Distortion Theorems for Almost Convex Mappings of Order α in Several Complex Variables

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Abstract In this paper, firstly, a sufficient condition for almost convex mappings of order α defined on the unit ball of complex Hilbert spaces and another sufficient condition for almost quasi-convex mappings of order α defined on the unit ball of complex Banach spaces are given. Secondly, the distortion theorem of the Fréchet derivative for almost convex mappings of order α on the unit ball of complex Banach spaces, the homogeneous ball of complex Banach spaces, and the unit ball of complex Hilbert spaces are established respectively. Finally, the distortion theorem of the Jacobi determinant for almost convex mappings of order α on the Euclidean unit ball in $\mathbb{C}^n$ are obtained. Our results generalize many known results.

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1 Introduction

Throughout this paper, let X be a complex Banach space with the norm $\| \, . \, \|$, X^* be the dual space of X, B be the open unit ball in X, and U be the Euclidean open unit disk in $\mathbb{C}$. We denote ∂B by the boundary of B, and $\overline{B}$ by the closure of B. Also, let U^n denote the open unit polydisk in $\mathbb{C}^n$, and $\mathbb{N}$ denote the set of all positive integers. Let the symbol $'$ mean transpose. For each $x \in X\backslash\{0\}$, we define

$$T(x) = \{T_x \in X^* : \|T_x\| = 1, T_x(x) = \|x\|\}.$$

By the Hahn-Banach theorem, $T(x)$ is nonempty.

Let $H(B)$ be the set of all holomorphic mappings from B into X. We know that if $f \in H(B)$, then

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x)((y-x)^n)$$

for all y in some neighborhood of $x \in B$, where $D^n f(x)$ is the nth-Fréchet derivative of f at x, and for $n \geq 1$,

$$D^n f(x)((y-x)^n) = D^n f(x)(\underbrace{y-x, \cdots, y-x}_{n}).$$

Furthermore, $D^n f(x)$ is a bounded symmetric n-linear mapping from $\prod_{j=1}^{n} X$ into X.

We say that a holomorphic mapping $f : B \to X$ is biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(B)$. A mapping $f \in H(B)$ is said to be locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in B$. If $f : B \to X$ is a holomorphic mapping, then we say that f is normalized if $f(0) = 0$ and $Df(0) = I$, where I represents the identity operator from X into X. Let $Aut(G)$ be the automorphism group of biholomorphic mapping of domain $G \subset X$ onto itself. A domain is called homogeneous if for any $x, y \in G$, there exists a mapping $\varphi \in Aut(G)$ such that $\varphi(x) = y$.

We now introduce some definitions as follow.

Definition 1.1 *Suppose that $f : B \to X$ is a normalized locally biholomorphic mapping. If $\alpha \in [0, 1)$, and*

(1.1)

$$\Re e\left\{T_x\left[(Df(x))^{-1}(f(x)-f(y))\right]\right\} \geq \alpha \Re e(\|x\| - T_x(y)), x, y \in B, \|y\| \leq \|x\|, T_x \in T(x),$$

then we say that f is an almost convex mapping of order α.

Especially, when B is the unit ball of complex Hilbert spaces, then (1.1) is replaced by the following condition.

(1.2) $$\Re e\langle Df(x))^{-1}(f(x)-f(y)), x\rangle \geq \alpha(\|x\|^2 - \Re e\langle y, x\rangle), x, y \in B, \|y\| \leq \|x\|.$$

When $B = B^n$, then (1.1) reduces to the following condition.

(1.3) $$\Re e\langle Df(z))^{-1}(f(z)-f(w)), z\rangle \geq \alpha(\|z\|^2 - \Re e\langle w, z\rangle), z, w \in B^n, \|w\| \leq \|z\|.$$

Letting $y = 0$ in Definition 1.1, one obtains that f is an almost starlike mapping of order α, is the sense introduced by Xu and Liu [22].

Definition 1.2 *[16] Suppose that $f : B \to X$ is a normalized locally biholomorphic mapping. If $\alpha \in [0,1)$, and*

$$\Re e\left\{T_x\left[(Df(x))^{-1}(f(x)-f(\xi x))\right]\right\} \geq \alpha \Re e(1-\xi)\|x\|\,, \xi \in \overline{U}, x \in B\,,$$

then we say that f is an almost quasi-convex mapping of order α.

From Definitions 1.1 and 1.2, we easily see that almost convex mappings of order α is the subclass of almost quasi-convex mappings of order α.

Definition 1.3 *[10] Suppose that $f : B \to X$ is a normalized locally biholomorphic mapping. If $\alpha \in (0,1)$ and*

$$\left|\frac{1}{\|x\|}T_x[(Df(x))^{-1}f(x)] - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}, \quad x \in B \setminus \{0\}\,,$$

then we say that f is a starlike mapping of order α on B.

We denote by $K(B)$ the set of normalized biholomorphic mappings on B, $AK_\alpha(B)$ the set of almost convex mappings of order α on B, $AQ_\alpha(B)$ the set of almost quasi-convex mappings of order α on B, $AS^*_\alpha(B)$ the set of almost starlike mappings of order α on B, and $S^*_\alpha(B)$ the set of starlike mappings of order α on B respectively.

