Geometries Induced by Logarithmic Oscillations as Examples of Gromov Hyperbolic Spaces

Wladimir G. Boskoff, Bogdan D. Suceavă

June 16, 2014

Dedicated to Professor Bang-Yen Chen's 70th Birthday

Abstract

We explore the connection between the geometries generated by logarithmic oscillations and the class of metric spaces satisfying the condition of Gromov hyperbolicity. We start our discussion with the most fundamental examples, inspired from classical geometries, e.g. the Euclidean distance on the infinite strip or Hilbert's distance on the unit disk, and we continue our study with Barbilian's distance, which historically appeared as a natural extension of a model of hyperbolic geometry. We introduce a new metric, called the stabilizing metric, and study its properties. Continuing this study, we explore a class of extensions of this distance and show that, under some analytic conditions, infinitely many new examples of Gromov hyperbolic metric spaces can be constructed. Using similar procedures, we construct Vuorinen's stabilizing metric j_G and its extensions and we discuss their Gromov hyperbolicity.

Keywords: metric spaces, logarithmic oscillations, Barbilian spaces, Gromov hyperbolic metrics, Vuorinen's metric

MSC Primary: 53C23, 54E25. Secondary: 51F99, 51K05, 53C99, 54E35.

1 Introduction

M. Gromov's influential work [15] inspired many investigations on classes of metrics that are Gromov hyperbolic (see e.g. [7, 18, 19, 20]). An important class of examples of Gromov hyperbolic spaces are the so-called CAT(κ) spaces, with $\kappa < 0$ (see [11, 12]), and the fundamental geometric properties of Gromov hyperbolic spaces parallel key facts in the geometry of CAT(κ) spaces. A recent comprehensive treatment of the topic is [11] and new results and viewpoints are still produced, e.g. [19]. Perhaps the natural context where the geometry of the topic is revealed is the study of coarse geometries, as described in [21]. That's why we believe it's particularly important to enrich the array of examples which arise naturally from properties studied in the classical non-Euclidean geometry.

If we are looking for the best description of the questions we are studying in the present article, we should remind here Mikhail Gromov's view that: "It is hardly possible to find a convincing definition of the curvature for an arbitrary metric space X, but one can distinguish certain classes of metric spaces corresponding to Riemannian manifolds with curvatures of a given type. This can be done, for example, by imposing inequalities between mutual distances of finite configurations of points in X" (see [16]).

Quite independently from the developments in global topology, the geometries induced by metrics given by a logarithmic oscillation originate in a paper written in 1934 by Dan Barbilian [1]. We have described the historical context in which the works [2, 3, 4] and [5] have been written in [9]. It is quite interesting that the so-called Apollonian metric was rediscovered independently by A.F. Beardon [6], and his work lead to a series of advances in the study of quasiconformal mappings that attracted the attention of many authors. However, we need to point out that our present work is not motivated by these developments, but rather by several geometric reasons, as we will show in detail below.

In the present work we are seeing these two theories naturally merging. The examples developed here are showing *how a unified treatment of the two directions is the most natural standpoint that captures the whole nature of the underlying geometric process.* This thought motivates the structure of our whole paper and we felt we should start our study with several examples that, by every means, could be described as classical.

To remind the standard terminology, most of it established in [15], we consider a metric space (M, d), where d satisfies the usual definition of a distance. Given $x, y, z \in M$, the quantity $(x|y)_z = \frac{1}{2}[d(x,z) + d(y,z) - d(x,y)]$ is called the *Gromov product* of x and y with respect to z (see p.76 in [15]). Denote $a \wedge b = \min\{a, b\}$. The metric space (M, d) is called *Gromov hyperbolic* if there exists some constant $\delta \geq 0$ such that (see relation (*) at p.76 in [15]):

$$(x|y)_w \geq (x|z)_w \wedge (z|y)_w - \delta,$$

for all $x, y, w, z \in M$. M. Gromov points out that this condition is inspired by the "well known properties of manifold of negative curvature" (also p.76 in [15]). Using the fact that $a \vee b = \max\{a, b\}$, the Gromov hyperbolic condition can be rewritten in the following way. (M, d) is Gromov hyperbolic if there exists a constant $\delta \geq 0$ such that

$$d(x,z) + d(y,w) \leq [d(z,w) + d(y,z)] \vee [d(x,y) + d(z,w)] + 2\delta,$$

$\forall x, y, w, z \in M$. The notions before are available for semidistances, also.

2 A Fundamental Exploration

We start our array of examples with an exploration of several fundamental geometries. As J. Roe points out ([21], p.87), "many interesting properties of metric spaces X depend on the distance properties of *finite* sets of points in X."

For the beginning, we claim that the minimum number of points we need for our study is four, as we can see from the following example. Consider the Euclidean plane endowed with the Euclidean distance. Denote this metric space by $(\mathbb{R}^2, |\cdot|)$. The distance between the points x and y is $|x-y|$. Consider the same set $\mathbb{R}^2$ endowed with another distance, $d(x, y) := \ln(1 + |x - y|)$. Looking only at the length of the sides, one cannot distinguish between an equilateral triangle constructed in the Euclidean plane $(\mathbb{R}^2, |\cdot|)$ and another one constructed in $(\mathbb{R}^2, d)$. The two triangles can have the same length for sides. That is, if the length of the side of the Euclidean triangle is a, there exists a positive number b such that $a = \ln(1 + b)$. No metric differences can be figured out in this case. Now let us choose two four-point configurations, one of them in the Euclidean plane and the other one in $(\mathbb{R}^2, d)$. Both of them can be described in the same way. Each configuration is made from two equal equilateral triangles with a common side.

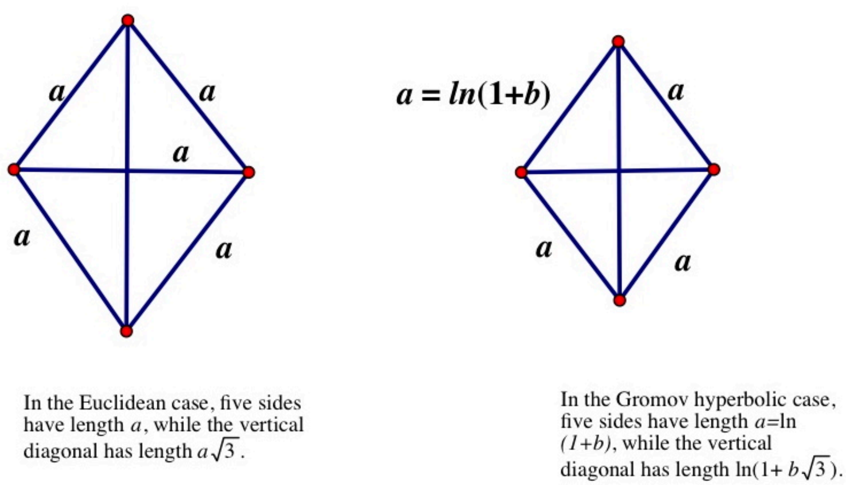

Figure 1: Four-point configuration comparison

We remark that the lengths of four sides and the length of one diagonal are equal in the two configurations. However, while in the Euclidean case the other diagonal is $a\sqrt{3}$, in $(\mathbb{R}^2, d)$ the length of the other diagonal is $\ln(1 + b\sqrt{3})$. If we compare the lengths of the two diagonals, we see that $a\sqrt{3} = \sqrt{3} \cdot \ln(1 + b) \geq \ln(1 + b\sqrt{3})$. Therefore, in four-point configurations there are metric differences between the two diagonals. We may expect to discover other inequalities when we compare sums of other geometric elements from the two quadrilaterals.

The first classical example belongs to Gromov and is related to our previous discussion. Let $|x - y|$ be the Euclidean distance between the points x and y from $\mathbb{R}^n$. Denote by $d(x, y) := \ln(1 + |x - y|)$.

Theorem 2.1. *The metric space $(\mathbb{R}^n, d)$ is Gromov hyperbolic.*

Proof: Obviously, d is a distance. To establish its Gromov hyperbolicity, let us observe that for any four points in the plane represented by the complex numbers z, z_1, z_2, z_3, we have the identity

$$(z - z_1)(z_2 - z_3) = (z - z_2)(z_1 - z_3) + (z - z_3)(z_2 - z_1).$$

Using the properties of the moduli of complex numbers, we have

$$|z - z_1| \cdot |z_2 - z_3| \leq |z - z_2| \cdot |z_1 - z_3| + |z - z_3| \cdot |z_2 - z_1|.$$

The four points z, z_1, z_2, z_3, are the vertices of a quadrilateral. Denoting by $a; c$, and by $b; d$ the length of the opposite sides, respectively, and by $e; f$ the length of diagonals, the previous inequality can be rewritten in three different ways: $e \cdot f \leq a \cdot c + b \cdot d$; $a \cdot c \leq e \cdot f + b \cdot d$; $b \cdot d \leq a \cdot c + e \cdot f$.

