Critical Curves for Multidimensional Nonlinear Diffusion Equations Coupled by Nonlinear Boundary Sources

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Abstract

This paper is concerned with a class of nonlinear diffusion equations coupled by the nonlinear boundary sources on the exterior domain of the unit ball in $\mathbb{R}^N$. We are interested in the critical curves which can describe the large time behavior of the solutions. It is shown that the critical global existence curve coincides with the critical Fujita curve for the system we considered.

Keywords: Global existence; Blow up; Critical curves
2000 MR Subject Classification: 35K65 35B33

1 Introduction

In this paper, we deal with a class of nonlinear diffusion equations on the exterior domain of the unit ball in $\mathbb{R}^N$, i.e.,

$$\frac{\partial u}{\partial t} = \Delta u^m, \qquad \frac{\partial v}{\partial t} = \Delta v^n, \qquad x \in \mathbb{R}^N \backslash B_1(0), t > 0,, \tag{1.1}$$

$$\nabla u^m \cdot \vec{\nu} = v^\alpha(x,t), \qquad \nabla v^n \cdot \vec{\nu} = u^\beta(x,t), \qquad x \in \partial B_1(0), t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad x \in \mathbb{R}^N \setminus B_1(0), \tag{1.3}$$

where $m, n > 1$, $\alpha, \beta \geq 0$, $N > 2$, $B_1(0)$ is the unit ball in $\mathbb{R}^N$ with boundary $\partial B_1(0)$, $\vec{\nu}$ is the inward normal vector on $\partial B_1(0)$, and $u_0(x), v_0(x)$ are nonnegative, suitably smooth and bounded functions with compact supports.

As well known that the equations in (1.1) are Newtonian filtration equations, they degenerate at the points where $u = 0$. The local existence of solutions can be established by the standard method, see [3, 8, 14]. In this paper we mainly investigate the large time behavior of solutions, such as the global existence in time and blow-up in a finite time. In particular, we are interested in the critical exponents that may describe the large time behavior of solutions.

Since the begining work on critical exponent done by Fujita [4] in 1966, lots of Fujita type results are established for various of problems, see the survey papers [2, 6]. It was Glalaktionov

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and Levine who first discussed the critical exponents for the one-dimensional nonlinear diffusion equations with boundary sources in [6]:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2}, \qquad x > 0, t > 0, \tag{1.4}$$

$$\frac{\partial u^m}{\partial x}(0,t) = u^\alpha(0,t), \qquad t > 0, \tag{1.5}$$

$$u(x,0) = u_0(x), \qquad x \in (0,+\infty) \tag{1.6}$$

here $m > 1, \alpha \geq 0$. For the problems (1.4)-(1.6), they proved that $\alpha_0 = (m+1)/2, \alpha_c = m+1$. Here, we call α_0 as the critical global existence exponent and α_c as the critical Fujita exponent respectively,

(i) if $0 < \alpha < \alpha_0$, then every nontrivial nonnegative solution is global in time;

(ii) if $\alpha_0 < \alpha < \alpha_c$, then the nontrivial nonnegative solutions blow up in a finite time;

(iii) if $\alpha > \alpha_c$, then the solutions exist globally for the small initial data and blow up in a finite time for the large initial data.

The results obtained by Galakionov and Levine were extended to the problems with fast diffusion, i.e., $0 < m < 1$ in [5, 10]. However, for the multi-dimentional case, the critical exponents for the nonlinear diffusion equation with the boundary sources remained open. In [12], Wang *et al* considered this problem on the exterior domain $\mathbb{R}^N \backslash B_1(0)$, namely, the system (1.1)-(1.3) with $m = n, \alpha = \beta$, $N > 2$, they obtained that $\alpha_0 = \alpha_c = m$, which indicates that the critical global existence exponent coincides with the critical Fujita exponent for the multi-dimensional case.

