COEFFICIENT ESTIMATES AND BLOCH'S CONSTANT IN SOME CLASSES OF HARMONIC MAPPINGS

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Abstract. Following Clunie and Sheil-Small denote by $\mathcal{S_H}$ the class of normalized univalent harmonic mappings in the unit disk. The aim of a paper is to study the properties of a subclass of $\mathcal{S_H}$, such that the analytic part is a convex function. We establish estimates of some functionals and bounds of the Bloch constant for co-analytic part.

1. Introduction

A complex-valued harmonic function f that is harmonic in a simply connected domain $\Omega \subset \mathbb{C}$ has the canonical representation

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are analytic in Ω with $g(z_0) = 0$ for some prescribed point $z_0 \in \Omega$. According to a theorem of Lewy [17], f is *locally univalent*, if and only if its Jacobian $J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2$ does not vanish, and is *sense-preserving* if the Jacobian is positive. Then $h'(z) \neq 0$ and the analytic function $\omega = g'/h'$, called the *second complex dilatation* of f, has the property $|\omega| < 1$ in Ω. Throughout this paper we will assume that f is locally univalent and sense-preserving, and we call f *a harmonic mapping*. Also, we assume $\Omega = \mathbb{D} \subset \mathbb{C}$, and $z_0 = 0$, where $\mathbb{D}$ is the open unit disk on the complex plane. The class of all sense-preserving univalent harmonic mappings of $\mathbb{D}$ with $h(0) = g(0) = h'(0) - 1 = 0$ is denoted by $\mathcal{S_H}$, and its subclass for that $g'(0) = 0$ by $\mathcal{S}^0_{\mathcal{H}}$ (cf. [8]). Fundamental informations about harmonic mappings in the plane can be found in [11]. Note that each f satisfying (1.1) in $\mathbb{D}$ is uniquely determined by coefficients of the following power

2010 *Mathematics Subject Classifcation:* 30C75; 30C45; 30D45
Keywords: Univalent harmonic mappings, convex functions, Bloch constant, normal family

series expansions

$$h(z)=\sum_{n=0}^{\infty}a_nz^n,\quad g(z)=\sum_{n=1}^{\infty}b_nz^n\quad (z\in\mathbb{D}),\tag{1.2}$$

with $a_n\in\mathbb{C}$, $n=0,1,2,...,$ and $b_n\in\mathbb{C}$, $n=1,2,3,....$ When $f\in\mathcal{S}_{\mathcal{H}}$ then $a_0=0, a_1=1$.

In [14] authors studied properties of a subclass $\overline{\mathcal{S}}^{\alpha}_{\mathcal{H}}$ of $\mathcal{S}_{\mathcal{H}}$, consisting of all univalent anti-analytic perturbations of the identity in the unit disc with $|b_1|=\alpha$, and in [15] there was studied the class $\widehat{\mathcal{S}}^{\alpha}$ of all $f\in\mathcal{S}_{\mathcal{H}}$, such that $|b_1|=\alpha\in(0,1)$ and $h\in\mathcal{CV}$, where $\mathcal{CV}$ denotes the well-known family of normalized, univalent functions which are convex.

The classical Schwarz-Pick estimate for an analytic function ω which is bounded by one on the unit disk of the complex plane is the inequality

$$|\omega'(z)|\le\frac{1-|\omega(z)|^2}{1-|z|^2}\quad (|z|<1).\tag{1.3}$$

Ruscheweyh [21] has obtained the best-possible estimates of higher order derivatives of bounded analytic functions on the disk. Similar estimates were derived by other methods and for different classes of analytic functions in one and several variables by Anderson and Rovnyak [1]

$$(1-|z|^2)^{n-1}\left|\frac{\omega^{(n)}(z)}{n!}\right|\le\frac{1-|\omega(z)|^2}{1-|z|^2}\quad (n=1,2,...).\tag{1.4}$$

The case $z=0$ in (1.4) asserts that if

$$\omega(z)=c_0+c_1z+c_2z^2+\cdots,\tag{1.5}$$

then

$$|c_n|\le 1-|c_0|^2,\tag{1.6}$$

for every $n\ge1$. This result is classical and due to Wiener; see [2], [16].