Concerning the distortion theorem in several complex variables, the results of convex mappings are numerous, such as the estimate for $\det Df(z)$ for convex mappings defined on the Euclidean unit ball in $\mathbb{C}^2$ were first shown by Barnard, FitzGerald and Gong [1] in 1994, and after that the above work was extended to the general case by Liu and Zhang [17]. Indeed, these are distortion theorems for the Jacobi determinant. On the other hand, with respect to the distortion theorem for the Frećhet derivative, Gong, Wang and Yu [6] first established the estimates of $Df(z)\overline{Df(z)}'$ for convex mappings, and Gong and Liu [5] then extended their work to the bounded convex circular domain in $\mathbb{C}^n$. Subsequently, Liu and Zhang [18], Zhu and Liu [23] independently applied different methods to extend it to the unit ball of the complex Banach spaces. Recently, Chu, Hamada, Honda and Kohr [2] established the distortion theorem for convex mappings on homogeneous balls. For the unit ball of a complex Hilbert space, Hamada and Kohr [9] gave the stronger upper bounds estimate of the distortion theorem for convex mappings. We next recall convex functions of order α in one complex variable, there are a number of classical results hold for them, such as the growth, covering and distortion theorem. Further, the well-known Alexander theorem remains to be valid for convex functions of order α(see [7]). However, in view of the difference between one complex variable and several complex variables, there seem to be a few of analytical forms for almost convex mappings of order α in several complex variables. Unfortunately, there are almost no geometric properties for most of them except for Liu and Zhu [19] establish the radius of convexity and the sufficient condition for starlike mappings.

In this paper, we will first introduce one definition of almost convex mappings of order α in several complex variables. Subsequently one sufficient condition and the distortion theorems (include the Frećhet derivative and the Jacobi determinant) for the above mappings are discussed. The derived results generalize many known results.

2 A sufficient condition for almost convex mappings of order α

We begin with two concrete examples to state the criteria for almost convex mappings of order α and almost quasi-convex mappings of order α.

Example 2.1 *Let* $f(z) = (z_1 + az_2^2, z_2)'$, *where* $z = (z_1, z_2)' \in B^2$. *Then* $f \in AK_\alpha(B^2)$ *if and only if* $|a| \leq (1-\alpha)/2$.

Proof Let $w = (w_1, w_2)' \in B^2$ with $\|w\|^2 \leq \|z\|^2$. By similar reasoning of [20, Example 7], we conclude that

$$\begin{aligned}
& \Re e\langle (Df(z))^{-1}(f(z) - f(w)), z\rangle - \alpha(\|z\|^2 - \Re e\langle z, w\rangle) \\
= \;& (1-\alpha)(\|z\|^2 - \Re e\langle z, w\rangle) - \Re e\{a\overline{z_1}(z_2 - w_2)^2\} \\
\geq \;& (\|z\|^2 - \Re e\langle z, w\rangle)(1 - \alpha - 2|a||z_1|) + |a||z_1||z_1 - w_1|^2 \geq 0
\end{aligned}$$

for $|a| \leq (1-\alpha)/2$. Hence we know that $f \in AK_\alpha(B^2)$ from (1.3).

Conversely, if $a > (1-\alpha)/2$, then there exists z_1 which satisfies $\overline{z_1}a > (1-\alpha)/2, w_1 = z_1, w_2 = -z_2 \in \mathbb{R}$. Consequently, with the analogous method of [20, Example 7], we also have

$$\begin{aligned}
& \Re e\langle (Df(z))^{-1}(f(z) - f(w)), z\rangle - \alpha(\|z\|^2 - \Re e\langle z, w\rangle) \\
= \;& (1-\alpha)(\|z\|^2 - \Re e\langle z, w\rangle) - \Re e\{a\overline{z_1}(z_2 - w_2)^2\} \\
< \;& (1-\alpha)(\|z\|^2 - \Re e(z_1\overline{z_1} - z_2\overline{z_2})) - \frac{1-\alpha}{2}(2z_2)^2 = 0\,.
\end{aligned}$$

This completes the proof.

Example 2.2 *[16] Let* $f(z) = (z_1 + az_2^2, z_2, \cdots, z_n)'$, *where* $z = (z_1, z_2, \cdots, z_n)' \in B^n$. *If* $|a| \leq (1-\alpha)/2$, *then* $f \in AQ_\alpha(B^n)$.

It is easy to show that the following proposition holds from Examples 2.1 and 2.2 (we omit the proof here).

Proposition 2.1

$$AK_\alpha(B) \subsetneqq K(B)\,, AK_\alpha(B) \subsetneqq AQ_\alpha(B) \subsetneqq AS^*_\alpha(B)\,. \tag{2.1}$$

We will use the following two lemmas to prove the desired theorems in this section.