Instead of the critical global existence exponent and the critical Fujita exponent, there are the critical global existence curve and the critical Fujita curve for the coupled system, they were proposed by Deng *et al* in the study of the heat equations coupled by nonlinear boundary sources, see [1]. For the one-dimensional nonlinear diffusion equations, Quirós and Rossi [11] considered the coupled Newtonian filtration equations as follows

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2}, \quad \frac{\partial v}{\partial t} = \frac{\partial^2 v^n}{\partial x^2}, && x > 0, t > 0,\\
&-\frac{\partial u^m}{\partial x}(0,t) = v^\alpha(0,t), \quad -\frac{\partial v^n}{\partial x}(0,t) = u^\beta(0,t), && t > 0,\\
&u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), && x > 0.
\end{aligned}$$

They showed that the critical global existence curve is given by $\alpha\beta = (m+1)(n+1)/4$ and the critical Fujita curve is given by $\min\{\alpha_1 + \beta_1, \alpha_2 + \beta_2\} = 0$, where

$$\alpha_1 = \frac{2\alpha + n + 1}{(m+1)(n+1) - 4\alpha\beta}, \quad \beta_1 = \frac{\alpha(m-1-2\beta) + (n+1)m}{(m+1)(n+1) - 4\alpha\beta},$$
$$\alpha_2 = \frac{2\beta + m + 1}{(m+1)(n+1) - 4\alpha\beta}, \quad \beta_2 = \frac{\beta(n-1-2\alpha) + (m+1)n}{(m+1)(n+1) - 4\alpha\beta}.$$

The similar results were established in [9]. We see that these two critical curves don't coincide by some calculations.

The above papers motivate us to investigate the critical curves for the multi-dimensional system (1.1)-(1.3). The purpose of this present paper is to verify the phenomenon that the critical exponents α_0 and α_c coincide for the single nonlinear diffuison equation (see [12]) also occurs for the system (1.1)–(1.3). Namely, we prove that both the critical global existence curve and the critical Fujita curve are determined by $\alpha\beta = mn$ for the system (1.1)–(1.3).

Furthermore, by virtue of the radial symmetry of the exterior domain of the unit ball, we can extend our results to the following more general equations

$$\frac{\partial}{\partial t}(|x|^{\lambda_1}u) = \operatorname{div}(|x|^{\lambda_1}\nabla u^m), \quad \frac{\partial}{\partial t}(|x|^{\lambda_2}v) = \operatorname{div}(|x|^{\lambda_2}\nabla v^n), \quad x \in \mathbb{R}^N \backslash B_1(0), t > 0 \tag{1.7}$$

with $\lambda_1, \lambda_2 > 2 - N$, $N \geq 1$.

The rest of this paper is organized as follows. Section 2 is devoted to the large time behavior of solutions to the nonlinear boundary problem for the Newonian filtration equations, namely (1.1)–(1.3) and (1.7),(1.2),(1.3).

2 Main results and their proofs

In this section, we first introduce our results on the system of Newtonian filtration equations coupled by nonlinear boundary sources, then we give the proofs.

Theorem 2.1 *For the system (1.1)–(1.3) with $N > 2$, both the critical global existence curve and the critical Fujita curve are given by $\alpha\beta = mn$. Namely, if $0 \leq \alpha\beta < mn$, then every nonnegative solution (u, v) of the system* (1.1)-(1.3) *exists globally in time; if $\alpha\beta > mn$, then the solutions with large initial data blow up in a finite time while the solutions with small initial data exist globally in time.*

Theorem 2.2 *Assume $\lambda_1, \lambda_2 > 2 - N$, $N \geq 1$. For the equation (1.7) with the initial and boundary conditions (1.2),(1.3), the critical global existence curve and the critical Fujita curve both are given by $\alpha\beta = mn$.*

Before we give the proofs of Theorem2.1 and Theorem 2.2, we consider the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial r^2} + \frac{\tilde{\lambda}_1}{r}\frac{\partial u^m}{\partial r}, \quad \frac{\partial v}{\partial t} = \frac{\partial^2 v^n}{\partial r^2} + \frac{\tilde{\lambda}_2}{r}\frac{\partial v^n}{\partial r}, \qquad r > 1, t > 0, \tag{2.1}$$

$$-\frac{\partial u^m}{\partial r}(1,t) = v^\alpha(1,t), \quad -\frac{\partial v^n}{\partial r}(1,t) = u^\beta(1,t), \qquad t > 0, \tag{2.2}$$

