On the universal $\alpha$-central extension of the semi-direct product of Hom-Leibniz algebras

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Abstract We introduce Hom-actions, semidirect product and establish the equivalence between split extensions and the semi-direct product extension of Hom-Leibniz algebras. We analyze the functorial properties of the universal ($\alpha$)-central extensions of ($\alpha$)-perfect Hom-Leibniz algebras. We establish under what conditions an automorphism or a derivation can be lifted in an $\alpha$-cover and we analyze the universal $\alpha$-central extension of the semi-direct product of two $\alpha$-perfect Hom-Leibniz algebras.

Key words: universal ($\alpha$)-central extension, Hom-action, semi-direct product, derivation.

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1 Introduction

Hom-Lie algebras were introduced in [11] as Lie algebras whose Jacobi identity is twisted by means of a map. This fact occurs in different applications in models of quantum phenomena or in analysis of complex systems and processes exhibiting complete or partial scaling invariance.

From the introductory paper, the investigation of several kinds of Hom-structures is in progress (for instance, see [1, 2, 3, 4, 15, 17, 21, 22] and references given therein). Naturally, the non-skew-symmetric version of Hom-Lie algebras, the so called Hom-Leibniz algebras, was considered as well (see [2, 7, 9, 12, 16, 17, 19]). A Hom-Leibniz algebra is a triple $(L, [\cdot, \cdot], \alpha_L)$ consisting of a $\mathbb{K}$-vector space $L$, a bilinear map $[\cdot, \cdot] : L \times L \to L$ and a homomorphism of $\mathbb{K}$-vector spaces $\alpha_L : L \to L$ satisfying the Hom-Leibniz identity:

$$[[\alpha_L(x), y, z]] = [[x, y], \alpha_L(z)] - [[x, z], \alpha_L(y)]$$
for all $x, y, z \in L$. When $\alpha_L = Id$, the definition of Leibniz algebra [13] is recovered. If the bracket is skew-symmetric, then we recover the definition of Hom-Lie algebra [11].

Lie and Leibniz algebras have found important applications in Mathematics and Physics, in particular degenerations, contractions and deformations (see [8, 18] and references given therein). The analysis of these properties in the Hom-Lie setting [10] have led to deal with universal central extensions.

The goal of the present paper is to continue with the investigations on universal $(\alpha)$-central extensions of $(\alpha)$-perfect Hom-Leibniz algebras initiated in [7]. In concrete, we consider the extension of results about the universal central extension of the semi-direct product of Leibniz algebra in [6] to the framework of Hom-Leibniz algebras.

To do so, we organize the paper as follows: an initial section recalling the background material on Hom-Leibniz algebras. We introduce the concepts of Hom-action and semi-direct product and we prove a new result (Lemma 2.11) that establishes the equivalence between split extensions and the semi-direct product extension. Section 3 is devoted to analyze the functorial properties of the universal $(\alpha)$-central extensions of $(\alpha)$-perfect Hom-Leibniz algebras. In section 4 we establish under what conditions an automorphism or a derivation can be lifted in an $\alpha$-cover (a central extension $f : (L', \alpha_{L'}) \to (L, \alpha_L)$ where $(L', \alpha_{L'})$ is $\alpha$-perfect ($L' = [\alpha_{L'}(L'), \alpha_{L'}(L')]$). Final section is devoted to analyze the relationships between the universal $\alpha$-central extension of the semi-direct product of two $\alpha$-perfect Hom-Leibniz algebras, such that one of them Hom-acts over the other one, and the semi-direct product of the universal $\alpha$-central extensions of both of them.

2 Preliminaries on Hom-Leibniz algebras

In this section we introduce necessary material on Hom-Leibniz algebras which will be used in subsequent sections.

**Definition 2.1** [17] A Hom-Leibniz algebra is a triple $(L, [-, -], \alpha_L)$ consisting of a $K$-vector space $L$, a bilinear map $[-, -] : L \times L \to L$ and a $K$-linear map $\alpha_L : L \to L$ satisfying:

$$[\alpha_L(x), [y, z]] = [[x, y], \alpha_L(z)] - [[x, z], \alpha_L(y)]$$ (Hom - Leibniz identity) \hspace{1cm} (1)

for all $x, y, z \in L$.

A Hom-Leibniz algebra $(L, [-, -], \alpha_L)$ is said to be multiplicative [19] if the $K$-linear map $\alpha_L$ preserves the bracket, that is, if $\alpha_L[x, y] = [\alpha_L(x), \alpha_L(y)]$, for all $x, y \in L$.