2. Bounds of the Fekete-Szegö and other functionals

Theorem 2.1. *Let* $f\in\widehat{S}^{\alpha}$, $f=h+\bar{g}$ *with the power series* (1.2). *Then*

$$|b_n|\le\alpha+\frac{(1-\alpha^2)(n-1)}{2}\quad (n=2,3,...).\tag{2.1}$$

Proof. Making use of a relation $g'=\omega h'$ and the power series expansions (1.2), (1.5) we obtain

$$nb_n=\sum_{p=0}^{n-1}(p+1)a_{p+1}c_{n-p-1}\quad (n=2,3,...).\tag{2.2}$$

Since $h\in\mathcal{CV}$, then $|a_k|\le1$ $(k=1,2,...)$. Applying this for (2.2) we have

$$|b_n|\ \le\ \frac{1}{n}\sum_{p=0}^{n-1}(p+1)|c_{n-p-1}|.$$

The fact $g' = \omega h'$, for the case $z = 0$, implies that $c_0 = b_1$, so that by (1.6) we obtain $|c_{n-p-1}| \leq 1 - |b_1|^2 = 1 - \alpha^2$. Therefore

$$\begin{aligned} |b_n| &\leq \alpha + \frac{1}{n}\sum_{p=0}^{n-2}(p+1)(1-\alpha^2) \\ &= \alpha + \frac{(1-\alpha^2)(n-1)}{2}. \end{aligned} \tag{2.3}$$

Specially, we get

$$|b_2| \leq \alpha + \frac{1-\alpha^2}{2}, \quad |b_3| \leq 1 + \alpha - \alpha^2.$$

For the case, when $n = 2$ the inequality is sharp, with the equality realized by the function

$$f(z) = \frac{z}{1-z} + \overline{\frac{z}{1-z} - \frac{1-\alpha}{1+\alpha}\log\frac{1+\alpha z}{1-z}}.$$

We note that for α close to 1 the above bounds are better than obtained in [15]. $\square$

In conclusion, we obtain:

Corollary 2.2. *Let $f \in \widehat{S}^{\alpha}$, $f = h + \bar{g}$ with the power series* (1.2). *Then*

$$|b_n| \leq \min\left\{\alpha + \frac{(1-\alpha^2)(n-1)}{2}, \frac{\alpha + \sqrt{(n-\alpha^2)(n-1)}}{n}\right\} \quad (n = 2, 3, \ldots). \tag{2.4}$$

Theorem 2.3. *Let $f \in \widehat{S}^{\alpha}$, $f = h + \bar{g}$ with the power series* (1.2). *Then for $\mu \in \mathbb{R}$*

$$|b_3 - \mu b_2^2| \leq \frac{1-\alpha^2}{3}\left\{1 + \frac{3}{4}|\mu|(1-\alpha^2) + |2 - 3\mu b_1|\right\} + \alpha \max\left\{\frac{1}{3}, |1 - \mu b_1|\right\}, \tag{2.5}$$

and

$$|b_{n+1} - b_n| \leq 2\alpha + (1-\alpha^2)\frac{2n-1}{2}. \tag{2.6}$$

Proof. From the relation (2.2) we have

$$2b_2 = c_1 + 2a_2c_0, \quad 3b_3 = c_2 + 2a_2c_1 + 3a_3c_0.$$

Then

$$\begin{aligned} |b_3 - \mu b_2^2| &= \left|\tfrac{1}{3}c_2 + \tfrac{2}{3}a_2c_1 + a_3c_0 - \mu\left(\tfrac{1}{2}c_1 + a_2c_0\right)^2\right| \\ &= \left|\left(\tfrac{1}{3}c_2 - \tfrac{\mu}{4}c_1^2\right) + a_2c_1\left(\tfrac{2}{3} - \mu c_0\right) + c_0(a_3 - \mu c_0a_2^2)\right| \\ &\leq \tfrac{1}{3}\left|c_2 - \tfrac{3}{4}\mu c_1^2\right| + |a_2||c_1|\left|\tfrac{2}{3} - \mu c_0\right| + \alpha|a_3 - \mu c_0a_2^2|. \end{aligned}$$