$$u(r,0) = u_0(r), \qquad v(r,0) = v_0(r), \qquad r > 1, \tag{2.3}$$

where $r = |x|$, $m, n > 1$, $\alpha, \beta \geq 0$, $N \geq 1$, $\tilde{\lambda}_1, \tilde{\lambda}_2 > 1$, and $u_0(r), v_0(r)$ are nonnegative, nontrivial functions with compact supports. Clearly, the solution (u, v) of the system (2.1)–(2.3) with $\tilde{\lambda}_1 = \tilde{\lambda}_2 = N - 1$ is also the solution of the system (1.1)–(1.3) if $u_0(x), v_0(x)$ are radially symmetrical. If $\tilde{\lambda}_1 = \lambda_1 + N - 1, \tilde{\lambda}_2 = \lambda_2 + N - 1$, the same facts also hold valid between the system (2.1)-(2.3) with the system (1.7), (1.2), (1.3). In order to obtain Theorem 2.1, Theorem 2.2, we firstly show the following results on the system (2.1)–(2.3).

Proposition 2.1 *If $0 \leq \alpha\beta < mn$, then all nonnegative solutions of the system* (2.1)–(2.3) *exist globally in time.*

Proposition 2.2 *If $\alpha\beta > mn$, then the nonnegative solution of the system* (2.1)–(2.3) *with large initial data blows up in a finite time.*

Proposition 2.3 *If $\alpha\beta > mn$, then every nonnegative nontrivial solution of the problem* (2.1)–(2.3) *with small initial data exists globally.*

Remark 2.1 *Proposition 2.1, 2.2 and 2.3 make us to conclude that both the crtical global existence curve and the critical Fujita curve for the system* (2.1)–(2.3) *are determined by $\alpha\beta = mn$.*

Now, we prove Propostion 2.1 – Proposition 2.3.

Proof of Proposition 2.1. We prove this proposition by constructing a kind of global upper solutions. Let

$$u_1(r,t) = (T+t)^{k_1} h_1(\xi_1), \quad \xi_1 = \frac{r-1}{(T+t)^{l_1}}, \tag{2.4}$$

$$v_1(r,t) = (T+t)^{k_2} h_2(\xi_2), \quad \xi_2 = \frac{r-1}{(T+t)^{l_2}}, \tag{2.5}$$

where $T > 0$, $k_2 = kk_1$ with k, k_1 being the positive constants to be determined and

$$l_1 = \frac{1+k_1(m-1)}{2}, \quad l_2 = \frac{1+k_2(n-1)}{2}.$$

For $\xi_1, \xi_2 > 0$, we take

$$h_1(\xi_1) = \Big(\frac{m-1}{m}(1-\xi_1)_+\Big)^{1/(m-1)}, \quad h_2(\xi_2) = \Big(\frac{n-1}{n}(1-\xi_2)_+\Big)^{1/(n-1)}.$$

Denote

$$u_2(r,t) = r^{-\alpha_1/m} u_1(r,t), \quad v_2(r,t) = r^{-\alpha_2/n} v_1(r,t), \quad r > 1, t > 0, \tag{2.6}$$

where α_1, α_2 are given by the following:

$$\alpha_1 = \begin{cases} \tilde{\lambda}_1 - 1, & 1 < \tilde{\lambda}_1 < 2, \\ \frac{1}{2}\tilde{\lambda}_1, & \tilde{\lambda}_1 \geq 2; \end{cases} \qquad \alpha_2 = \begin{cases} \tilde{\lambda}_2 - 1, & 1 < \tilde{\lambda}_2 < 2, \\ \frac{1}{2}\tilde{\lambda}_2, & \tilde{\lambda}_2 \geq 2. \end{cases}$$

We claim that (u_2, v_2) is a upper solution to the problem (2.1)-(2.3). Firstly, we verify $u_2(r,t), v_2(r,t)$ satisfy the boundary conditions

$$-\frac{\partial u_2^m}{\partial r}(1,t) \geq v_2^\alpha(1,t), \quad -\frac{\partial v_2^n}{\partial r}(1,t) \geq u_2^\beta(1,t). \tag{2.7}$$