**Example 2.2**
a) Taking $\alpha = \text{Id}$ in Definition 2.1 we obtain the definition of Leibniz algebra [13]. Hence Hom-Leibniz algebras include Leibniz algebras as a full subcategory, thereby motivating the name "Hom-Leibniz algebras" as a deformation of Leibniz algebras twisted by a homomorphism. Moreover it is a multiplicative Hom-Leibniz algebra.

b) Hom-Lie algebras [11] are Hom-Leibniz algebras whose bracket satisfies the condition $[x, x] = 0$, for all $x$. So Hom-Lie algebras can be considered as a full subcategory of Hom-Leibniz algebras category. For any multiplicative Hom-Leibniz algebra $(L, [-, -], \alpha_L)$ it is associated the Hom-Lie algebra $(L_{\text{Lie}}, [-, -], \tilde{\alpha})$, where $L_{\text{Lie}} = L/L^{\text{ann}}$, the bracket is the canonical bracket induced on the quotient and $\tilde{\alpha}$ is the homomorphism naturally induced by $\alpha$. Here $L^{\text{ann}} = \langle \{ [x, x] : x \in L \} \rangle$.

c) Let $(D, \dashv, \triangleright, \alpha_D)$ be a Hom-dialgebra. Then $(D, \dashv, \triangleright, \alpha_D)$ is a Hom-Leibniz algebra with respect to the bracket $[x, y] = x \dashv y - y \triangleright x$, for all $x, y \in A$ [20].

d) Let $(L, [-, -])$ be a Leibniz algebra and $\alpha_L : L \to L$ a Leibniz algebra endomorphism. Define $[-, -]_\alpha : L \otimes L \to L$ by $[x, y]_\alpha = [\alpha(x), \alpha(y)]$, for all $x, y \in L$. Then $(L, [-, -], \alpha_L)$ is a multiplicative Hom-Leibniz algebra.

e) Abelian or commutative Hom-Leibniz algebras are $K$-vector spaces $L$ with trivial bracket and any linear map $\alpha_L : L \to L$.

Definition 2.3 A homomorphism of Hom-Leibniz algebras $f : (L, [-, -], \alpha_L) \to (L', [-, -], \alpha_{L'})$ is a $K$-linear map $f : L \to L'$ such that

a) $f([x, y]) = [f(x), f(y)]'$,

b) $f \circ \alpha_L(x) = \alpha_{L'} \circ f(x),$

for all $x, y \in L$.

A homomorphism of multiplicative Hom-Leibniz algebras is a homomorphism of the underlying Hom-Leibniz algebras.

In the sequel we refer to Hom-Leibniz algebra as a multiplicative Hom-Leibniz algebra and we shall use the shortened notation $(L, \alpha_L)$ when there is not confusion with the bracket operation.

Definition 2.4 Let $(L, [-, -], \alpha_L)$ be a Hom-Leibniz algebra. A Hom-Leibniz subalgebra $(H, \alpha_H)$ is a linear subspace $H$ of $L$, which is closed for the bracket and invariant by $\alpha_L$, that is,

a) $[x, y] \in H$, for all $x, y \in H$,
b) $\alpha_L(x) \in H$, for all $x \in H$ ($\alpha_H = \alpha_L$).

A Hom-Leibniz subalgebra $(H, \alpha_H)$ of $(L, \alpha_L)$ is said to be a two-sided Hom-ideal if $[x, y], [y, x] \in H$, for all $x \in H, y \in L$.

If $(H, \alpha_H)$ is a two-sided Hom-ideal of $(L, \alpha_L)$, then the quotient $L/H$ naturally inherits a structure of Hom-Leibniz algebra with respect to the endomorphism $\bar{\alpha} : L/H \to L/H, \bar{\alpha}(l) = \alpha_L(l)$, which is said to be the quotient Hom-Leibniz algebra.

So we have defined the category $\text{Hom-Leib}$ (respectively, $\text{Hom-Leib}_{\text{mult}}$) whose objects are Hom-Leibniz (respectively, multiplicative Hom-Leibniz) algebras and whose morphisms are the homomorphisms of Hom-Leibniz (respectively, multiplicative Hom-Leibniz) algebras. There is an obvious inclusion functor $\text{inc} : \text{Hom-Leib}_{\text{mult}} \to \text{Hom-Leib}$. This functor has as left adjoint the multiplicative functor $(-)_{\text{mult}} : \text{Hom-Leib} \to \text{Hom-Leib}_{\text{mult}}$ which assigns to a Hom-Leibniz algebra $(L, [-, -], \alpha_L)$ the multiplicative Hom-Leibniz algebra $(L/I, [-, -], \tilde{\alpha})$, where $I$ is the two-sided ideal of $L$ spanned by the elements $\alpha_L(x, y) - [\alpha_L(x), \alpha_L(y)]$, for all $x, y \in L$.