Lemma 2.1 *If $f \in H(B)$, then*

$$\begin{aligned} & D^m f(0)(x^m) - D^m f(0)(y^m) - mD^m f(0)(x^{m-1}, x-y) \\ = & \begin{cases} -D^2 f(0)((x-y)^2), & m=2; \\ -D^3 f(0)((x-y)^2, y) - 2D^3 f(0)((x-y)^2, x), & m=3; \\ -D^m f(0)((x-y)^2, y^{m-2}) - 2D^m f(0)((x-y)^2, x, y^{m-3}) - \cdots & m \geqslant 4; \\ -(m-1)D^m f(0)((x-y)^2, x^{m-2}). & \end{cases} \end{aligned}$$

Proof In view of $D^m f(0)(m \geqslant 2)$ is a bounded symmetric m-linear mapping from $\prod_{j=1}^n X$ into X, when $m=2$, it yields that

$$\begin{aligned} & D^2 f(0)(x^2) - D^2 f(0)(y^2) - 2D^2 f(0)(x, x-y) \\ = & D^2 f(0)(x-y, x) + D^2 f(0)(x-y, y) - 2D^2 f(0)(x, x-y) \\ = & -D^2 f(0)((x-y)^2)\,; \end{aligned}$$

when $m=3$, we have

$$\begin{aligned} & D^3 f(0)(x^3) - D^3 f(0)(y^3) - 3D^3 f(0)(x^2, x-y) \\ = & D^3 f(0)(x-y, x^2) + D^3 f(0)(x-y, x, y) \\ & + D^3 f(0)(x-y, y^2) - 3D^3 f(0)(x^2, x-y) \\ = & -2D^3 f(0)(x-y, x^2) + D^3 f(0)(x-y, x, y) + D^3 f(0)(x-y, y^2) \\ = & -D^3 f(0)((x-y)^2, x) - [D^3 f(0)((x-y)^2, x) + D^3 f(0)((x-y)^2, y)] \\ = & -D^3 f(0)((x-y)^2, y) - 2D^3 f(0)((x-y)^2, x)\,; \end{aligned}$$

when $m \geqslant 4$, we obtain

$$\begin{aligned} & D^m f(0)(x^m) - D^m f(0)(y^m) - mD^3 f(0)(x^{m-1}, x-y) \\ = & D^m f(0)(x-y, x^{m-1}) + D^m f(0)(x-y, x^{m-2}, y) + \cdots \\ & + D^m f(0)(x-y, y^{m-1}) - mD^m f(0)(x-y, x^{m-1}) \\ = & -(m-1)D^m f(0)(x-y, x^{m-1}) + D^m f(0)(x-y, x^{m-2}, y) \\ & + \cdots + D^m f(0)(x-y, y^{m-1}) \\ = & -D^m f(0)((x-y)^2, x^{m-2}) - [D^m f(0)((x-y)^2, x^{m-2}) \\ & + D^m f(0)((x-y)^2, x^{m-3}, y)] - \cdots - [D^m f(0)((x-y)^2, x^{m-2}) \\ & + D^m f(0)((x-y)^2, x^{m-3}, y) + \cdots + D^m f(0)((x-y)^2, y^{m-2})] \\ = & -D^m f(0)((x-y)^2, y^{m-2}) - 2D^m f(0)((x-y)^2, x, y^{m-3}) - \cdots \\ & -(m-1)D^m f(0)((x-y)^2, x^{m-2})\,. \end{aligned}$$

This completes the proof.

Lemma 2.2 *If $\alpha \in [0,1)$, $f \in H(B), f(0)=0, Df(0)=I$ and $\Sigma_{m=2}^{\infty} m(m-\alpha)/(m!)$ $\|D^m f(0)\| \leqslant 1-\alpha$, where $\|D^m f(0)\| = \sup_{\|x^{(k)}\|=1, 1\leqslant k\leqslant m} \|D^m f(0)(x^{(1)}, x^{(2)}, \cdots, x^{(m)})\|$, then*

$$\|(Df(x))^{-1}\| \leqslant \frac{1}{1-\sum\limits_{m=2}^{\infty}\frac{m\|D^m f(0)\|\|x\|^{m-2}}{m!}}, \tag{2.2}$$

(2.3)

$$\|f(x)-f(y)-Df(x)(x-y)\| \leqslant (1-\alpha)\frac{\|x-y\|^2}{2}\left(1-\sum_{m=2}^{\infty}\frac{m\|D^m f(0)\|\|x\|^{m-2}}{m!}\right), \|y\| \leqslant \|x\|.$$

Proof Since$\Sigma_{m=2}^{\infty} m(m-\alpha)/(m!)\|D^m f(0)\| \leqslant 1-\alpha$, then with the similar argument in the proof of [14, Theorem 1.1], it yields that

$$\|(Df(x))^{-1}\| \leqslant \frac{1}{1-\sum\limits_{m=2}^{\infty}\frac{m\|D^m f(0)\|\|x\|^{m-2}}{m!}}.$$