On the one hand, due to that

$$\frac{\partial u_1^m}{\partial r} = (T+t)^{mk_1 - l_1}(h_1^m)'(\xi_1) \leq 0, \quad \frac{\partial u_2^n}{\partial r} = (T+t)^{nk_2 - l_2}(h_2^n)'(\xi_2) \leq 0,$$

we have

$$\begin{aligned} -\frac{\partial u_2^m}{\partial r}(1,t) =& \alpha_1 u_1^m(1,t) - \frac{\partial u_1^m}{\partial r}(1,t) \\ \geq& \alpha_1 u_1^m(1,t) = \alpha_1 (T+t)^{k_1 m}\Big(\frac{m-1}{m}\Big)^{m/(m-1)}, \end{aligned}$$

$$
\begin{aligned}
-\frac{\partial v_2^n}{\partial r}(1,t) =&\alpha_2 v_1^n(1,t) - \frac{\partial v_1^n}{\partial r}(1,t)\\
\geq&\alpha_2 v_1^n(1,t) = \alpha_2 (T+t)^{k_2 n}\Big(\frac{n-1}{n}\Big)^{n/(n-1)}.
\end{aligned}
$$

On the other hand, it is clear that

$$
\begin{aligned}
v_2^\alpha(1,t) = v_1^\alpha(1,t) = (T+t)^{k_2\alpha}\Big(\frac{n-1}{n}\Big)^{\alpha/(n-1)},\\
u_2^\alpha(1,t) = u_1^\alpha(1,t) = (T+t)^{k_1\alpha}\Big(\frac{m-1}{m}\Big)^{\alpha/(m-1)}.
\end{aligned}
$$

Thus, the inequalities in (2.7) are valid if

$$
\alpha_1\Big(\frac{m-1}{m}\Big)^{m/(m-1)}(T+t)^{k_1 m} \geq \Big(\frac{n-1}{n}\Big)^{\alpha/(n-1)}(T+t)^{k_2\alpha}, \tag{2.8}
$$

$$
\alpha_2\Big(\frac{n-1}{n}\Big)^{n/(n-1)}(T+t)^{k_2 n} \geq \Big(\frac{m-1}{m}\Big)^{\alpha/(m-1)}(T+t)^{k_1\beta}. \tag{2.9}
$$

Since $\alpha\beta < mn$, there exists a constant $k > 0$, such that $\beta/n < k < m/\alpha$. For the above k and $k_2 = k_1 k$, (2.8) and (2.9) hold for the large enough T.

Secondly, we verify that u_2, v_2 are the upper solutions to the equations in (2.1). A direct computation yields that

$$
\begin{aligned}
(h_1^m)''(\xi_1) + \xi_1 h_1'(\xi_1) - \frac{1}{m-1}h_1(\xi_1) = 0, \qquad \xi_1 > 0,\\
(h_2^n)''(\xi_2) + \xi_2 h_2'(\xi_2) - \frac{1}{n-1}h_2(\xi_2) = 0, \qquad \xi_2 > 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1^m}{\partial r^2} &= (T+t)^{k_1-1}\Big(k_1 h_1(\xi_1) - l_1\xi_1 h_1'(\xi_1) - (h_1^m)''(\xi_1)\Big)\\
&= (T+t)^{k_1-1}\Big[(k_1 - \frac{1}{m-1})h_1(\xi_1) - (l_1-1)\xi_1 h_1'(\xi_1)\Big]\\
\frac{\partial v_1}{\partial t} - \frac{\partial^2 v_1^n}{\partial r^2} &= (T+t)^{k_2-1}\Big(k_2 h_2(\xi_2) - l_2\xi_2 h_2'(\xi_2) - (h_2^n)''(\xi_2)\Big)\\
&= (T+t)^{k_2-1}\Big[(k_2 - \frac{1}{n-1})h_2(\xi_2) - (l_2-1)\xi_2 h_2'(\xi_2)\Big].
\end{aligned}
$$