In the case when the points are the vertices of a tetrahedron we apply an inversion having a vertex x as pole and arbitrary power, k. The other vertices y, z, w are transformed into y', z', w' which belong to a plane. This plane is the transformed of the sphere circumscribing the tetrahedron.

We have $|y' - z'| = k\frac{|y-z|}{|y-x| \cdot |z-x|}$, $|y' - w'| = k\frac{|y-w|}{|y-x| \cdot |w-x|}$, $|w' - z'| = k\frac{|w-z|}{|w-x| \cdot |z-x|}$. Since $|y' - z'| \leq |y' - w'| + |w' - z'|$, we derive that $|y - z| \cdot |w - x| \leq |y - x| \cdot |z - w| + |y - w| \cdot |z - x|$, i.e. $e \cdot f \leq a \cdot c + b \cdot d$. Therefore, the three previous inequalities hold also in space. These inequalities are known as Ptolemy's inequalities.

The triangle inequality yields three relations: $e + f \leq a + c + b + d$; $a + c \leq e + f + b + d$; $b + d \leq a + c + e + f$.

Using the appropriate inequalities in each group of terms, we have:

$$(1+e)\cdot(1+f) \leq (1+a)\cdot(1+c)+(1+b)\cdot(1+d) \leq 2\cdot[(1+a)\cdot(1+c)\vee(1+b)\cdot(1+d)],$$

$$(1+a)\cdot(1+c) \leq (1+e)\cdot(1+f)+(1+b)\cdot(1+d) \leq 2\cdot[(1+e)\cdot(1+f)\vee(1+b)\cdot(1+d)],$$

$$(1+b)\cdot(1+d) \leq (1+a)\cdot(1+c)+(1+e)\cdot(1+f) \leq 2\cdot[(1+a)\cdot(1+c)\vee(1+e)\cdot(1+f)],$$

that is

$$d(x, z) + d(y, w) \leq [d(z, w) + d(y, z)] \vee [d(x, y) + d(z, w)] + 2 \ln \sqrt{2}.$$

As a consequence, we have obtained the Gromov hyperbolicity of the distance $d(x, y) := \ln(1 + |x - y|)$ for $\delta = \ln \sqrt{2}$. □

However, what can we say about the Euclidean plane endowed with the Euclidean distance? This is the question that motivates our approach in the following section.

3 Classical Geometries and Gromov Hyperbolicity

The starting point of our discussion is the following.

Theorem 3.1. *The Euclidean metric space* $(\mathbb{R}^2, |\cdot|)$ *is not Gromov hyperbolic.*

Proof: Let x be the length of a side of a square in the Euclidean plane. Therefore the length of a diagonal is $x\sqrt{2}$. Suppose by contrary that $(\mathbb{R}^2, |\cdot|)$ is Gromov hyperbolic. Then there exists a constant $\delta \geq 0$ such that the Gromov's inequality holds. On the other hand, $x\sqrt{2} \leq x + \delta$ does not hold when x approaches infinity, i.e. the Euclidean two dimensional metric space cannot be Gromov hyperbolic. □

Theorem 3.2. *The Euclidean distance on the infinite strip* $\{(x, y) \in \mathbb{R}^2 : 0 < y < 1\}$ *is Gromov hyperbolic.*

Sketch of the proof: Consider the quadrilateral $ABCD$ and denote by $\{O\} = AC \cap BD$. The one of the angles $\angle AOD$ and $\angle DOC$ is not acute.

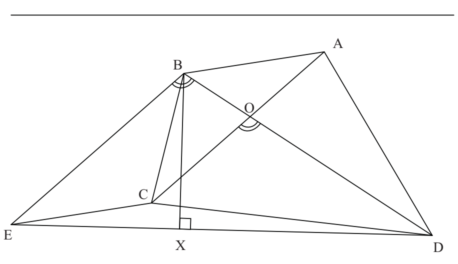

Figure 2: Gromov hyperbolicity for the Euclidean distance on the infinite strip.

Suppose this is $\angle DOC$. Consider the point E lying on the parallel to AC through B, satisfying $|BE| = |AC|$. There are two possible positions for E. We choose E such that the points C and E lie in the same half plane of the line BD. The quadrilateral $ABEC$ is a parallelogram and all the points of the entire figure lie in a strip twice wider than the width of the original strip. Consider the triangle BED and its height through B, which intersects DE in X. Since $\angle EBD = \angle DOC$, X belongs to the interior of segment DE.

We have $|ED| \leq |EC| + |CD|$ and $|BE| + |BD| \leq |ED| + 2 \cdot |BX|$. Therefore

$$|AC| + |BD| = |BE| + |BD| \leq |ED| + 2 \cdot |BX| \leq |EC| + |CD| + 2 \cdot |BX|.$$

However,

$$|EC| + |CD| + 2 \cdot |BX| = |AB| + |CD| + 2 \cdot |BX|.$$

Since angle $\angle EBD$ is not acute, the following inequality holds: $2 \cdot |BX| \leq 3$. This implies $|AC| + |BD| \leq |AB| + |CD| + 3$. In the same way, we obtain the inequality $|AC| + |BD| \leq |AD| + |BC| + 3$. We conclude that

$$|AC| + |BD| \leq [|AD| + |BC|] \vee [|AB| + |CD|] + 3$$

for any quadrilateral constructed in the interior of the strip. This means that the Euclidean infinite strip satisfies the Gromov hyperbolicity condition. □

We are studying next the Gromov hyperbolic property in the unit disk endowed with Hilbert's distance. Consider two points X and Y in the interior of the unit disk $\mathbf{D}$ centered at the origin of the Euclidean plane. Denote by $\{s, S\}$ the intersection of the circle $\partial\mathbf{D}$ with the line XY, such that the order of the points on the line is s, X, Y, S. The Hilbert distance between X and Y is given by

$$h(X,Y) = \ln \frac{SX}{SY} \cdot \frac{sY}{sX},$$

where SX is the Euclidean distance between the points S and X(see [20]). (Although the next theorem appears in [20], for the sake of unitary exposition and since we are using a different argument, we include its proof below.)

Theorem 3.3. *The unit disk $\boldsymbol{D}$ endowed with Hilbert's distance is Gromov hyperbolic.*

Proof: Consider four points $A, B, C, D \in \mathbf{D}$. Denote by

$$L(B,D) = h(B,D) + h(A,C) - h(A,B) - h(C,D)$$

and

$$L(D,B) = h(D,B) + h(A,C) - h(A,D) - h(C,B).$$

The Gromov hiperbolicity condition may be restated in the following way. $(\mathbf{D}, h)$ is Gromov hyperbolic if there exists a constant $\delta \geq 0$ such that

$$\min(L(B,D), L(D,B)) \leq \delta.$$

Using the notations from figure 2 we have

$$L(D,B) = \ln \frac{f+x'}{x'} \cdot \frac{f+x}{x} \cdot \frac{e+m'}{m'} \cdot \frac{e+m}{m} \cdot \frac{y'}{b+y'} \cdot \frac{y}{b+y} \cdot \frac{w'}{d+w'} \cdot \frac{w}{d+w}.$$

From the power of the point in the circle, we derive the following relations: $m \cdot (e+m') = w \cdot (d+w')$, $y \cdot (b+y') = x \cdot (f+x')$, $m \cdot (e+m') = w \cdot (d+w')$, $m' \cdot (e+m) = y' \cdot (b+y)$, $w' \cdot (d+w) = x' \cdot (f+x)$. Replacing in $L(D,B)$ formula we have

$$L(D,B) = 2 \cdot \ln \left(\frac{f+x'}{b+y'}\right) \cdot \left(\frac{f+x}{d+w}\right) \cdot \left(\frac{e+m'}{d+w'}\right) \cdot \left(\frac{e+m}{b+y}\right).$$

Since A is a fixed point and $\frac{e+m'}{d+w'}, m+e, d+w$ are bounded both from above and from below, we have

$$L(D,B) \leq 2 \cdot \ln M + 2 \cdot \ln \left(\frac{f+x'}{b+y'}\right) \cdot \left(\frac{f+x}{b+y}\right).$$

It remains to show that $\left(\frac{f+x'}{b+y'}\right) \cdot \left(\frac{f+x}{b+y}\right)$ is bounded from above. Because of symmetry we may suppose $x \leq x'$.