Apply now the estimate that holds for the coefficients of convex functions: $|a_n| \leq 1$ $(n = 2, 3, ...)$, $|a_3 - \nu a_2^2| \leq \max\{1/3, |1-\nu|\}$ $(\nu \in \mathbb{R})$, and the relation (1.6). We obtain then

$$\begin{array}{rcl} |b_3 - \mu b_2^2| & \leq & \frac{1}{3}\left[|c_2| + \frac{3}{4}\left|\mu c_1^2\right|\right] + |a_2||c_1|\left|\frac{2}{3} - \mu c_0\right| + \alpha|a_3 - \mu b_1 a_2^2| \\ & \leq & \frac{1-\alpha^2}{3}\left[1 + \frac{3}{4}|\mu|(1-\alpha^2)\right] + (1-\alpha^2)\left|\frac{2}{3} - \mu b_1\right| \\ & + & \alpha \max\{1/3, |1 - \mu b_1|\}. \end{array}$$

Next, by (2.2), we have

$$\begin{array}{rcl} |b_{n+1} - b_n| & = & \left| \frac{1}{n+1}\sum\limits_{p=1}^{n+1} p a_p c_{n+1-p} - \frac{1}{n}\sum\limits_{p=1}^{n} p a_p c_{n-p} \right| \\ & \leq & |a_{n+1}c_0| + |a_n c_0| + \frac{1}{n+1}\sum\limits_{p=1}^{n} p|a_p c_{n+1-p}| + \frac{1}{n}\sum\limits_{p=1}^{n-1} p|a_p c_{n-p}| \\ & \leq & 2\alpha + \frac{1-\alpha^2}{n+1}\sum\limits_{p=1}^{n} p + \frac{1-\alpha^2}{n}\sum\limits_{p=1}^{n-1} p \\ & = & 2\alpha + (1-\alpha^2)\frac{2n-1}{2}. \end{array}$$

The proof is now complete, however the results are not sharp, for example the function that realizes the accuracy of $|b_2|$ in the previous theorem, gives $|b_2 - b_1| = (1-\alpha^2)/2 \leq 2\alpha + (1-\alpha^2)/2$, for any $\alpha \in (0, 1)$. The right hand side is obtained from (2.6) for the case when $n = 1$. □

Theorem 2.4. *For $f \in \widehat{S}^\alpha$, $f = h + \bar{g}$ and $|z| = r < 1$ it holds*

$$\left| \frac{r}{1+r} - \frac{1-\alpha}{1+\alpha}\log\frac{1+r}{1-\alpha r} \right| \leq |g(z)| \leq \frac{r}{1-r} + \frac{1-\alpha}{1+\alpha}\log\frac{1-r}{1+\alpha r}, \tag{2.7}$$

$$\frac{2r}{1+r} - \frac{r(1-\alpha^2)}{|r-\alpha|(1-\alpha r)} \leq \left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{2r}{1-r} + \frac{r(1-\alpha^2)}{|r-\alpha|(1-\alpha r)}, \tag{2.8}$$

$$\mathrm{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > \frac{r(\alpha^2-1)}{|\alpha - r|(1-\alpha r)} + \frac{1-r}{1+r}. \tag{2.9}$$

Proof. Applying the relation $g' = \omega h'$ we estimate $|g'(z)|$ as follows [15]:

$$\frac{|\alpha - r|}{(1-\alpha r)(1+r)^2} \leq |g'(z)| \leq \frac{\alpha + r}{(1+\alpha r)(1-r)^2} \quad (|z| = r < 1).$$

Then integrating along a radial line $\zeta = te^{i\theta}$ the right hand side of (2.7) is obtained immediately [15].