For the fixed $k \in (\beta/n, m/\alpha)$ and $k_2 = kk_1$, we choose $k_1 > 0$ such that

$$
k_1 > \max\{\frac{1}{m-1}, \frac{1}{k(n-1)}\},
$$

which yields $l_1 > 1, l_2 > 1$. Combing with that $h_1'(\xi_1) \leq 0, h_2'(\xi_2) \leq 0$, we get

$$
\frac{\partial u_1}{\partial t} \geq \frac{\partial^2 u_1^m}{\partial r^2} \geq \frac{\partial^2 u_1^m}{\partial r^2} + \frac{\tilde{\alpha}_1}{r}\frac{\partial u_1^m}{\partial r}, \qquad \frac{\partial v_1}{\partial t} \geq \frac{\partial^2 v_1^n}{\partial r^2} \geq \frac{\partial^2 v_1^n}{\partial r^2} + \frac{\tilde{\alpha}_2}{r}\frac{\partial v_1^n}{\partial r} \tag{2.10}
$$

with

$$
\tilde{\alpha}_1 = \begin{cases} 2-\tilde{\lambda}_1, & 1 < \tilde{\lambda}_1 < 2,\\ 0, & \tilde{\lambda}_1 \geq 2; \end{cases} \qquad \tilde{\alpha}_2 = \begin{cases} 2-\tilde{\lambda}_2, & 1 < \tilde{\lambda}_2 < 2,\\ 0, & \tilde{\lambda}_2 \geq 2. \end{cases}
$$

Note that

$$\frac{\partial u_1}{\partial t} = r^{\alpha_1/m}\frac{\partial u_2}{\partial t},$$
$$\frac{\partial u_1^m}{\partial r} = \alpha_1 r^{\alpha_1-1}u_2^m + r^{\alpha_1}\frac{\partial u_2^m}{\partial r},$$
$$\frac{\partial^2 u_1^m}{\partial r^2} = r^{\alpha_1}\frac{\partial^2 u_2^m}{\partial r^2} + 2\alpha_1 r^{\alpha_1-1}\frac{\partial u_2^m}{\partial r} + \alpha_1(\alpha_1-1)r^{\alpha_1-2}u_2^m.$$

Substituting the above equalities into (2.10), we obtain that

$$\begin{aligned} r^{\alpha_1/m}\frac{\partial u_2}{\partial t} &\geq r^{\alpha_1}\frac{\partial^2 u_2^m}{\partial r^2} + (2\alpha_1+\tilde{\alpha}_1)r^{\alpha_1-1}\frac{\partial u_2^m}{\partial r} + \alpha_1(\alpha_1-1-\tilde{\alpha}_1)r^{\alpha_1-2}u_2^m \\ &\geq r^{\alpha_1}\Big(\frac{\partial^2 u_2^m}{\partial r^2} + \frac{\tilde{\lambda}_1}{r}\frac{\partial u_2^m}{\partial r}\Big) \end{aligned}$$

due to that $\alpha_1 - 1 - \tilde{\alpha}_1 \geq 0$ and $2\alpha_1 + \tilde{\alpha}_1 = \tilde{\lambda}_1$. The above inequality implies that for $r > 1$

$$\frac{\partial u_2}{\partial t} \geq r^{\alpha_1(1-1/m)}\Big(\frac{\partial^2 u_2^m}{\partial r^2} + \frac{\tilde{\lambda}}{r}\frac{\partial u_2^m}{\partial r}\Big) \geq \frac{\partial^2 u_2^m}{\partial r^2} + \frac{\tilde{\lambda}}{r}\frac{\partial u_2^m}{\partial r}.$$

Similarly,

$$\frac{\partial v_2}{\partial t} \geq \frac{\partial^2 v_2^n}{\partial r^2} + \frac{\tilde{\lambda}}{r}\frac{\partial v_2^n}{\partial r}.$$

Finally, we verify the initial conditions that

$$u_2(r,0) \geq u_0(r), \quad v_2(r,0) \geq v_0(r). \tag{2.11}$$

Let

$$M_1 = \max_{r>1} u_0(r), \quad M_2 = \max_{r>1} v_0(r), \qquad \text{supp} u_0 = [1, R_1], \quad \text{supp} v_0 = [1, R_2].$$

Then, (2.11) holds provided with

$$u_2(R_1, 0) \geq M_1, \qquad v_2(R_2, 0) \geq M_2, \tag{2.12}$$

since that $u_2(r,0)$ and $v_2(r,0)$ are decreasing with respect to r. In fact, we can choose T to be large such that

$$R_1^{-\alpha_1/m}T^{k_1}\Big(\frac{m-1}{m}(1-\frac{R_1-1}{T^{l_1}})_+\Big)^{1/(m-1)} \geq M_1,$$
$$R_2^{-\alpha_2/n}T^{k_2}\Big(\frac{n-1}{n}(1-\frac{R_2-1}{T^{l_2}})_+\Big)^{1/(n-1)} \geq M_2.$$