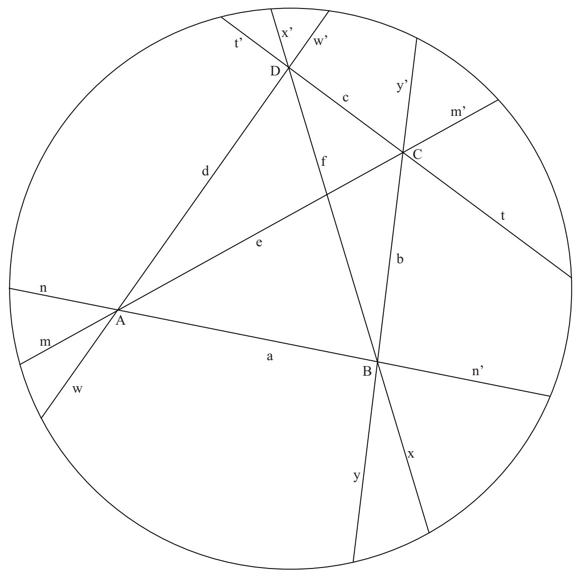

Figure 3: Gromov hyperbolicity for the Hilbert distance in the disk

Case 1. $b \geq x'$ or $c \geq x'$. If $b \geq c$ then

$$\left(\frac{f+x'}{b+y'}\right) \cdot \left(\frac{f+x}{b+y}\right) \leq \left(\frac{b+c+x'}{b}\right) \cdot \left(\frac{b+c+x}{b}\right) \leq \frac{9b^2}{b^2} = 9.$$

If $b \leq c$ we perform the same computations and we show that

$$L(B,D) \leq 2 \cdot \ln M' + 2 \cdot \ln \left(\frac{f+x'}{c+t'}\right) \cdot \left(\frac{f+x}{c+t}\right) \leq 2 \ln M' + \ln 9.$$

Case 2. $b \leq x'$ and $c \leq x'$. In this case, we show that $L(B,D)$ is bounded from above. But

$$\frac{f+x'}{c+t'} \cdot \frac{f+x}{c+t} = \frac{t'}{x'} \cdot \frac{f+x'}{c+t'} \leq \frac{t'}{c+t'} \cdot \frac{b+c+x'}{x'} \leq \frac{3x'}{x'} = 3$$

$\square$

Our next goal is to study the Gromov hyperbolicity of Barbilian's classical distance [1, 2]. Consider a circle K centered at O and let I_1 and I_2 be arbitrary points in the region enclosed by K, that is denoted J. Denote by $\{S, S'\} := I_1I_2 \cap K$, such that the order is S', I_1, I_2, S. Define the half Hilbert distance in the interior of the disk in terms of anharmonic ratio by

$$d^H(I_1, I_2) = \frac{1}{2} \ln [I_1 I_2 S S'].$$

Consider the arc of circle $g := SS'$ orthogonal to K and denote by $\{F_i\} := OI \cap g$, for $i = \overline{1,2}$. Let $[F_1F_2SS']_g$ be the anharmonic ratio on the orthogonal arc g. For $P \in K$, consider the formula $\dfrac{\max_{P\in K} \dfrac{PF_2}{PF_1}}{\min_{P\in K} \dfrac{PF_2}{PF_1}}$. Then we obtain the following.

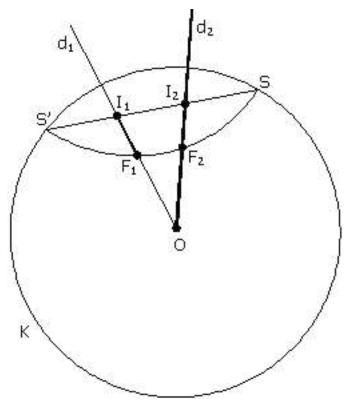

Figure 4: Before inversion

Theorem 3.4. *With the notations specified above, the following relations hold:*

(i) $[I_1I_2SS'] = [F_1F_2SS']_g^2$.

(ii)

$$[F_1F_2SS']_g = \frac{\max_{P\in K} \dfrac{PF_2}{PF_1}}{\min_{P\in K} \dfrac{PF_2}{PF_1}}.$$

Proof (i) Consider an inversion with pole S' and power $\mu = (SS')^2$. For this inversion, S is a fixed point, that is $S \to S$, but $S' \to \infty$. The circle K is transformed into the line $\bar{K}$, $S \in \bar{K}$, the orthogonal arc $g \to$ into the line $\bar{g}$, and $\bar{g} \perp \bar{K}$.

The line d_1 is transformed into the circle $\bar{d}_1, S' \in \bar{d}_1, \bar{F}_1 \in \bar{d}_1, \bar{I}_1 \in \bar{d}_1$, where $\bar{F}_1$ and $\bar{I}_1$ are the inverse of F_1 and I_1. Since d_1 is orthogonal to K, it results $\bar{d}_1$ and $\bar{K}$ are orthogonal, i.e. the circle $\bar{d}_1$ has the line $\bar{K}$ as a diameter. For the line d_2, we obtain similar results. To refer to our figures, we can say that the figure 4 transforms into figure 5.

Remark that

$$\begin{aligned} [I_1I_2SS'] &= \left[\bar{I}_1\bar{I}_2S\infty\right] = \frac{S\bar{I}_1}{S\bar{I}_2}, \\ [F_1F_2SS']_g &= \left[\bar{F}_1\bar{F}_3S\infty\right] = \frac{S\bar{F}_1}{S\bar{F}_2}. \end{aligned}$$

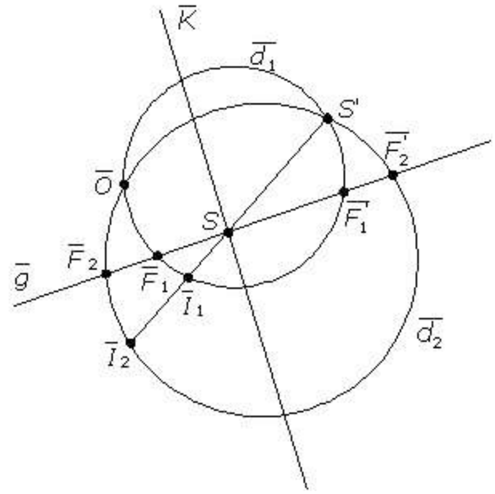

Figure 5: After the first inversion

By using the power of the point S with respect the circle $\bar{d}_1$, we obtain

$$SS' \cdot S\bar{I}_1 = S\bar{F}_1 \cdot S\bar{F}_1' = S\bar{F}_1^2.$$

Similarly,

$$SS' \cdot S\bar{I}_2 = S\bar{F}_2 \cdot S\bar{F}_2' = S\bar{F}_2^2$$

that is

$$\frac{S\bar{I}_1}{S\bar{I}_2} = \left(\frac{S\bar{F}_1}{S\bar{F}_2}\right)^2.$$

It results $[I_1 I_2 SS'] = [F_1 F_2 SS']_g^2$, which concludes the first part of the proof.

(ii) Consider an inversion of the configuration presented in figure 4, with pole F_1 and power μ', where μ' is the power of F_1 with respect the circle K. The circle K is preserved after this inversion, the orthogonal arc g transforms into the line $\bar{g}$ which is a diameter in K. Additionally, it turns out that $F_2 \to \bar{F}_2' \in \bar{g}$, such that $F_1F_2 \cdot F_1F_2' = \mu$ and $F_1 \to \infty$.

Furthermore, $P \in K \to P' \in K$, such that $F_1P \cdot F_1P' = \mu'$ and $P' \in K \cap F_1P$.

We have

$$[F_1F_2SS']_g = \left[\infty F_2'\bar{S}\bar{S}'\right] = \frac{\bar{S}'F_2'}{\bar{S}F_2'} = \frac{\max_{P' \in K} P'F_2'}{\min_{P' \in K} P'F_2'}.$$

Since

$$P'F_2' = \mu' \cdot \frac{PF_2}{F_1P \cdot F_1F_2} = \frac{\mu'}{F_1F_2} \cdot \frac{PF_2}{PF_1},$$

it results that $P'F_2'$ reaches its maximum, respectively its minimum, when the ratio $\dfrac{PF_2}{PF_1}$ reaches its corresponding extrema.

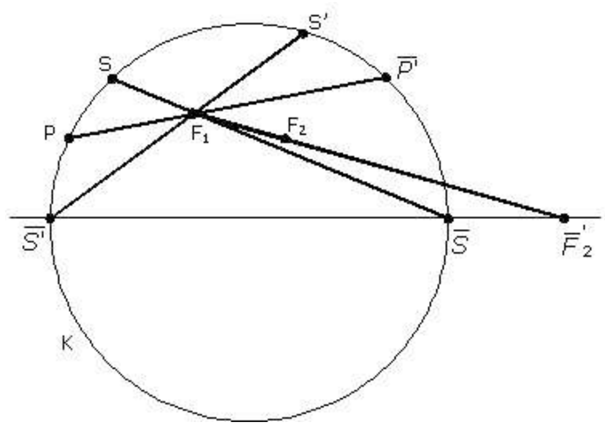

Figure 6: After the second inversion

This conclusion can be rewritten as

$$[F_1F_2SS']_g = \frac{\max_{P\in K} \dfrac{PF_2}{PF_1}}{\min_{P\in K} \dfrac{PF_2}{PF_1}},$$

which is the claim we had to prove. $\square$

Consider both the so-called Poincaré distance defined by

$$d^P(F_1, F_2) := \ln [F_1F_2SS']_g$$

and the Barbilian distance in J, defined by

$$d^*(F_1, F_2) = \ln \frac{\max_{P\in K} \dfrac{PF_2}{PF_1}}{\min_{P\in K} \dfrac{PF_2}{PF_1}}.$$

An important consequence of the previous theorem is the following.