In order to prove the left side of (2.7) we note first that g is univalent. Let $\Gamma = g(\{z : |z| = r\})$ and let $\xi_1 \in \Gamma$ be the nearest point to the origin. By a rotation we may assume that $\xi_1 > 0$. Let γ be the line segment $0 \leq \xi \leq \xi_1$ and suppose that $z_1 = g^{-1}(\xi_1)$ and $L = g^{-1}(\gamma)$. With ζ as the variable of integration on L we have that $d\xi = g'(\zeta)d\zeta >$

0 on L. Hence

$$\xi_1 = \int_0^{\xi_1} d\xi = \int_0^{z_1} g'(\zeta)d\zeta = \int_0^{z_1} |g'(\zeta)||d\zeta| \geq \int_0^{r} |g'(te^{i\theta})|dt$$

$$\geq \int_0^{r} \frac{|\alpha - r|}{(1-\alpha r)(1+r)^2}dr = \left|\frac{r}{1+r} - \frac{1-\alpha}{1+\alpha}\log\frac{1+r}{1-\alpha r}\right|.$$

From the relation $g' = \omega h'$ we obtain

$$\frac{zg''(z)}{g'(z)} = \frac{z\omega'(z)}{\omega(z)} + \frac{zh''(z)}{h'(z)}. \tag{2.10}$$

Since h is convex, so is univalent, then it holds [13, p. 118]

$$\frac{2r}{1+r} \leq \left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{2r}{1-r} \quad (|z| = r). \tag{2.11}$$

Moreover, ω satisfies [12, p. 320]

$$\left|\frac{\omega(z) - \omega(0)}{1 - \overline{\omega(0)}\omega(z)}\right| \leq |z| \quad (|z| = r), \tag{2.12}$$

from which it follows

$$\left|\omega(z) - \frac{\omega(0)(1-r^2)}{1-|\omega(0)|^2r^2}\right| \leq \frac{r(1-|\omega(0)|^2)}{1-|\omega(0)|^2r^2}. \tag{2.13}$$

We note that $|\omega(0)| = |c_0| = |b_1| = \alpha$, so that by (2.13) we have

$$\frac{|r-\alpha|}{1-\alpha r} \leq |\omega(z)| \leq \frac{r+\alpha}{1+\alpha r}. \tag{2.14}$$

Taking into account (2.10), (2.11), (2.14) and the Schwarz-Pick inequality (1.3) we obtain for $|z| = r < 1$

$$\begin{aligned}
\left|\frac{zg''(z)}{g'(z)}\right| &\leq \left|\frac{z\omega'(z)}{\omega(z)}\right| + \left|\frac{zh''(z)}{h'(z)}\right| \\
&\leq \frac{r(1-|\omega(z)|^2)}{|\omega(z)|(1-r^2)} + \frac{2r}{1-r} \\
&\leq \frac{r(1-r^2)(1-\alpha^2)}{(1-r^2)|r-\alpha|(1-\alpha r)} + \frac{2r}{1-r} \\
&= \frac{r(1-\alpha^2)}{|r-\alpha|(1-\alpha r)} + \frac{2r}{1-r}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left|\frac{zg''(z)}{g'(z)}\right| &\geq \left|\frac{zh''(z)}{h'(z)}\right| - \left|\frac{z\omega'(z)}{\omega(z)}\right| \\ &\geq \frac{2r}{1+r} - \frac{r(1-|\omega(z)|^2)}{|\omega(z)|(1-r^2)} \\ &\geq \frac{2r}{1+r} - \frac{r(1-r^2)(1-\alpha^2)}{(1-r^2)|r-\alpha|(1-\alpha r)} \\ &= \frac{2r}{1+r} - \frac{r(1-\alpha^2)}{|r-\alpha|(1-\alpha r)}.\end{aligned}$$