From the above, for large T satisfying (2.8) and (2.12), (u_2, v_2) given by (2.6) is a global upper solution of the problem (2.1)-(2.3) in the distribution sense. Then the solutions to (2.1)-(2.3) are global in time by using the comparison principle. The proof is completed. □

Proof of Proposition 2.2. The proposition is proved by constructing a kind of blow-up lower solutions of the system (2.1)-(2.3). For $r > 1, 0 < t < T$, set

$$\underline{u}(r,t) = (T-t)^{-\mu_1}f_1(\eta), \qquad \underline{v}(r,t) = (T-t)^{-\mu_2}f_2(\eta), \qquad \eta = (r-1)(T+t),$$

where $T > 0$ is a given constant,

$$\mu_1 = \frac{1}{m-1} + \frac{\alpha}{m(n-1)}, \qquad \mu_2 = \frac{m}{\alpha}\mu_1 = \frac{m}{\alpha(m-1)} + \frac{1}{n-1}.$$

Assume that f_1, f_2 satisfy $f_1, f_2 \geq 0, f_1', f_2' \leq 0, (f_1^m)'', (f_2^n)'' \geq 0$, by computation we can see that $(\underline{u}, \underline{v})$ with $\underline{u}(r,0) \leq u_0(r), \underline{v}(r,0) \leq v_0(r)$ is a lower solution of the system (2.1)–(2.3) , if the functions $f_1(\eta), f_2(\eta)$ satisfy

$$T^2(f_1^m)'' + 2T\frac{\tilde{\lambda}_1}{r}(f_1^m)' - \mu_1(T-t)^{\mu_1 m-\mu_1-1}f_1(\eta) \geq 0, \tag{2.13}$$

$$T^2(f_2^n)'' + 2T\frac{\tilde{\lambda}_2}{r}(f_2^n)' - \mu_2(T-t)^{\mu_2 n-\mu_2-1}f_2(\eta) \geq 0, \tag{2.14}$$

$$-2T(T-t)^{\mu_2\alpha-\mu_1 m}(f_1^m)'(0) \leq f_2^\alpha(0), \tag{2.15}$$

$$-2T(T-t)^{\mu_1\beta-\mu_2 n}(f_2^n)'(0) \leq f_1^\beta(0). \tag{2.16}$$

Note that $r > 1$ and

$$\mu_1 m - \mu_1 - 1 = \frac{\alpha(m-1)}{m(n-1)} > 0, \qquad \mu_2 n - \mu_2 - 1 = \frac{m}{\alpha(m-1)} > 0,$$

$$\mu_2\alpha - \mu_1 m = \frac{m}{\alpha}\mu_1\alpha - \mu_1 m = 0, \qquad \mu_1\beta - \mu_2 n = \mu_1(\beta - \frac{m}{\alpha}n) > 0.$$

Thus, (2.13)-(2.16) hold if

$$T^2(f_1^m)'' + 2T\tilde{\lambda}_1(f_1^m)' - \mu_1 T^{\mu_1 m-\mu_1-1}f_1(\eta) \geq 0, \tag{2.17}$$

$$T^2(f_2^n)'' + 2T\tilde{\lambda}_2(f_2^n)' - \mu_2 T^{\mu_2 n-\mu_1-1}f_2(\eta) \geq 0, \tag{2.18}$$

$$-2T(f_1^m)'(0) \leq f_2^\alpha(0), \tag{2.19}$$

$$-2T^{\mu_1\beta-\mu_2 n+1}(f_2^n)'(0) \leq f_1^\beta(0). \tag{2.20}$$

Take

$$f_1(\eta) = A_1(B-\eta)_+^{1/(m-1)}, \qquad f_2(\eta) = A_2(B-\eta)_+^{1/(n-1)}, \qquad \eta > 0, \tag{2.21}$$

where A_1, A_2 are positive constants to be determined, and

$$B = \min\Big\{\frac{T}{4(m-1)\tilde{\lambda}_1}, \frac{T}{4(n-1)\tilde{\lambda}_2}, \frac{mT^{3+\mu_1-\mu_1 m}}{2(m-1)^2\mu_1}, \frac{nT^{3+\mu_2-\mu_2 n}}{2(n-1)^2\mu_2}\Big\}.$$