Theorem 3.5. *Barbilian distance, Poincaré distance and the half Hilbert distance coincide on the unit disk* $\mathbb{D}$, *i.e.*

$$d^*(F_1, F_2) = d^P(F_1, F_2) = d^H(I_1, I_2).$$

A consequence of Theorem 3.3 is the following.

Theorem 3.6. *Barbilian distance and Poincaré distance are Gromov hyperbolic on the disk.*

The results established here allow us extensions to other classical models of hyperbolic metrics. We use as starting point for this process the unit disk $\mathbb{D}$.

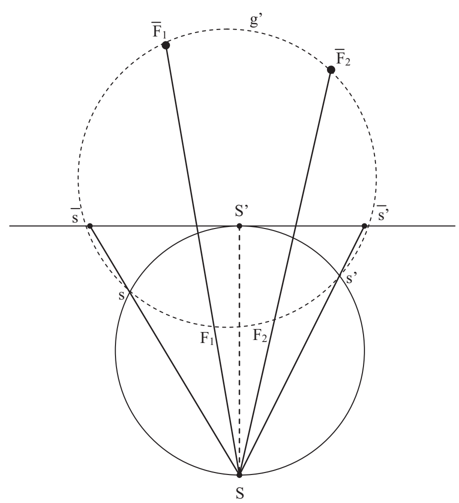

Figure 7: Poincare half plane

Consider a diameter SS' and the inversion with pole S and power SS'^2. This inversion transforms the interior of the disk into the half-plane $\mathbb{H}$ in which S doesn't lie. An orthogonal arc of circle in J is transformed into a half circle g' in $\mathbb{H}$. The Poincaré distance in J for $F_1, F_2 \in g$ is preserved for the transformed points $\bar{F}_1, \bar{F}_2$, that is $d^P(F_1, F_2) = d_H\left(\bar{F}_1, \bar{F}_2\right)$. Based on these considerations and according to Theorem 3.6, we obtain the following.

Theorem 3.7. *Poincaré's half plane $\mathbb{H}$ endowed with the Poincaré distance is Gromov hyperbolic.*

We conclude the present overview of the classical content with another important distance, namely the chordal distance. Consider a sphere of radius $1/2$ tangent to the Euclidean plane in its origin. Let N be the north pole of the sphere. For x, y in $\mathbb{R}^2$ denote by $\pi(x), \pi(y)$ the intersections of the sphere with the straight lines Nx, Ny. Stereographic projection defines the chordal distance by

$$q(x,y) := |\pi(x) - \pi(y)| = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}$$

for $x, y \in \mathbb{R}^2$. First, remark that q is a distance. The first two axioms are clearly satisfied. The triangle inequality reduces to

$$|x-y|\sqrt{1+|z|^2} \leq |x-z|\sqrt{1+|y|^2} + |z-y|\sqrt{1+|x|^2},$$

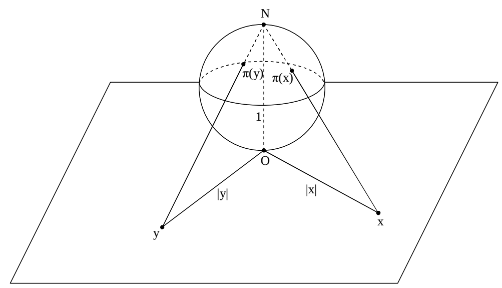

Figure 8: The chordal metric

which is nothing else but Ptolomy's inequality for the tetrahedron $Nxyz$.

It's easy to see that the Riemannian metric corresponding to the chordal distance is

$$ds^2 = \frac{1}{(1+x^2+y^2)^2}(dx^2+dy^2).$$

By a direct computation, the Gaussian curvature of this metric is $K = 4$, which means that this Riemannian metric is elliptic. Hence, the following.

Theorem 3.8. *The sphere of radius* $1/2$ *endowed with the chordal distance is not Gromov hyperbolic.*

Proof: Consider $\pi(x), \pi(y), \pi\ (z), \pi(w)$ on the sphere such that $x, y, z, w \in \mathbb{R}^2$ are the vertices of a square with its center in the origin of $\mathbb{R}^2$. Let l be the length of a side, therefore the length of a diagonal is $l\sqrt{2}$. Suppose by contrary that the chordal distance q is Gromov hyperbolic. Then, there exists a constant $\delta \geq 0$ such that the Gromov's inequality holds. However, $l\sqrt{2} \leq l + \delta$ does not hold when l approaches infinity, which concludes the argument. $\square$

4 Barbilian's Logarithmic Oscillation and the Stabilizing Distance

In this section we are bridging the gap between the developments in the study of geometries generated by logarithmic oscillation and the study of Gromov hyperbolic spaces. We start by adapting Barbilian's metrization procedure to our present goal.

We should start by pointing out why we refer to $d_B(x,y)$ as to a semi-distance of Barbilian type. Consider $x, y \in \mathbb{R}^n - \{0\}$ with the property that $|x| = |x-0| = \delta_x > \delta_y = |y-0| = |y|$. Consider a set $K \subset \mathbb{R}^n$ and $p \in K$. The influence function (for terminology, see [2]) $f : K \times (\mathbb{R}^n \ \{0\}) \to \mathbb{R}$ defined by $f(p,x) = |x-p|$ for the particular case when $K = \{0\}$ leads us to

$$M_1 = \max_{p\in K} \frac{f(p,x)}{f(p,y)} = \max_{p\in K} \frac{|x-p|}{|y-p|} = \frac{|x-0|}{|y-0|} = \frac{|x|}{|y|} = \frac{\delta_x}{\delta_y},$$

$$m_1 = \min_{p\in K} \frac{f(p,x)}{f(p,y)} = \min_{p\in K} \frac{|x-p|}{|y-p|} = \frac{|x-0|}{|y-0|} = \frac{|x|}{|y|} = \frac{\delta_x}{\delta_y}.$$

Since we are in the framework of the Barbilian's work presented in [2] and consistently used in works like e.g. [8, 9], the logarithmic oscillation is:

$$\ln \frac{M_1}{m_1} = \ln \left(\frac{\delta_x}{\delta_y}\right)^2 = 2\ln\frac{\delta_x}{\delta_y} = 2\ln\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} = 2d_B(x,y).$$

We obtain that $d_B(x,y)$ is actually half of Barbilian's classical distance induced by logarithmic oscillation.

Theorem 4.1. *Let $G = \mathbb{R}^n/\{0\}$ and $M = \{0\}$. Denote by $\delta_x = |x-0| = |x|$, the Euclidean distance between x and the origin. Then:*
i) $d_B(x,y) = \ln\frac{\delta_x\vee\delta_y}{\delta_x\wedge\delta_y}$ is a semi-distance on $G = \mathbb{R}^n/\{0\}$;
ii) The semi-distance d_B is Gromov hyperbolic on G, with $\delta = 0$.

Proof: We need to prove only the second claim. To this goal, is sufficient to show that

$$d_B(x,y)+d_B(z,w) \le (d_B(x,z)+d_B(y,w)) \vee (d_B(x,w)+d_B(y,z)), \forall x,y,z \in \mathbb{R}^n/\{0\},$$

i.e. the Gromov hyperbolicity condition is satisfied for $\delta = 0$. The condition we have to show rephrases as:

$$\frac{\delta_x\vee\delta_y}{\delta_x\wedge\delta_y}\cdot\frac{\delta_z\vee\delta_w}{\delta_z\wedge\delta_w} \le \left(\frac{\delta_x\vee\delta_z}{\delta_x\wedge\delta_z}\cdot\frac{\delta_y\vee\delta_w}{\delta_y\wedge\delta_w}\right) \vee \left(\frac{\delta_x\vee\delta_w}{\delta_x\wedge\delta_w}\cdot\frac{\delta_y\vee\delta_z}{\delta_y\wedge\delta_z}\right). \tag{1}$$

Without any loss of generality, assume that $\delta_x \le \delta_y \vee \delta_z \vee \delta_w$. Under this assumption, (1) becomes:

$$\frac{\delta_y}{\delta_x}\cdot\frac{\delta_z\vee\delta_w}{\delta_z\wedge\delta_w} \le \left(\frac{\delta_z}{\delta_x}\cdot\frac{\delta_y\vee\delta_w}{\delta_y\wedge\delta_w}\right) \vee \left(\frac{\delta_w}{\delta_x}\cdot\frac{\delta_y\vee\delta_z}{\delta_y\wedge\delta_z}\right). \tag{2}$$

By using
$$\frac{\delta_z \vee \delta_w}{\delta_z \wedge \delta_w} = \frac{\delta_z}{\delta_w} \vee \frac{\delta_w}{\delta_z},$$
and after a simplification both sides by $\delta_x > 0$, the inequality (2) turns into:
$$\frac{\delta_y \cdot \delta_z}{\delta_w} \vee \frac{\delta_y \cdot \delta_w}{\delta_z} \leq \left(\frac{\delta_y \cdot \delta_z}{\delta_w} \vee \frac{\delta_w \cdot \delta_z}{\delta_y}\right) \vee \left(\frac{\delta_y \cdot \delta_w}{\delta_z} \vee \frac{\delta_w \cdot \delta_z}{\delta_y}\right),$$
which is certainly true.