Again by Lemma 2.1, for $\|y\| \leqslant \|x\|$, we obtain

$$\begin{aligned}
&\|f(x)-f(y)-Df(x)(x-y)\| \\
= &\left\|\sum_{m=2}^{\infty}\frac{D^m f(0)(x^m)-D^m f(0)(y^m)-mD^m f(0)(x^{m-1},x-y)}{m!}\right\| \\
\leqslant &\sum_{m=2}^{\infty}\frac{\|D^m f(0)(x^m)-D^m f(0)(y^m)-mD^m f(0)(x^{m-1},x-y)\|}{m!} \\
\leqslant &\sum_{m=2}^{\infty}\frac{(1+2+\cdots+m-1)\|D^m f(0)\|\|x-y\|^2\|x\|^{m-2}}{m!} \\
= &\sum_{m=2}^{\infty}\frac{\frac{m(m-\alpha)-m(1-\alpha)}{2}\|D^m f(0)\|\|x-y\|^2\|x\|^{m-2}}{m!} \\
\leqslant &(1-\alpha)\frac{\|x-y\|^2}{2}\left(1-\sum_{m=2}^{\infty}\frac{m\|D^m f(0)\|\|x\|^{m-2}}{m!}\right).
\end{aligned}$$

This completes the proof.

Theorem 2.1 *If $\alpha \in [0,1)$, $f \in H(B), f(0)=0, Df(0)=I$ and $\Sigma_{m=2}^{\infty} m(m-\alpha)/(m!) \cdot$ $\|D^m f(0)\| \leqslant 1-\alpha$, where $\|D^m f(0)\| = \sup_{\|x^{(k)}\|=1, 1\leqslant k\leqslant m} \|D^m f(0)(x^{(1)}, x^{(2)}, \cdots, x^{(m)})\|$, then $f \in AQ_\alpha(B)$.*

Proof In view of the condition of Theorem 2.1, (2.2) and (2.3), we have

$$\begin{aligned}
& \Re e\left\{T_x\left[(Df(x))^{-1}(f(x)-f(\xi x))-\alpha(1-\xi)x\right]\right\} \\
= & \Re e\left\{T_x\left[(Df(x))^{-1}(f(x)-f(\xi x)-Df(x)(1-\xi)x)\right]\right\}+(1-\Re e\xi)\|x\|-\alpha(1-\Re e\xi)\|x\| \\
\geqslant & (1-\alpha)(1-\Re e\xi)\|x\|-\left|T_x\left[(Df(x))^{-1}(f(x)-f(\xi x)-Df(x)(1-\xi)x)\right]\right| \\
\geqslant & (1-\alpha)(1-\Re e\xi)\|x\|-\|(Df(x))^{-1}(f(x)-f(\xi x)-Df(x)(1-\xi)x)\| \\
\geqslant & (1-\alpha)(1-\Re e\xi)\|x\|-\frac{(1-\alpha)\frac{|1-\xi|^2\|x\|^2}{2}\left(1-\sum\limits_{m=2}^{\infty}\frac{m\|D^m f(0)\|\|x\|^{m-2}}{m!}\right)}{1-\sum\limits_{m=2}^{\infty}\frac{m\|D^m f(0)\|\|x\|^{m-2}}{m!}} \\
\geq & (1-\alpha)(1-\Re e\xi)\|x\|-(1-\alpha)(1-\Re e\xi)\|x\|=0\,.
\end{aligned}$$

Thus, it yields that $f\in AQ_\alpha(B)$ by Definition 1.2. This completes the proof.

Especially, when B is the unit ball of complex Hilbert spaces, we have the following

Theorem 2.2 *If $\alpha\in[0,1)$, $f\in H(B), f(0)=0, Df(0)=I$ and $\Sigma_{m=2}^{\infty}m(m-\alpha)/(m!)$ $\|D^m f(0)\|\leqslant 1-\alpha$, where $\|D^m f(0)\|=\sup_{\|x^{(k)}\|=1,1\leqslant k\leqslant m}\|D^m f(0)(x^{(1)},x^{(2)},\cdots,x^{(m)})\|$, then $f\in AK_\alpha(B)$.*

Proof According to the hypothesis of Theorem 2.2, (1.2), (2.2) and (2.3), we have

$$\begin{aligned}
& \Re e\{<(Df(x))^{-1}(f(x)-f(y),x>-\alpha<x-y,x>\} \\
= & \Re e\{<(Df(x))^{-1}(f(x)-f(y)-Df(x)(x-y)),x>+<x-y,x>-\alpha<x-y,x>\} \\
\geqslant & (1-\alpha)(\|x\|^2-\Re e<x,y>)-|<(Df(x))^{-1}(f(x)-f(y)-Df(x)(x-y)),x>| \\
\geqslant & (1-\alpha)(\|x\|^2-\Re e<x,y>)-\|(Df(x))^{-1}(f(x)-f(y)-Df(x)(x-y))\|\|x\| \\
\geqslant & (1-\alpha)(\|x\|^2-\Re e<x,y>)-\frac{(1-\alpha)\frac{\|x-y\|^2}{2}\left(1-\sum\limits_{m=2}^{\infty}\frac{m\|D^m f(0)\|\|x\|^{m-2}}{m!}\right)}{1-\sum\limits_{m=2}^{\infty}\frac{m\|D^m f(0)\|\|x\|^{m-2}}{m!}} \\
= & \frac{1-\alpha}{2}(2\|x\|^2-2\Re e<x,y>-\|x\|^2-\|y\|^2+2\Re e<x,y>) \\
= & \frac{1-\alpha}{2}(\|x\|^2-\|y\|^2)\geqslant 0\,.
\end{aligned}$$

Therefore, $f\in AK_\alpha(B)$. This completes the proof.

