element of $\mathscr{V}$. The open covering $\mathscr{U}$ of X is said to be *locally finite* if each point of X has a neighborhood in X which meets only a finite number of elements from $\mathscr{U}$. A regular space X is called *paracompact* if for every open covering $\mathscr{U}$ of X there exists an open covering of X which refines $\mathscr{U}$. Paracompactness is viewed as the simultaneous generalization of metrizability and compactness, for every metrizable space as well as every compact space is paracompact.

The converse statement in Theorem 2.4 remains true if one replaces the metrizability requirement of Y by the paracompactness requirement; this is because paracompactness is hereditary with respect to closed subspaces and a paracompact space with a dense σ-compact subspace is σ-compact. (See Problem 3.8.C(b), Theorem 5.1.25 and Corollary 5.1.29 of [4].)

Question 2.7. It is shown in [2] that for locally compact spaces X and Y if $\mathrm{C}_0(X)$ and $\mathrm{C}_0(Y)$ are isomorphic (as rings) then X and Y are homeomorphic (as spaces). Does the existence of an isomorphism between $\mathrm{C}_0^\sigma(X)$ and $\mathrm{C}_0^\sigma(Y)$, in which X and Y are locally compact spaces, imply the existence of a homeomorphism between X and Y?

The above question is important, because, if answered in the positive, it shows that the space Y as introduced in Theorem 2.5 is indeed unique up to homeomorphism.

Acknowledgement

The author would like to thank Dr. M.R. Koushesh for useful communications.

References

1. S.K. Acharyya and S.K. Ghosh, Functions in $C(X)$ with support lying on a class of subsets of X. *Topology Proc.* **35** (2010), 127–148.
2. A.R. Aliabad, F. Azarpanah and M. Namdari, Rings of continuous functions vanishing at infinity. *Comment. Math. Univ. Carolin.* **45** (2004), 519–533.
3. S. Afrooz and M. Namdari, $C_\infty(X)$ and related ideals. *Real Anal. Exchange* **36** (2010), 45–54.
4. R. Engelking, General Topology. Second edition. Heldermann Verlag, Berlin, 1989.
5. L. Gillman and M. Jerison, Rings of Continuous Functions. Springer–Verlag, New York–Heidelberg, 1976.
6. M.R. Koushesh, Compactification-like extensions. *Dissertationes Math. (Rozprawy Mat.)* **476** (2011), 88 pp.
7. M.R. Koushesh, The partially ordered set of one-point extensions. *Topology Appl.* **158** (2011), 509–532.
8. M.R. Koushesh, A pseudocompactification. *Topology Appl.* **158** (2011), 2191–2197.
9. M.R. Koushesh, The Banach algebra of continuous bounded functions with separable support. *Studia Math.* **210** (2012), 227–237.
10. M.R. Koushesh, Representation theorems for Banach algebras. *Topology Appl.* **160** (2013), 1781–1793.
11. M.R. Koushesh, Topological extensions with compact remainder. *J. Math. Soc. Japan* (47 pp.) In press.
12. M.R. Koushesh, Representation theorems for normed algebras. *J. Aust. Math. Soc.* (22 pp.) In press.
13. M.R. Koushesh, Continuous mappings with null support. (40 pp.) Manuscript, available at: `arXiv:1302.2235 [math.FA]`
14. A. Taherifar, Some generalizations and unifications of $C_K(X)$, $C_\psi(X)$ and $C_\infty(X)$. (13 pp.) Manuscript, available at: `arXiv:1210.6521 [math.GN]`

Ragheb Isfahani Institute of Higher Education, Isfahan, Iran
E-mail address: `mehryar@raghebisf.ac.ir`

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia
E-mail address: `mm11_mah062@student.usm.my`