□

Consider G an arbitrary subset in $\mathbb{R}^n$. For any x and y in $\mathbb{R}^n$ denote by $|x-y|$ the Euclidean distance between x and y. Let $M \in \mathbb{R}^n$ such that $M \cap G = \emptyset$. Denote $\delta_x = \min_{z \in M} |x - z|$, with $x \in G$. We introduce *the stabilizing distance* on G by the following expression:
$$d_{G,M}(x, y) = \ln\left(\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + |x - y|\right). \tag{3}$$
The set M is called *the supporting set.* Note that this distance is different than the j_G and $\tilde{j}_G$ metrics introduced by F. Gehring and B. Osgood [14], and by M. Vuorinen [23], respectively, and whose Gromov hyperbolicity is studied by P. Hästö in [18]. We are proposing this terminology since in the expression of the logarithmic oscillation there is added further information inherited from the Euclidean metric; this thought led us to this terminology.

Theorem 4.2. *(i) With the notations specified above, $d_{G,M}$ is a distance on G.*

(ii) If G is the punctured open unit ball $D = \{x \in \mathbb{R}^n - \{0\} | |x| < 1\}$ and $M = \{0\}$ is the supporting set, then the stabilizing distance $d_{D,M}$ defined by (3) is Gromov hyperbolic, with $\delta = \frac{1}{2}\ln 9$.

For the proof of Theorem 4.2, we need the following.

Lemma 4.1. *In the conditions above we have:*
$$\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \leq \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + |x - y| \leq 3 \cdot \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \tag{4}$$

Proof: The triangle inequality yields $|x - y| < \delta_x \vee \delta_y + \delta_x \wedge \delta_y$. Hence:
$$\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \leq \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + |x - y| \leq \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + \delta_x \vee \delta_y + \delta_x \wedge \delta_y,$$
which immediately yields
$$\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \leq \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + |x - y| \leq \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + 2 \cdot \delta_x \vee \delta_y.$$
Since $0 < \delta_x \wedge \delta_y < 1$, then $\frac{1}{\delta_x \wedge \delta_y} > 1$. Therefore, $\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} > \delta_x \vee \delta_y$, which gives (4). □

We are ready now to prove Theorem 4.2.

Proof: We need to show that the stabilizing distance satisfies on D the Gromov hyperbolicity condition with $2\delta = \ln 9$, when the supporting set M is just the singleton consisting of the origin. We have to prove that for any $x, y, z, w \in D$ the following inequality holds true:

$$\frac{1}{9}\left(\frac{\delta_x \vee \delta_z}{\delta_x \wedge \delta_z} + |x-z|\right) \cdot \left(\frac{\delta_w \vee \delta_y}{\delta_w \wedge \delta_y} + |w-y|\right) \leq \tag{5}$$

$$\left(\frac{\delta_x \vee \delta_w}{\delta_x \wedge \delta_w} + |x-w|\right) \cdot \left(\frac{\delta_y \vee \delta_z}{\delta_y \wedge \delta_z} + |y-z|\right) \vee \left(\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + |x-y|\right) \cdot \left(\frac{\delta_w \vee \delta_z}{\delta_w \wedge \delta_z} + |w-z|\right).$$

Without loss of generality we can assume

$$\delta_x \leq \delta_y \wedge \delta_z \wedge \delta_w. \tag{6}$$

We also use in our argument

$$\frac{\delta_y \vee \delta_z}{\delta_y \wedge \delta_z} = \frac{\delta_y}{\delta_z} \vee \frac{\delta_z}{\delta_y}. \tag{7}$$

The inequality (5) is certainly true if when we maximize the left hand side and we minimize the right hand side we obtain a true statement. By using (4) we need to prove:

$$\frac{\delta_x \vee \delta_z}{\delta_x \wedge \delta_z} \cdot \frac{\delta_y \vee \delta_w}{\delta_y \wedge \delta_w} \leq \frac{\delta_x \vee \delta_w}{\delta_x \wedge \delta_w} \cdot \frac{\delta_y \vee \delta_z}{\delta_y \wedge \delta_z} \vee \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \cdot \frac{\delta_w \vee \delta_z}{\delta_w \wedge \delta_z}.$$

By using (6) and (7) all that's left to prove it

$$\frac{\delta_z}{\delta_x} \cdot \left(\frac{\delta_y}{\delta_w} \vee \frac{\delta_w}{\delta_y}\right) \leq \frac{\delta_w}{\delta_x} \cdot \left(\frac{\delta_y}{\delta_z} \vee \frac{\delta_z}{\delta_y}\right) \vee \frac{\delta_y}{\delta_x} \cdot \left(\frac{\delta_w}{\delta_z} \vee \frac{\delta_z}{\delta_w}\right).$$

By simplifying $\delta_x > 0$ both sides, we obtain

$$\frac{\delta_z \cdot \delta_y}{\delta_w} \vee \frac{\delta_z \cdot \delta_w}{\delta_y} \leq \left(\frac{\delta_w \cdot \delta_y}{\delta_z} \vee \frac{\delta_w \cdot \delta_z}{\delta_y}\right) \vee \left(\frac{\delta_z \cdot \delta_y}{\delta_w} \vee \frac{\delta_w \cdot \delta_y}{\delta_z}\right).$$

This last inequality is clearly true. $\square$

Remark 4.1. Consider now $D \subset \mathbb{R}^2$, the punctured open unit disk centered at the origin (same notation as in the previous section, taking now $n = 2$). We compute the Riemannian metric corresponding to the stabilizing distance. To this goal, let $x = (x_1, y_1)$, $y = (x_2, y_2) = (x_1 + dx, y_1 + dy)$. We have $|x - y| = \sqrt{dx^2 + dy^2}$. Bearing in mind that $\frac{\delta_x}{\delta_y} \vee \frac{\delta_y}{\delta_x} = \frac{|y|}{|x|}$ and the stabilizing distance

$$d_{D,M}(x,y) \approx ds = \ln\left(1 + \frac{|y|}{|x|} - 1 + |x-y|\right) = \ln\left(1 + \frac{|y| - |x| + |x| \cdot |x-y|}{|x|}\right)$$

$$\approx \frac{|y| - |x| + |x| \cdot |x - y|}{|x|} \approx \frac{1 + |x|}{|x|} \cdot |x - y|,$$

where in the last step we have used $|y| - |x| \approx |y - x|$. We derive the expression of the Riemannian metric

$$ds^2 = \frac{(1 + \sqrt{x^2 + y^2})^2}{x^2 + y^2} \cdot (dx^2 + dy^2).$$

Remark 4.2. As one may intuitively expect, the Gaussian curvature of the Riemannian metric corresponding to the stabilizing distance is everywhere negative on the punctured disk:

$$K(x, y) = -\frac{\sqrt{x^2 + y^2}}{1 + 6(x^2 + y^2) + x^4 + y^4 + 2x^2y^2 + 4\sqrt{x^2 + y^2} + 4(x^2 + y^2)^{3/2}}.$$

In polar coordinates the stabilizing metric

$$ds^2 = \frac{(1 + \sqrt{x^2 + y^2})^2}{x^2 + y^2} \cdot (dx^2 + dy^2)$$

becomes

$$ds^2 = \frac{a^2}{(a - 1)^2} da^2 + a^2 d\theta^2.$$

Consider the surface of revolution

$$h(a, \theta) = (a \cos\theta, a \sin\theta, f(a)), a \in (1, 1 + r), r > 0, \theta \in (0, 2\pi).$$

Since

$$\frac{\partial h}{\partial a} = (\cos\theta, \sin\theta, f'(a)); \frac{\partial h}{\partial \theta} = (-a \sin\theta, a \cos\theta, 0),$$

we have $g_{11} = (1 + (f'(a))^2); g_{12} = g_{21} = 0; g_{22} = a^2$.

The Riemannian corresponding metric is

$$ds^2 = (1 + (f'(a))^2) da^2 + a^2 d\theta^2.$$

A condition for the existence of a surface of revolution having the same metric is

$$f'(a) = \frac{\sqrt{2a - 1}}{a - 1},$$

that is

$$f(a) = 2\sqrt{2a - 1} + \ln \frac{\sqrt{2a - 1} - 1}{\sqrt{2a - 1} + 1} + C.$$

Then, the surface of revolution

$$h(a, \theta) = (a \cos\theta, a \sin\theta, 2\sqrt{2a - 1} + \ln \frac{\sqrt{2a - 1} - 1}{\sqrt{2a - 1} + 1}),$$

$a \in (1, 1+r), r > 0, \theta \in (0, 2\pi)$, is a spatial representation, a submanifold having the Riemannian metric coincident to the Riemannian metric determined by the stabilizing distance.