When $\alpha=0, B=B^n$, Theorem 2.1 reduces to [20, Theorem 2.1], and our proof seems to be more concise.

The following theorem provides a sufficient condition for starlike mappings of order α (the case $\alpha\in[1/2,1)$) due to Liu and Liu [14].

Theorem 2.3 *If $\alpha\in[1/2,1)$, $f\in H(B), f(0)=0, Df(0)=I$ and $\sum\limits_{m=2}^{\infty}(m-\alpha)/(m!)\|D^m f(0)\|$ $\leqslant 1-\alpha$, where $\|D^m f(0)\|=\sup_{\|x^{(k)}\|=1,1\leqslant k\leqslant m}\|D^m f(0)(x^{(1)},x^{(2)},\cdots,x^{(m)})\|$, then $f\in S^*_\alpha(B)$.*

Remark 2.1 *Noticing that the fact* $Df(x)x = x + \Sigma_{m=2}^{\infty} mD^m f(0)(x^m)/(m!)$ *if* $f(x) = x + \Sigma_{m=2}^{\infty} D^m f(0)(x^m)/(m!), x \in B$, *Theorems 2.1 and 2.3 show that the classical Alexander theorem is valid with some restricted conditions.*

3 Distortion theorems of the Frechet derivative for almost convex mappings of order α

In this section, let $H(B,U)$ be the set of all holomorphic mappings from B into U. For each $x \in B, \xi \in X$, the infinitesimal form of Carathéodory metric on B is defined by

$$F_C(x,\xi) = \sup\{|D\varphi(x)\xi| : \varphi \in H(B,U), \varphi(x) = 0\}\,.$$

the infinitesimal form of Carathéodory metric on B has some properties as follows.

$$F_C(x,\xi) \geq \frac{|T_x(\xi)|}{1-\|x\|^2},\ x \in B, \xi \in X, \tag{3.1}$$

$$\frac{\|\xi\|}{1+\|x\|} \leq F_C(x,\xi) \leq \frac{\|\xi\|}{1-\|x\|},\ x \in B, \xi \in X, \tag{3.2}$$

$$F_C(x,x) = \frac{\|x\|}{1-\|x\|^2}, x \in B \tag{3.3}$$

((3.1) see [15], (3.2) and (3.3) see [4]).

Lemma 3.1 *[16] If* $f \in AQ_\alpha(B)$, *then*

$$\frac{\|x\|}{1+(1-\alpha)\|x\|} \leq \|f(x)\| \leq \frac{\|x\|}{1-(1-\alpha)\|x\|}\,, x \in B\,.$$

We first begin to establish the distortion theorem of Fréchet derivative for almost convex mappings of order α on the unit ball of complex Banach spaces.

Theorem 3.1 *If* $f \in AK_\alpha(B)$, *then for* $x \in B, \xi \in X$,

$$\begin{aligned} \frac{(1-\|x\|)\|\xi\|}{(1+\|x\|)(1+(1-\alpha)\|x\|)} &\leq \frac{1-\|x\|}{1+(1-\alpha)\|x\|} F_C(x,\xi) \leq \|Df(x)\xi\| \\ &\leq \frac{1+\|x\|}{1-(1-\alpha)\|x\|} F_C(x,\xi) \leq \frac{(1+\|x\|)\|\xi\|}{(1-\|x\|)(1-(1-\alpha)\|x\|)} \end{aligned}$$

and

$$\|Df(x)\xi\| \geq \frac{1-\|x\|}{1+(1-\alpha)\|x\|} F_C(x,\xi) \geq \frac{|T_x(\xi)|}{(1+\|x\|)(1+(1-\alpha)\|x\|)}\,.$$

Proof By the same arguments as the proof of [4, Theorem 3.1.2], we deduce that

$$\frac{1}{\lambda} \le \frac{1+\|x\|}{1-(1-\alpha)\|x\|}, x \in B$$

from (2.1) and Lemma 3.1, and

$$\|Df(x)\xi\| \le \frac{1}{\lambda}F_C(x,\xi) \le \frac{1+\|x\|}{1-(1-\alpha)\|x\|}F_C(x,\xi) \le \frac{(1+\|x\|)\|\xi\|}{(1-\|x\|)(1-(1-\alpha)\|x\|)}$$

from (3.2). On the other hand, with arguments similar to those in the proof of [4, Theorem 3.1.2], we obtain

$$1-\mu \ge \frac{1-\|x\|}{1+(1-\alpha)\|x\|}, x \in B$$

by Lemma 3.1,

$$\|Df(x)\xi\| \ge (1-\mu)F_C(x,\xi) \ge \frac{1-\|x\|}{1+(1-\alpha)\|x\|}F_C(x,\xi) \ge \frac{(1-\|x\|)\|\xi\|}{(1+\|x\|)(1+(1-\alpha)\|x\|)},$$

and

$$\|Df(x)\xi\| \ge (1-\mu)F_C(x,\xi) \ge \frac{1-\|x\|}{1+(1-\alpha)\|x\|}F_C(x,\xi) \ge \frac{|T_x(\xi)|}{(1+\|x\|)(1+(1-\alpha)\|x\|)}$$

from (3.1) and (3.2). This completes the proof.