Moreover

$$1+\frac{zg''(z)}{g'(z)} = \frac{z\omega'(z)}{\omega(z)} + 1 + \frac{zh''(z)}{h'(z)} \tag{2.15}$$

and h is convex, therefore

$$\begin{aligned}\operatorname{Re}\left(1+\frac{zg''(z)}{g'(z)}\right) &= \operatorname{Re}\frac{z\omega'(z)}{\omega(z)} + \operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) \\ &> \operatorname{Re}\frac{z\omega'(z)}{\omega(z)} + \frac{1-r}{1+r}.\end{aligned} \tag{2.16}$$

By the above, (1.3) and (2.14), we have

$$\operatorname{Re}\left(1+\frac{zg''(z)}{g'(z)}\right) > \frac{r(\alpha^2-1)}{|\alpha-r|(1-\alpha r)} + \frac{1-r}{1+r}, \tag{2.17}$$

as asserted. □

3. Estimates of the Bloch constant

A harmonic function f is called *the Bloch function* if

$$\mathfrak{B}_f = \sup_{z,w\in\mathbb{D}, z\neq w} \frac{|f(z)-f(w)|}{\varrho(z,w)} < \infty, \tag{3.1}$$

where

$$\varrho(z,w) = \frac{1}{2}\log\left(\frac{1+\left|\frac{z-w}{1-\bar{z}w}\right|}{1-\left|\frac{z-w}{1-\bar{z}w}\right|}\right) = \operatorname{artanh}\left|\frac{z-w}{1-\bar{z}w}\right|$$

denotes the hyperbolic distance in $\mathbb{D}$, and $\mathfrak{B}_f$ is called *the Bloch constant of f*. The harmonic Bloch constant was studied by Colonna [9]. Colonna established that the Bloch constant $\mathfrak{B}_f$ of a harmonic mapping $f = h + \bar{g}$ can be expressed in terms of moduli of the derivatives of h and g

$$\begin{aligned}\mathfrak{B}_f &= \sup_{z\in\mathbb{D}}(1-|z|^2)\left(|h'(z)|+|g'(z)|\right) \\ &= \sup_{z\in\mathbb{D}}(1-|z|^2)|h'(z)|(1+|\omega(z)|),\end{aligned} \tag{3.2}$$

which agrees with the well known notion of the Bloch constant for analytic functions. Moreover, the function f is Bloch if and only if h and g are, and

$$\max(\mathfrak{B}_h, \mathfrak{B}_g) \leq \mathfrak{B}_f \leq \mathfrak{B}_h + \mathfrak{B}_g.$$

Colonna also obtained the best possible estimate of the Bloch constant for the family of harmonic mappings of $\mathbb{D}$ into itself. Recently, the Bloch constant was studied by many authors, see, for example [3, 4, 19]. Very interesting results in this direction were obtained in [5, 6, 7, 18, 20, 22]. Our aim is to determine bounds for the Bloch constant in the class $\overline{S}^{\alpha}$ and $\widehat{S}^{\alpha}$.

Theorem 3.1. *Let $f = h + \bar{g}$ with $h(z) = z/(1 - Bz), -1 < B < 1$, and let $|B| = A,\ 0 \le A < 1$. Then the Bloch constant $\mathfrak{B}_f$ is bounded by*

$$\mathfrak{B}_f \le (1+\alpha)\frac{(1+r_0)^3(1-r_0)^2}{(1-Ar_0)^2(1+\alpha r_0)}, \tag{3.3}$$

where r_0 is given by

$$r_0 = \frac{\alpha(1+3A) - 3 - A + \sqrt{(1+\alpha)(1+A)(9 - 7A + \alpha(-7+9A))}}{4\alpha + 2A(\alpha - 1)}. \tag{3.4}$$