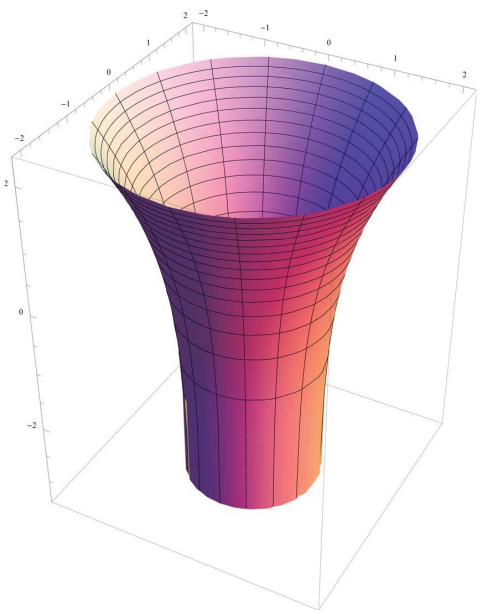

Figure 9: The surface having the same metric as the stabilizing distance

Remark 4.3. Since in Remark 4.2 we've discussed the Gaussian curvature of the Riemannian metric generated by the stabilizing metric, we should see how much this is for Barbilian's semi-distance d_B. We obtain the following:

$$ds^2 = \frac{1}{x^2 + y^2}(dx^2 + dy^2)$$

on $D - \{0\} \subset \mathbb{R}^2$. This metric has Gaussian curvature everywhere vanishing $K \equiv 0$, since if one switches to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, we get

$$ds^2 = \frac{1}{r^2}dr^2 + d\theta^2.$$

If in this last form we swap $r \to \frac{1}{r_1}$, we get $ds^2 = dr_1^2 + d\theta^2$, which is the flat Euclidean metric.

5 Extensions of the Stabilizing Metric

We are ready now to discuss possible extensions of the stabilizing metrics. More precisely, we'll study a construction of the following type:

$$d_f(x, y) = \ln\left(\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + f(|x - y|)\right),$$

where f is a appropriate function.

To explore this idea, we start with the following.

Question 1: Let $a, b, c \in [0, \infty)$. Are there any polynomial functions f with vanishing free term satisfying both $f(|a-b|) + f(|b-c|) \geq f(|a-c|), \quad a, b, c \in [0, \infty)$ and $f(x) > 0$ for $x \geq 0$? Suppose such polynomial exists. Then the relation holds true in particular for $c = 0$, hence $f(|a-b|) \geq f(|a|) - f(|b|), \ \forall a, b \in [0, \infty)$. That is, we have $f(|a-b|) \geq f(a) - f(b), \ \forall a, b \in [0, \infty)$. Suppose now that $a > b$. Divide this last relation by $a - b$ both sides and obtain:

$$\frac{f(a-b)}{a-b} \geq \frac{f(a) - f(b)}{a-b}.$$

Bearing in mind that our class of polynomials f satisfies the condition $f(0) = 0$, this last relation can be viewed as:

$$\frac{f(a-b) - f(0)}{a-b-0} \geq \frac{f(a) - f(b)}{a-b}.$$

Fixing b and letting a approach b, this last relation turns out into $f'(0) := \alpha \geq f'(b), \ \forall b \in [0, \infty)$. This means $f(b) - f(0) \leq \alpha b, \forall b \in [0, \infty)$. According to our initial assumptions we can write $0 \leq f(b) \leq f(0) + \alpha b, \forall b \in [0, \infty)$. Suppose $f(b) = \alpha_1 b + ...\alpha_k b^k, k \geq 2$. Dividing by b we obtain, $\alpha_1 + ...\alpha_k b^{k-1} \leq \frac{f(0)}{b} + \alpha, \forall b \geq 0$. Since $\lim_{b \to \infty} \frac{f(b)}{b^k} = \alpha_k$ it results $\alpha_k \geq 0$ and if $k \geq 2$, the right hand of the previous inequality approaches ∞ while the second is constant. It means that the polynomial could have degree one. In conclusion, we got the answer to the question stated above. Should the polynomial f exist, then it is $f(b) = \alpha b$, with $\alpha > 0$.

Question 2: Are there any functions $f : [0, \infty) \to [0, \infty)$ satisfying the following four properties

(i) $f(x) = 0$ if and only if $x = 0$;

(ii) f is increasing on $[0, \infty)$;

(iii) f is subadditive on $[0, \infty)$;

(iv) $\beta x \leq f(x) \leq \alpha x, \ \forall x \geq 0, \alpha > \beta > 0$?

The answer to Question 2 is yes, and it's suggested by our exploration in the answer to Question 1. We can construct many exemples, all as piecewise-defined functions. Take, for example, the piecewise-defined function f defined as follows. $f(x) = \alpha x$, for $x \in [0, x_0)$; $f(x) = \alpha_1(x - x_0) + \alpha x_0$, for $x \in [x_0, x_1)$, with $\beta < \alpha_1 < \alpha$; $f(x) = \alpha_2(x - x_1) + \alpha_1(x_1 - x_0) + \alpha x_0$, for $x \in [x_1, x_2), \beta < \alpha_2 < \alpha_1 < \alpha$, and so on. The image of this function is like an inclined "broken line" lying between the lines $y = \beta x$ and $y = \alpha x$.

The reason to address first these elementary details is the following.

Theorem 5.1. *(a) Let f be a function satisfying the conditions (i)-(iv) listed above. Consider $d_f : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ defined by*

$$d_f(x, y) = \ln\left(\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + f(|x-y|)\right),$$

where δ_x expresses the distance from to the support set $M = \{0\}$. Then d_f is a distance.

(b) On the punctured open unit disk $D - \{0\} \subset \mathbb{R}^n$, d_f is Gromov hyperbolic with $\delta = \frac{1}{2}\ln(1+2\alpha)$.

Proof: The equality $d_f(x,y) = 0$ is equivalent to

$$\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + f(|x-y|) = 1.$$

Since $\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \geq 1$ and $f(|x-y|) \geq 0$, we obtain that $d_f(x,y) = 0$ is ultimately equivalent to $x = y$.

The symmetry condition holds true: $d_f(x,y) = d_f(y,x), \quad \forall x, y \in \mathbb{R}^n$.

To prove the triangle inequality, remark that since f is increasing on $[0,\infty)$ (by condition (ii)), from

$$|x-z| \leq |x-y| + |y-z|$$

we obtain:

$$f(|x-z|) \leq f(|x-y|+|y-z|) \leq f(|x-y|) + f(|y-z|), \quad \forall x,y,z \in [0,\infty), \tag{8}$$

the last inequality being due to the subadditive condition (iii). Therefore

$$d_f(x,y) + d_f(y,z) \geq d_f(x,z), \quad \forall x,y,z \in \mathbb{R}^n.$$

This concludes the proof of part (a).

To prove (b), remark that for $\delta_x < 1$ and $\delta_y < 1$, we have

$$\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} < \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + f(|x-y|) <$$

$$< \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + f(2\delta_x \vee \delta_y) < (1+2\alpha)\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y},$$

where the last inequality uses condition (iv), namely $f(x) \leq \alpha x, \ \forall x \geq 0, \alpha > 0$. This concludes the argument for (b). $\square$

6 Extensions of Vuorinen's Metric

In this section we focus our study on Vuorinen's metric [23] defined by

$$j_G(x,y) = \ln\left(1 + \frac{|x-y|}{\delta_x \wedge \delta_y}\right).$$

The construction we propose below is similar to the construction of our stabilizing metric. If for the stabilizing metric we had as starting point a metric studied by M. Gromov, by considering instead of 1 the geometric quantity $\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y}$, in the present section we study a construction of the following type:

$$d_{VH,s}(x,y) = \ln\left(\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + \frac{|x-y|}{\delta_x \wedge \delta_y}\right).$$

For this metric, we prove the following.

Theorem 6.1. *(a) Consider* $d_{VH,s} : \mathbb{R}^n \backslash M \times \mathbb{R}^n \backslash M \to [0,\infty)$ *defined by*

$$d_{VH,s}(x,y) = \ln\left(\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + \frac{|x-y|}{\delta_x \wedge \delta_y}\right),$$

where δ_x *expresses the distance from* x *to the support set* M*, i.e.* $\delta_x = \min_{z\in M} |x-z|$*, with* $x \in \mathbb{R}^n \backslash M$*. Then* $d_{VH,s}$ *is a distance.*

(b) If $M = \{0\}$*, then* $d_{VH,s}$ *is Gromov hyperbolic with* $\delta = \frac{1}{2}\ln 9$.