Applying (3.3) to Theorem 3.1, we immediately deduce the following

Corollary 3.1 *If $f \in AK_\alpha(B)$, then*

$$\frac{\|x\|}{(1+\|x\|)(1+(1-\alpha)\|x\|)} \le \|Df(x)x\| \le \frac{\|x\|}{(1-\|x\|)(1-(1-\alpha)\|x\|)}, x \in B.$$

When $\alpha = 0$, Theorem 3.1 and Corollary 3.1 reduce to [18, Theorem 3.3].

Banach spaces with a homogeneous open unit ball are precisely the JB^*-triplles [11]. They are the complex Banach spaces X equipped with a triple product $\{.,.,. : X^3 \to X\}$ which is conjugate linear in the middle variable, but linear and symmetric in the other variables such that

(*i*) $\{a,b,\{x,y,z\}\} = \{\{a,b,x\},y,z\} - \{x,\{b,a,y\},z\} + \{x,y,\{a,b,z\}\}$;

(*ii*) the map$a\Box a : x \in X \mapsto \{a,a,x\} \in X$ is hermitian with nonnegative spectrum;

(*iii*) $\{a,a,a\} = \|a\|^3$

for $a,b,x,y,z \in X$. On the open unit ball B of a JB^*-triple, each point $a \in B$ induces the Möbius transformation $g_a \in Aut(B)$ given by

$$g_a(x) = a + B(a,a)^{1/2}(I + x\Box a)^{-1}x(x \in X),$$

where $x\Box a$ is the box operator $(x\Box a)(y) = \{x, a, y\}$. Moreover, $g_a(0) = a, g_a^{-1} = g_{-a}$ and

$$Dg_a(0) = B(a,a)^{1/2},\ Dg_a(a) = B(a,a)^{-1/2}.$$

In any dimension, we have

$$\|B(a,a)^{-1/2}\| = \frac{1}{1-\|a\|^2}$$

from [12, Corollary 3.6]. More details of JB^*-triple may consult [21] or [8].

We next establish the distortion theorem of Fréchet derivative for almost convex mappings of order α on the unit ball B of a JB^*-triple X.

Theorem 3.2 *If $f \in AK_\alpha(B)$, then for $x \in B, \xi \in X$,*

$$\frac{(1-\|x\|)\|\xi\|}{(1+(1-\alpha)\|x\|)\|B(x,x)^{1/2}\|} \le \|Df(x)\xi\| \le \frac{\|\xi\|}{(1-\|x\|)(1-(1-\alpha)\|x\|)}.$$

Proof For any $x \in B, \xi \in X$, we have

$$F_C(x,\xi) = F_C(0, Dg_{-x}(x)\xi) = \|B(x,x)^{-1/2}\xi\| \le \frac{\|\xi\|}{1-\|x\|^2}$$

for the Möbius transformation $g_{-x} \in Aut(B)$ such that $g_{-x}(0) = 0$(see [2]). Note that $\|B(a,a)^{-1/2}\| = \frac{1}{1-\|a\|^2}, a \in B$. Take into account $\frac{1-\|x\|}{1+(1-\alpha)\|x\|}F_C(x,\xi) \le \|Df(x)\xi\| \le \frac{1+\|x\|}{1-(1-\alpha)\|x\|}$ $F_C(x,\xi)$. By the same arguments as the proof of [2, Lemmas 2.5 and 2.6], it follows the desired results. This completes the proof.

In particular, when B is the unit ball of complex Hilbert spaces, we have the following theorem.

Theorem 3.3 *If $f \in AK_\alpha(B)$, then for $x \in B, \xi \in X$,*

$$\begin{aligned}
&\frac{\sqrt{1-\|x\|^2}\|\xi\|}{(1+\|x\|)(1+(1-\alpha)\|x\|)} \le \frac{\sqrt{(1-\|x\|^2)\|\xi\|^2+|\langle x,\xi\rangle|^2}}{(1+\|x\|)(1+(1-\alpha)\|x\|)}\\
\le\ &\frac{1-\|x\|}{1+(1-\alpha)\|x\|}F_C(x,\xi) \le \|Df(x)\xi\| \le \frac{1+\|x\|}{1-(1-\alpha)\|x\|}F_C(x,\xi)\\
\le\ &\frac{\sqrt{(1-\|x\|^2)\|\xi\|^2+|\langle x,\xi\rangle|^2}}{(1-\|x\|)(1-(1-\alpha)\|x\|)} \le \frac{\|\xi\|}{(1-\|x\|)(1-(1-\alpha)\|x\|)}.
\end{aligned}$$

Proof For any $x \in B, \xi \in X$, it is shown that

$$F_C(x,\xi) = \frac{\sqrt{(1-\|x\|^2)\|\xi\|^2+|\langle x,\xi\rangle|^2}}{1-\|x\|^2}$$

(see [3]). Hence, the desired result holds from Theorem 3.1.