In the following, we verify that the above f_1, f_2 satisfy (2.17)–(2.20). Substituting (2.21) into (2.17) and (2.18) yields that

$$A_1^m\frac{m}{(m-1)^2}T^2 - A_1^m\tilde{\lambda}_1\frac{m}{m-1}2T(B-\eta)_+ - A_1\alpha_1 T^{\alpha_1 m-\alpha_1-1}(B-\eta)_+ \geq 0,$$

$$A_2^n\frac{n}{(n-1)^2}T^2 - A_2^n\tilde{\lambda}_2\frac{n}{n-1}2T(B-\eta)_+ - A_2\alpha_2 T^{\alpha_2 n-\alpha_2-1}(B-\eta)_+ \geq 0,$$

which can be obtained by the following

$$A_1^m\frac{m}{(m-1)^2}T^2 \geq \frac{1}{2}A_1^m\tilde{\lambda}_1\frac{m}{m-1}2TB, \quad A_1^m\frac{m}{(m-1)^2}T^2 \geq \frac{1}{2}A_1\mu_1 T^{\mu_1 m-\mu_1-1}B,$$

$$A_2^n\frac{n}{(n-1)^2}T^2 \geq \frac{1}{2}A_2^n\tilde{\lambda}_2\frac{n}{n-1}2TB, \qquad A_2^n\frac{n}{(n-1)^2}T^2 \geq \frac{1}{2}A_2\alpha_2 T^{\mu_2 n-\mu_2-1}B.$$

The choice of B makes us to conclude that the above inequalities hold for $A_1, A_2 > 1$. This indicated that f_1, f_2 satisfy (2.17) and (2.18).

Next, substitute (2.21) into (2.19) and (2.20), we have

$$2T\frac{m}{m-1}B^{1/(m-1)}A_1^m \leq B^{\alpha/(n-1)}A_2^\alpha, \tag{2.22}$$

$$2T^{\mu_1\beta-\mu_2 n+1}\frac{n}{n-1}B^{1/(n-1)}A_2^n \leq B^{\beta/(m-1)}A_1^\beta. \tag{2.23}$$

Thanks to the assumption $\alpha\beta > mn$, there exists a constant k, such that $m/\alpha < k < \beta/n$. Let $A_2 = A_1^k$, then (2.22) and (2.23) are equivalent to the following

$$2T\frac{m}{m-1}B^{1/(m-1)}A_1^m \leq B^{\alpha/(n-1)}A_1^{k\alpha}, \tag{2.24}$$

$$2T^{\mu_1\beta-\mu_2 n+1}\frac{n}{n-1}B^{1/(n-1)}A_1^{kn} \leq B^{\beta/(m-1)}A_1^\beta. \tag{2.25}$$

Choose A_1 large enough to satisfy (2.24) and (2.25), then we get (2.19) and (2.20).

Therefore, the solution (u, v) of the system (2.1)–(2.3) blows up in a finite time if (u_0, v_0) is large enough such that

$$u_0(r) \geq \underline{u}(r, 0), \qquad v_0(r) \geq \underline{v}(r, 0), \qquad r > 1.$$

The proof is completed. $\square$

Proof of Proposition 2.3. In fact, we can show the precent proposition is valid for all $\alpha\beta \neq mn$. Let

$$\bar{u}(r,t) = (B_1 r^{1-\tilde{\lambda}_1})^{1/m}, \quad \bar{v}(r,t) = (B_2 r^{1-\tilde{\lambda}_2})^{1/n}, \quad r > 1, t > 0,$$

where

$$B_1 = (\tilde{\lambda}_1 - 1)^{-mn/(mn-\alpha\beta)}(\tilde{\lambda}_2 - 1)^{-m\alpha/(mn-\alpha\beta)},$$
$$B_2 = (\tilde{\lambda}_1 - 1)^{-n\beta/(mn-\alpha\beta)}(\tilde{\lambda}_2 - 1)^{-mn/(mn-\alpha\beta)}.$$