Proof: (a) The equality $d_{VH,s}(x,y) = 0$ is equivalent to

$$\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + \frac{|x-y|}{\delta_x \wedge \delta_y} = 1.$$

Since $\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \geq 1$ and $\frac{|x-y|}{\delta_x \wedge \delta_y} \geq 0$, it results that $d_{VH,s}(x,y) = 0$ is equivalent to $x = y$.

Obviously, the symmetry condition holds true:

$$d_{VH,s}(x,y) = d_{VH,s}(y,x), \quad \forall x,y \in \mathbb{R}^n \backslash \{0\}.$$

It remains to prove

$$\left(\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + \frac{|x-y|}{\delta_x \wedge \delta_y}\right) \cdot \left(\frac{\delta_y \vee \delta_z}{\delta_y \wedge \delta_z} + \frac{|y-z|}{\delta_y \wedge \delta_z}\right) \geq \left(\frac{\delta_x \vee \delta_z}{\delta_x \wedge \delta_z} + \frac{|x-z|}{\delta_x \wedge \delta_z}\right).$$

Since this inequality is symmetric in x and z, we distinguish three cases.

Case 1. $\delta_x \leq \delta_y \wedge \delta_z$. The inequality becomes

$$|x-y|\cdot|y-z| + |x-y|\cdot\delta_y \vee \delta_z + \delta_y \cdot |y-z| + \delta_y \cdot (\delta_y \vee \delta_z) \geq \delta_y \wedge \delta_z \cdot |x-z| + \delta_z \cdot (\delta_y \wedge \delta_z)$$

which is true because $\delta_y^2 \vee \delta_y \delta_z \geq \delta_z^2 \wedge \delta_y \delta_z$ and

$$|x-y| \cdot \delta_y \vee \delta_z + \delta_y \cdot |x-y| \geq \delta_y \wedge \delta_z \cdot |x-z|.$$

Note that the last inequality is a version of the triangle inequality.

Case 2. $\delta_y \leq \delta_x \wedge \delta_z$. The inequality becomes

$$|x-y|\cdot|y-z|\cdot\delta_x \wedge \delta_z + |x-y|\cdot\delta_z \cdot(\delta_x \wedge \delta_z) + |y-z|\cdot\delta_x\cdot(\delta_x \wedge \delta_z) + \delta_x \cdot \delta_z \cdot (\delta_x \wedge \delta_z) \geq$$

$$\geq \delta_y^2 \cdot |x-z| + \delta_y^2 \cdot (\delta_x \vee \delta_z),$$

which is true because $\delta_x \cdot \delta_z \cdot (\delta_x \wedge \delta_z) \geq \delta_y^2 \cdot (\delta_x \vee \delta_z)$ and

$$|x-y| \cdot \delta_z \cdot (\delta_x \wedge \delta_z) + |y-z| \cdot \delta_x \cdot (\delta_x \wedge \delta_z) \geq \delta_y^2 \cdot (\delta_x \vee \delta_z).$$

As in the previous case, note that the last inequality is a version of the triangle inequality.

Case 3. $\delta_y \geq \delta_x \wedge \delta_z$ can be addressed by a similar argument.

(b). Start with the triangle inequality

$$|x-y| \leq \delta_x \vee \delta_y + \delta_x \wedge \delta_y \leq 2 \cdot \delta_x \vee \delta_y.$$

It follows

$$\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \leq \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + \frac{|x-y|}{\delta_x \wedge \delta_y} \leq 3 \cdot \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y}.$$

This means

$$\ln \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} - \ln 3 \leq \ln \left(\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + \frac{|x-y|}{\delta_x \wedge \delta_y} \right) \leq \ln \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + \ln 3.$$

Remark that $\ln \frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y}$ is the Barbilian hyperbolic semidistance, therefore the previous inequality becomes

$$d_B(x,y) - \ln 3 \leq d_{VH,s}(x,y) \leq d_B(x,y) + \ln 3,$$

which, according to the general theory (see e.g. P. Hästö's Theorem 1, [18], p. 1138) leads to the fact that $d_{VH,s}(x,y)$ is Gromov hyperbolic with $\delta = \frac{1}{2} \ln 3$.
□

We are calling this metric Vuorinen's stabilizing metric. (We are using the subscript H in our definition since P. Hästö obtained interesting results on Vuorinen's original metric in [18].)

Theorem 6.2. *If the Vuorinen stabilizing metric $d_{VH,s}$ is Gromov hyperbolic, then the support set M has exactly one point.*

Proof: Suppose that the Vuorinen stabilizing metric is Gromov hyperbolic on $\mathbb{R}^n \backslash M$. Suppose by contrary that M does not have exactly one point. Therefore, there exist at least two points u and v in M, $u \neq v$. Consider two spheres centered at u and v, $S_1(u)$ and $S_2(v)$, such that $S_1(u) \cap S_2(v) = \emptyset$.

Let $x \in \int S_1(u)$ and $w \in \int S_2(v)$ such that $\delta_x = |x-u| = \epsilon$, $\delta_w = |w-v| = \epsilon$, where $\epsilon < min\{1, |u-v|\}$.

On segments ux, vw, we consider y and z, respectively, such that $\delta_y = |y-u| = \epsilon^2$ and $\delta_z = |z-v| = \epsilon^2$. Then,

$$d_{VH,s}(x,y) = d_{VH,s}(z,w) = \ln \left(\frac{2}{\epsilon} - 1 \right)$$

$$d_{VH,s}(x,w) = \ln \left(1 + \frac{|x-w|}{\epsilon} \right); \quad d_{VH,s}(y,z) = \ln \left(1 + \frac{|y-z|}{\epsilon^2} \right);$$

$$d_{VH,s}(x,z) = \ln\left(\frac{1}{\epsilon} + \frac{|x-z|}{\epsilon^2}\right); \quad d_{VH,s}(y,w) = \ln\left(\frac{1}{\epsilon} + \frac{|y-w|}{\epsilon^2}\right).$$

From our hypothesis, it exists $2\delta := \ln a, a > 1$, such that the following inequality holds

$$d_{VH,s}(x,z)+d_{VH,s}(y,w) \leq [d_{VH,s}(x,w)+d_{VH,s}(y,z)]\vee[d_{VH,s}(x,y)+d_{VH,s}(z,w)]+2\delta.$$

However,

$$\ln\frac{|x-z|\cdot|y-w|}{\epsilon^4} \leq d_{VH,s}(x,z) + d_{VH,s}(y,w)$$

and

$$[d_{VH,s}(x,w) + d_{VH,s}(y,z)] \vee [d_{VH,s}(x,y) + d_{VH,s}(z,w)] + 2\delta \leq$$

$$\leq \ln\frac{A}{\epsilon^3} \vee \ln\frac{4B}{\epsilon^2} + \ln a,$$

where A and B are some real positive quantities which satisfy the above inequality. Pursuing this idea,

$$\frac{|x-z|\cdot|y-w|}{\epsilon^4} \leq \frac{aA}{\epsilon^3} \vee \frac{4aB}{\epsilon^2}, \quad \forall\epsilon > 0.$$

However, when ϵ approaches 0, the previous inequality does not hold any longer, therefore if Vuorinen stabilizing metric was Gromov hyperbolic, the support set M cannot have more then one point. This concludes our proof.

□

When we stated Question 2 in the previous section, we considered a class of functions that could support an extension of a given metric. We study the similar idea in the context of Vuorinen's metric.

Consider the functions $f : [0,\infty) \to [0,\infty)$ satisfying the following four properties:
(i) $f(x) = 0$ if and only if $x = 0$;
(ii) f is increasing on $[0,\infty)$;
(iii) f is subadditive on $[0,\infty)$;
(iv) $x \leq f(x) \leq \alpha x,\ \forall x \geq 0, \alpha > 1$.
In this context, we prove the following.

Theorem 6.3. *(a) Let f be a function satisfying the conditions (i)-(iv) listed above. Consider $d_{V,f} : \mathbb{R}^n\backslash\{0\} \times \mathbb{R}^n\backslash\{0\} \to [0,\infty)$ defined by*

$$d_{V,f}(x,y) = \ln\left(1 + \frac{f(|x-y|)}{\delta_x \wedge \delta_y}\right),$$

where δ_x expresses the distance from to the support set $M = \{0\}$. Then $d_{V,f}$ is a distance.

(b) On $\mathbb{R}^n\backslash\{0\}$, $d_{V,f}$ is Gromov hyperbolic with $\delta = \frac{1}{2}\ln(1+2\alpha)$.

Proof: (a) The equality $d_{V,f}(x,y)=0$ is equivalent to $\frac{f(|x-y|)}{\delta_x\wedge\delta_y}=0$, that is $f(|x-y|)=0$, i.e. $x=y$.

Obviuosly, the symmetry condition holds true: $d_{V,f}(x,y)=d_{V,f}(y,x), \quad \forall x,y\in\mathbb{R}^n\backslash\{0\}$.