When $\alpha = 0$, Theorem 3.3 reduces to [23, Theorem 2.4].

4 Distortion theorems of the Jacobi determinant for almost convex mappings of order α

In this section, we denote by $J_f(z)$ the Jacobi matrix of the holomorphic mapping $f(z)$, and $\det J_f(z)$ the Jacobi determinant of the holomorphic mapping $f(z)$. Let $H(\Omega,\Omega)$ be the holomorphic mappings from Ω into Ω, where Ω is a bounded domain in $\mathbb{C}^n$.

We will use the following lemma (see [13]).

Lemma 4.1 *Suppose that Ω is a bounded domain in $\mathbb{C}^n$. If $\varphi:\Omega\to\Omega$ is biholomorphic, then*

$$|\det J_\varphi(z)| = \sqrt{\frac{K(z,z)}{K(\varphi(z),\varphi(z))}}, z\in\Omega\,.$$

The following lemma is also applied to prove the desired theorem in this section.

Lemma 4.2 *Suppose that Ω is a bounded convex circular homogeneous domain in $\mathbb{C}^n$. If $\varphi\in H(\Omega,\Omega)$, then*

$$|\det J_\varphi(z)| \le \sqrt{\frac{K(z,z)}{K(\varphi(z),\varphi(z))}}, z\in\Omega\,.$$

In particular, when $\Omega = B^n$ (resp. U^n)

$$(4.1)\qquad |\det J_\varphi(z)| \le \left(\frac{1-\|\varphi(z)\|^2}{1-\|z\|^2}\right)^{\frac{n+1}{2}} \quad (\textit{resp.} |\det J_\varphi(z)| \le \prod_{j=1}^{n}\frac{1-\|\varphi_j(z)\|^2}{1-\|z_j\|^2})\,.$$

Proof We write $\tau_\varphi(z) = \sqrt{\frac{K(\varphi(z),\varphi(z))}{K(z,z)}}|\det J_\varphi(z)|$. Taking into account Lemma 4.1, a direct computation shows that

$$\begin{aligned}
&\tau_{\psi\circ\varphi\circ\omega}(z)\\
=&\sqrt{\frac{K(\psi(\varphi(\omega(z))),\psi(\varphi(\omega(z)))}{K(z,z)}}|\det J_\psi(\varphi(\omega(z)))||\det J_\varphi(\omega(z))||\det J_\omega(z)|\\
=&\sqrt{\frac{K(\varphi(\omega(z)),\varphi(\omega(z))}{K(z,z)}}|\det J_\varphi(\omega(z))||\det J_\omega(z)|\\
=&\sqrt{\frac{K(\varphi(\omega(z)),\varphi(\omega(z))}{K(\omega(z),\omega(z))}}|\det J_\varphi(\omega(z))| = \tau_\varphi(\omega(z))
\end{aligned}$$

for every $\psi,\omega\in Aut(\Omega)$. That is,

$$(4.2)\qquad \tau_{\psi\circ\varphi\circ\omega}(z) = \tau_\varphi(\omega(z)), z\in\Omega\,.$$

Let $\rho(z)$ be the Minkowski functional of Ω. Fix $\xi \in \mathbb{C}^n \setminus \{0\}$ and denote $\xi_0 = \frac{\xi}{\rho(\xi)}$. When $\varphi(0) = 0$ if $z = 0$, we may define

$$w(\lambda) = T_{J_\varphi(0)\xi_0}(\varphi(\lambda\xi_0)), \lambda \in U\,,$$

then $w \in H(U, U), w(0) = 0$. Hence, in view of Schwarz lemma, we conclude that

$$|w'(0)| = \rho(J_\varphi(0)\xi_0) \leq 1\,. \tag{4.3}$$

If λ is the eigenvalue of $J_\varphi(0)$, then

$$J_\varphi(0)\eta = \lambda\eta\,, \tag{4.4}$$

where η is the eigenvector of $J_\varphi(0)$. Letting $\eta_0 = \frac{\eta}{\rho(\eta)}$, then $\eta_0 \in \partial\Omega$. According to (4.3) and (4.4), we obtain

$$\rho(J_\varphi(0)\eta_0) = \rho(\lambda\eta_0) = |\lambda| \leq 1\,.$$

Therefore

$$\tau_\varphi(0) = |\det J_\varphi(0)| \leq 1\,. \tag{4.5}$$

For the general case, take $\psi, \omega \in Aut(\Omega)$ which satisfy $\omega(0) = z$ and $\psi(\varphi(z)) = 0$. It yields that

$$\tau_\varphi(z) = \tau_\varphi(\omega(0)) = \tau_{\psi\circ\varphi\circ\omega}(0) \leq 1$$

from (4.2) and (4.5). This implies that

$$|\det J_\varphi(z)| \leq \sqrt{\frac{K(z,z)}{K(\varphi(z), \varphi(z))}}, z \in \Omega\,.$$

We recall the fact that

$$K(z,z) = \frac{n!}{\pi^n(1 - \|z\|^2)^{n+1}}, z \in B^n,\ K(z,z) = \frac{1}{\pi^n}\prod_{j=1}^n \frac{1}{(1 - \|z_j\|^2)^2}, z \in U^n$$

(see [13]), then the estimates of (4.1) are valid. This completes the proof.