By some calculations, we get

$$\frac{\partial \bar{u}}{\partial t} - \frac{\partial^2 \bar{u}^m}{\partial r^2} - \frac{\tilde{\lambda}_1}{r}\frac{\partial \bar{u}^m}{\partial r} = 0, \qquad \frac{\partial \bar{v}}{\partial t} - \frac{\partial^2 \bar{v}^n}{\partial r^2} - \frac{\tilde{\lambda}_2}{r}\frac{\partial \bar{v}^n}{\partial r} = 0, \qquad r > 1, t > 0,$$
$$-\frac{\partial \bar{u}^m}{\partial r}(1,t) = \bar{v}^\alpha(1,t), \qquad -\frac{\partial \bar{v}^n}{\partial r}(1,t) = \bar{u}^\beta(1,t), \qquad t > 0.$$

That is to say that $(\overline{u}, \overline{v})$ is a stationary solution of the problem (2.1)–(2.3). Due to the comparison principle, for any initial value $(u_0(r), v_0(r))$ which is small enough to satisfy

$$u_0(r) \leq \bar{u}(r, 0), \quad v_0(r) \leq \bar{v}(r, 0), \qquad r > 1,$$

the solutions of the system (2.1)–(2.3) exist globally in time. $\square$

Now, we prove the main result for the system (1.1)–(1.3), i.e., Theorem 2.1.

Proof of Theorem 2.1 Noticing that the functions $u_0(x), v_0(x)$ have compact supports, we can choose two bounded, radially symmetrical functions, denoted by $u_1(x) = u_1(|x|) \geq$

$u_0(x), v_1(x) = v_1(|x|) \geq v_0(x)$. By using Proposition 2.1 and the comparison principle, we can obtain the global existence of solutions for the system (1.1)–(1.3). However, for the large and radially symmetric functions $\underline{u}(|x|,0), \underline{v}(|x|,0)$ defined in the proof of Proposition 2.2, if (u_0, v_0) is large enough such that $u_0(x) \geq \underline{u}(|x|,0)$, $v_0(x) \geq \underline{v}(|x|,0)$, then the solutions of the system (1.1)–(1.3) with $\alpha\beta > mn$ blow up by the comparison principle and Proposition 2.2. This clarifies that the critical global existence curve is $\alpha\beta = mn$ for the system (1.1)–(1.3).

On the other hand, using the comparison principle again, we conclude that the solution (u, v) of (1.1)–(1.3) with

$$u_0(x) \leq (B_1|x|^{2-N})^{1/m}, \quad v_0(x) \leq (B_2|x|^{2-N})^{1/n}, \quad x \in \mathbb{R}^N \backslash B_1(0) \tag{2.26}$$

where

$$B_1 = (N-2)^{-m(n+\alpha)/(mn-\alpha\beta)}, \qquad B_2 = (N-2)^{-n(m+\beta)/(mn-\alpha\beta)}$$

exists globally for $\alpha\beta > mn$ by Proposition 2.3. This combined with Proposition 2.2 indicates that the critical Fujita curve $\alpha\beta = mn$ for the system (1.1)–(1.3). □

Proof of Theorem 2.2 By virtue of the same discussion in the proof of Theorem 2.1, if we prove this theorem by taking $\tilde{\lambda}_1 = \lambda_1 + N - 1$, $\tilde{\lambda}_2 = \lambda_2 + N - 1$ in the system (2.1)–(2.3), and replacing (2.26) with

$$u_0(x) \leq (B_1|x|^{\lambda_1+2-N})^{1/m}, \quad v_0(x) \leq (B_2|x|^{\lambda_2+2-N})^{1/n},$$

where

$$B_1 = (\lambda_1 + N - 2)^{-mn/(mn-\alpha\beta)}(\lambda_2 + N - 2)^{-m\alpha/(mn-\alpha\beta)},$$
$$B_2 = (\lambda_1 + N - 2)^{-n\beta/(mn-\alpha\beta)}(\lambda_2 + N - 2)^{-mn/(mn-\alpha\beta)}.$$

The proof is complete. □

Acknowledgment. This work is supported by the National Natural Science of China under the grant number is 11361029. The authors are particular grateful to the reviewers for their valuable suggestions.

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