It remains to prove

$$\left(1+\frac{f(|x-y|)}{\delta_x\wedge\delta_y}\right)\left(1+\frac{f(|y-z|)}{\delta_y\wedge\delta_z}\right)\geq\left(1+\frac{f(|x-z|)}{\delta_x\wedge\delta_z}\right).$$

Case 1. $\delta_x\geq\delta_y\geq\delta_z$. The inequality becomes

$$\left(1+\frac{f(|x-y|)}{\delta_y}\right)\left(1+\frac{f(|y-z|)}{\delta_z}\right)\geq\left(1+\frac{f(|x-z|)}{\delta_z}\right).$$

It is enough to prove that

$$\frac{f(|x-y|)}{\delta_y}\left(1+\frac{f(|y-z|)}{\delta_z}\right)\geq\frac{f(|x-z|)}{\delta_z}-\frac{f(|y-z|)}{\delta_z}.$$

The previous inequality still holds true if

$$\frac{f(|x-y|)}{\delta_y}\left(1+\frac{f(|y-z|)}{\delta_z}\right)\geq\frac{f(|x-y|)}{\delta_z}.$$

Some computations show that the previos inequality is true because of the triangle inequality and (iv), i.e.

$$\delta_z+f(|y-z|)\geq\delta_z+|y-z|\geq\delta_y.$$

Case 2. $\delta_x\geq\delta_z\geq\delta_y$. The inequality becomes

$$\left(1+\frac{f(|x-y|)}{\delta_y}\right)\left(1+\frac{f(|y-z|)}{\delta_y}\right)\geq\left(1+\frac{f(|x-z|)}{\delta_z}\right),$$

and it may be reduced to

$$\frac{f(|x-y|)}{\delta_y}+\frac{f(|y-z|)}{\delta_y}\geq\frac{f(|x-z|)}{\delta_y}\geq\frac{f(|x-z|)}{\delta_z}.$$

Case 3. $\delta_y\geq\delta_x\geq\delta_z$. The inequality to prove becomes

$$\left(1+\frac{f(|x-y|)}{\delta_x}\right)\left(1+\frac{f(|y-z|)}{\delta_z}\right)\geq\left(1+\frac{f(|x-z|)}{\delta_z}\right).$$

After some computations, exactly as in the first case it remains to prove

$$\frac{f(|x-y|)}{\delta_x}\left(1+\frac{f(|y-z|)}{\delta_z}\right)\geq\frac{f(|x-z|)}{\delta_z}-\frac{f(|y-z|)}{\delta_z}.$$

Increasing the right hand side at $\frac{f(|x-y|)}{\delta_z}$, the inequality is solved using the same reasons as in Case 1, i.e.

$$\delta_z + f(|y-z|) \geq \delta_z + |y-z| \geq \delta_y \geq \delta_x.$$

(b) We start from the triangle inequality $\delta_x \vee \delta_y \leq |x-y| + \delta_x \wedge \delta_y$ by applying the function f which satisfies (i)-(iv) as in the statement. It follows

$$\delta_x \vee \delta_y \leq f(\delta_x \vee \delta_y) \leq f(|x-y|) + f(\delta_x \wedge \delta_y) \leq \alpha f(|x-y|) + \alpha\delta_x \wedge \delta_y$$

that is

$$\frac{1}{\alpha}\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \leq 1 + \frac{f(|x-y|)}{\delta_x \wedge \delta_y}.$$

Starting from $|x-y| \leq \delta_x \vee \delta_y + \delta_x \wedge \delta_y$, we obtain, exactly as before, another important inequality

$$1 + \frac{f(|x-y|)}{\delta_x \wedge \delta_y} \leq (1+2\alpha)\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y}.$$

The two previous inequalities lead to

$$\frac{1}{(2\alpha+1)}\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \leq \frac{1}{\alpha}\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} \leq 1 + \frac{f(|x-y|)}{\delta_x \wedge \delta_y} \leq (1+2\alpha)\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y},$$

that is

$$\ln\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} - \ln(2\alpha+1) \leq \ln(1 + \frac{f(|x-y|)}{\delta_x \wedge \delta_y}) \leq \ln\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y} + \ln(2\alpha+1).$$

Remark that $\ln\frac{\delta_x \vee \delta_y}{\delta_x \wedge \delta_y}$ is the Barbilian hyperbolic semidistace, therefore the previos inequality becomes

$$d_B(x,y) - \ln(2\alpha+1) \leq d_{V,f}(x,y) \leq d_B(x,y) + \ln(2\alpha+1),$$

which according to the general theory (see P. Hästö's Theorem 1, [18], p. 1138) yields that the extension of Vuorinen's metric is Gromov hyperbolic with $\delta = \frac{1}{2}\ln(1+2\alpha)$. $\square$

If $\alpha < 1$, then the formula above does not represent a metric. To see this, remark that any three radial points do not satisfy the triangle inequality. That's why the condition (iv) is needed.

The authors express their thanks to the editor and the referee for their useful suggestions in preparing the final form of the present paper.

References

[1] D. Barbilian, *Einordnung von Lobatschewskys Massenbestimmung in einer gewissen allgemeinen Metrik der Jordanschen Bereiche,* Časopis Mathematiky a Fysiky **64** (1934-35), 182-183.

[2] D. Barbilian, *Asupra unui principiu de metrizare,* Stud. Cercet. Mat. **10** (1959), 68 - 116.

[3] D. Barbilian, *Fundamentele metricilor abstracte ale lui Poincaré şi Carathéodory ca aplicaţie a unui principiu general de metrizare,* Stud. Cercet. Mat. **10** (1959), 273 - 306.

[4] D. Barbilian, *J-metricile naturale finsleriene,* Stud. Cercet. Mat. **11** (1960), 7 - 44.

[5] D. Barbilian and N. Radu, *J-metricile naturale finsleriene şi funcţia de reprezentare a lui Riemann,* Stud. Cercet. Mat. **12** (1962), 21 - 36.

[6] A. F. Beardon, *The Apollonian metric of a domain in* $\mathbb{R}^n$*,* in *Quasiconformal mappings and analysis,* Springer-Verlag, 1998, 91-108.

[7] M. Bonk, J. Heinonen and P. Koskela, *Uniformizing Gromov hyperbolic spaces,* Astérisque, **270** (2001), 99 p.

[8] W. G. Boskoff, M. G. Ciucă and B. D. Suceavă, *Distances obtained by Barbilians metrization procedure,* Houston J. Math. **33,** (2007), p. 709–717.

[9] W. G. Boskoff and B. D. Suceavă, *Barbilian Spaces: the History of a Geometric Idea*, Hist. Math. **34** (2007), 221–224.

[10] W. G. Boskoff and B. D. Suceavă, *A Barbilian-Type Semi-Distance, The Stabilizing Distance, and Gromov Hyperbolic Spaces*, to appear in Anal. Ştiin. Univ. *Al. I. Cuza*, Iaşi.

[11] M.R. Bridson and A. Häfliger - *Metric Spaces of Non-Positive Curvature (Grundlehren der mathematischen Wissenschaften)*, Springer-Verlag, 2010.

[12] M. M. Deza and E. Deza, *Encyclopedia of Distances,* Springer-Verlag, 2009.

[13] F. W. Gehring and K. Hag, *The Apollonian metric and quasiconformal mappings*, Contemp. Math., 256 (2000) 143-163.

[14] F.W. Gehring and B.S. Osgood, *Uniform domains and the quasihyperbolic metric,* J. Anal. Math., **36**(1979), 50–74.

[15] M. Gromov, *Hyperbolic groups*, in *Essays in group theory* (S. M. Gersten, ed.), MSRI Publ. **8**(1987), 75-263.

[16] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Birkhäuser, 2001.

[17] P. A. Hästö, *The Appollonian metric: uniformity and quasiconvexity*, Ann. Acad. Sci. Fennicae, **28** (2003), 385 - 414.

[18] P.A. Hästö, *Gromov Hyperbolicity of the j_G and $\tilde{j}_G$ metrics,* Proc. Amer. Math. Soc., **134** (2005), 1137–1142.

[19] Z. Ibragimov, *Hyperbolizing metric spaces*, Proc. Amer. Math. Soc., **139** (2011), 4401–4407.

[20] A. Karlsson and G. A. Noskov, *The Hilbert metric and Gromov hyperbolicity,* Enseign. Math. **48** (2002), 73–89.

[21] J. Roe, *Lectures on Coarse Geometries*, American Mathematical Society, 2003.

[22] B. D. Suceavă, *Distances generated by Barbilian's metrization procedure by oscillation of sublogarithmic functions*, Houston Journal of Mathematics, **37** (2011), 147–159.

[23] M. Vuorinen, *Conformal invariants and quasiregular mappings*, J. Anal. Math. **45** (1985), 69–115.

Authors adresses:
W. G. Boskoff, Department of Mathematics and Computer Science, University Ovidius, Constanţa, Romania; e-mail: boskoff@univ-ovidius.ro

B. D. Suceavă, California State Unversity Fullerton, Department of Mathematics, P.O. Box 6850, Fullerton, CA 92834-6850, U.S.A.; e-mail: bsuceava@fullerton.edu