We now begin to present the desired theorem in this section.

Theorem 4.1 *Suppose that Ω is a bounded convex circular homogeneous domain in $\mathbb{C}^n$. If $f \in AK_\alpha(\Omega)(\alpha \in [0,1))$, then*

$$\left(\frac{1 - \rho(z)}{1 + (1-\alpha)\rho(z)}\right)^n \sqrt{\frac{K(z,z)}{K(0,0)}} \leq |\det J_f(z)| \leq \left(\frac{1 + \rho(z)}{1 - (1-\alpha)\rho(z)}\right)^n \sqrt{\frac{K(z,z)}{K(0,0)}}, z \in \Omega\,.$$

In particular, when $\Omega = B^n$, we have

$$\frac{(1 - \|z\|)^{\frac{n-1}{2}}}{(1 + (1-\alpha)\|z\|)^n(1 + \|z\|)^{\frac{n+1}{2}}} \leq |\det J_f(z)| \leq \frac{(1 + \|z\|)^{\frac{n-1}{2}}}{(1 - (1-\alpha)\|z\|)^n(1 - \|z\|)^{\frac{n+1}{2}}}, z \in B^n\,. \tag{4.6}$$

Proof With arguments similar to those in the proof of [4, Theorem 3.1.2], we have

$$\frac{1}{\lambda} \le \frac{1+\rho(z)}{1-(1-\alpha)\rho(z)}, z \in \Omega \tag{4.7}$$

from Lemma 3.1. Define

$$\varphi(w) = f^{-1}(\lambda f(w) + (1-\lambda) f(z^*)), w \in \Omega\,, \tag{4.8}$$

where $z^* \in \partial\Omega$. It is shown that $f \in AK_\alpha(\Omega) \subset K(\Omega)$ from (2.1), then $\varphi(w)$ is well defined, further, $\varphi \in H(\Omega, \Omega)$ and $\varphi(z) = 0$. From (4.8), we have

$$f(\varphi(w)) = \lambda f(w) + (1-\lambda) f(z^*)\,.$$

Differentiating both sides of the above equality with respect to w, we obtain

$$J_f(\varphi(w)) J_\varphi(w) = \lambda J_f(w)\,.$$

Noticing that $\varphi(z) = 0$, especially taking $w = z$, we deduce that

$$J_f(0) J_\varphi(z) = \lambda J_f(z)\,.$$

This implies that

$$J_f(z) = \frac{1}{\lambda} J_f(0) J_\varphi(z) = \frac{1}{\lambda} J_\varphi(z)\,.$$

Using Lemma 4.2 and (4.7), we conclude that

$$\begin{aligned} |\det J_f(z)| &= \frac{1}{\lambda^n} |\det J_\varphi(z)| \\ &\le \left(\frac{1+\rho(z)}{1-(1-\alpha)\rho(z)}\right)^n \sqrt{\frac{K(z,z)}{K(0,0)}}\,. \end{aligned}$$

On the other hand, by the same arguments as in the proof of [4, Theorem 3.1.2], we conclude that

$$1 - \mu \ge \frac{1-\rho(z)}{1+(1-\alpha)\rho(z)}, z \in \Omega \tag{4.9}$$

from Lemma 3.1.

Let

$$\psi(w) = f^{-1}((1-\mu) f(w) + \mu f(\widetilde{z})), w \in \Omega\,, \tag{4.10}$$

where $\widetilde{z} \in \partial\Omega$. Since $f \in AK_\alpha(\Omega) \subset K(\Omega)$, then $\psi(w)$ is well defined, moreover, $\psi \in H(\Omega, \Omega)$ and $\psi(0) = z$. From (4.10), we obtain

$$f(\psi(w)) = (1-\mu) f(w) + \mu f(\widetilde{z})\,.$$

Differentiating both sides of the above equality with respect to w, we also obtain

$$J_f(\psi(w))J_\psi(w) = (1-\mu)J_f(w)\,.$$

Note that $\psi(0) = z$, especially take $w = 0$. We conclude that

$$J_f(z)J_\psi(0) = (1-\mu)J_f(0) = (1-\mu)I_n\,,$$

where I_n is a unit matrix of $n \times n$. Applying Lemma 4.2 and (4.9), we deduce that

$$\begin{aligned}|\det J_f(z)| &= (1-\mu)^n \frac{1}{|\det J_\psi(0)|} \\ &\geq \left(\frac{1-\rho(z)}{1+(1-\alpha)\rho(z)}\right)^n \sqrt{\frac{K(z,z)}{K(0,0)}}\,.\end{aligned}$$

In particular, if $\Omega = B^n$, taking into account the fact $K(z,z) = \frac{n!}{\pi^n(1-\|z\|^2)^{n+1}}$, $z \in B^n$, it is easy to see that the estimate of (4.6) holds.

When $\alpha = 0$, the estimate of (4.6) reduce to the corollary of [4, Theorem 3.1.1].

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