Proof. Applying the distortion theorem

$$|h'(z)| \le \frac{1}{(1-Ar)^2} \quad (|z| = r),$$

and (3.2), we find

$$\mathfrak{B}_f = \sup_{z\in\mathbb{D}}(1-|z|^2)|h'(z)|(1+|\omega(z)|) \le (1+\alpha) \sup_{0\le r<1} \frac{(1+r)(1-r^2)}{(1-Ar)^2(1+\alpha r)}.$$

Setting

$$q(r) = \frac{(1+r)(1-r^2)}{(1-Ar)^2(1+\alpha r)},$$

we observe that $q'(r) = 0$, if and only if

$$(1+r)[(2\alpha + \alpha A - A)r^2 + (3 + A - \alpha - 3\alpha A)r + \alpha - 1 - 2A] = 0.$$

The last equation has solution in the interval $(0,1)$ at the point r_0 given by (3.4), and the function q attains its maximum at r_0. □

Setting $B = 0$ in the above theorem, we obtain the estimate of $\mathfrak{B}_f$ in the class $\bar{S}^{\alpha}$, below.

Corollary 3.2. *For $f \in \bar{S}^{\alpha}$, $f = h + \bar{g}$, the Bloch constant $\mathfrak{B}_f$ is bounded by*

$$\mathfrak{B}_f \le (1+\alpha)\frac{(1+r_0)^3(1-r_0)^2}{(1+\alpha r_0)}, \tag{3.5}$$

where r_0 is given by

$$r_0 = \frac{\alpha - 3 + \sqrt{9 + 2\alpha - 7\alpha^2}}{4\alpha}. \tag{3.6}$$

Remark. By the fact that the Bloch constant is finite we already have that the family of harmonic mappings with $h(z) \equiv z$, and $|b_1| = \alpha$, is a normal family. A function f is *normal*, if the constant σ_f is finite, where

$$\sigma_f = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|f'(z)|}{1+|f(z)|},$$

see [10]. Indeed, since the quantity $|f(z)|$ in $\overline{\mathcal{S}}^{\alpha}$ is bounded [15]:

$$|f(z)| \geq \begin{cases} \left(1-\frac{1}{\alpha}\right) r - \left(1-\frac{1}{\alpha^2}\right)\log(1+\alpha r) & for \quad \alpha \neq 0, \\ r - \frac{r^2}{2} & for \quad \alpha = 0, \end{cases} \tag{3.7}$$

therefore, by (3.5) and (3.7), we obtain

$$\sigma_f \leq \begin{cases} (1+\alpha)\frac{(1+r)(1-r^2)}{1+\alpha r}\frac{1}{1+\left(1-\frac{1}{\alpha}\right)r-\left(1-\frac{1}{\alpha^2}\right)\log(1+\alpha r)} & for \quad \alpha \neq 0, \\ (1+\alpha)\frac{(1+r)(1-r^2)}{1+\alpha r}\frac{1}{1+r-\frac{r^2}{2}} & for \quad \alpha = 0, \end{cases} \tag{3.8}$$

where $r = r_0$ is given by (3.6), and we see that in both cases σ_f are finite.

Remark. The univalent Bloch functions can be described in terms of geometry of their images; they are precisely those functions whose images do not contain disks of arbitrarily large radius [10]. Therefore, we suppose that the functions from the class $\widehat{S}^{\alpha}$ may not be the Bloch functions. Indeed, reasoning similarly as in the Theorem 3.1 we note that in the class $\widehat{S}^{\alpha}$ we have $h(z) = z/(1-z)$, then $|h'(z)| \leq 1/(1-r)^2$. Thus

$$\mathfrak{B}_f = (1+\alpha) \sup_{0\leq r<1} \frac{(1+r)^2}{(1-r)(1+\alpha r)},$$

and the function $p(r) = (1+r)^2/[(1-r)(1+\alpha r)]$ increases in the whole interval $(0,1)$, with infinity as the supremum.

Acknowledgements

This work was partially supported by the Centre for Innovation and Transfer of Natural Sciences and Engineering Knowledge.

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