**Definition 2.5** Let $(H, \alpha_H)$ and $(K, \alpha_K)$ be two-sided Hom-ideals of a Hom-Leibniz algebra $(L, [-, -], \alpha_L)$. The commutator of $(H, \alpha_H)$ and $(K, \alpha_K)$, denoted by $([H, K], \alpha_{[H,K]})$, is the Hom-Leibniz subalgebra of $(L, \alpha_L)$ spanned by the brackets $[h, k], h \in H, k \in K$.

Obviously, $[H, K] \subseteq H \cap K$ and $[K, H] \subseteq H \cap K$. When $H = K = L$, we obtain the definition of derived Hom-Leibniz subalgebra. Let us observe that, in general, $([H, K], \alpha_{[H,K]})$ is not a Hom-ideal, but if $H, K \subseteq \alpha_L(L)$, then $([H, K], \alpha_{[H,K]})$ is a two-sided ideal of $(\alpha_L(L), \alpha_L)$.

**Definition 2.6** Let $(L, [-, -], \alpha_L)$ be a Hom-Leibniz algebra. The subspace $Z(L) = \{x \in L \mid [x, y] = 0 = [y, x], \text{for all } y \in L\}$ is said to be the center of $(L, [-, -], \alpha_L)$.

When $\alpha_L : L \to L$ is a surjective homomorphism, then $Z(L)$ is a Hom-ideal of $L$.

**2.1 Hom-Leibniz actions**

**Definition 2.7** Let $(L, \alpha_L)$ and $(M, \alpha_M)$ be Hom-Leibniz algebras. A (right) Hom-action of $(L, \alpha_L)$ over $(M, \alpha_M)$ consists of two bilinear maps, $\lambda : L \otimes M \to M$, $\lambda(l \otimes m) = l \cdot m$, and $\rho : M \otimes L \to M$, $\rho(m \otimes l) = m \cdot l$, satisfying the following identities:

a) $\alpha_M (m) \cdot [x, y] = (m \cdot x) \cdot \alpha_L (y) - (m \cdot y) \cdot \alpha_L (x)$;

b) $\alpha_L (x) \cdot (m \cdot y) = (x \cdot m) \cdot \alpha_L (y) - [x, y] \cdot \alpha_M (m)$;
c) $\alpha_L(x) \cdot (y \cdot m) = [x, y] \cdot \alpha_M(m) - (x \cdot m) \cdot \alpha_L(y)$;

d) $\alpha_L(x) \cdot [m, m'] = [x \cdot m, \alpha_M(m')] - [x \cdot m', \alpha_M(m)]$;

e) $[\alpha_M(m), m' \cdot x] = [m, m'] \cdot \alpha_L(x) - [m \cdot x, \alpha_M(m')]$;

f) $[\alpha_M(m), x \cdot m'] = [m, x, \alpha_M(m')] - [m, m'] \cdot \alpha_L(x)$;

g) $\alpha_M(x \cdot m) = \alpha_L(x) \cdot \alpha_M(m)$;

h) $\alpha_M(m \cdot x) = \alpha_M(m) \cdot \alpha_L(x)$;

for all $x, y \in L$ and $m, m' \in M$.

When $(M, \alpha_M)$ is an abelian Hom-Leibniz algebra, that is the bracket on $M$ is trivial, then the Hom-action is called Hom-representation.

**Example 2.8**

a) Let $M$ be a representation of a Leibniz algebra $L$ [14]. Then $(M, \text{Id}_M)$ is a Hom-representation of the Hom-Leibniz algebra $(L, \text{Id}_L)$.

b) Let $(K, \alpha_K)$ be a Hom-Leibniz subalgebra of a Hom-Leibniz algebra $(L, \alpha_L)$ (even $(K, \alpha_K) = (L, \alpha_L)$) and $(H, \alpha_H)$ a two-sided Hom-ideal of $(L, \alpha_L)$. There exists a Hom-action of $(K, \alpha_K)$ over $(H, \alpha_H)$ given by the bracket in $(L, \alpha_L)$.

c) An abelian sequence of Hom-Leibniz algebras is an exact sequence of Hom-Leibniz algebras $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$, where $(M, \alpha_M)$ is an abelian Hom-Leibniz algebra, $\alpha_K \circ i = i \circ \alpha_M$ and $\pi \circ \alpha_K = \alpha_L \circ \pi$.

An abelian sequence induces a Hom-representation structure of $(L, \alpha_L)$ over $(M, \alpha_M)$ by means of the actions given by $\lambda : L \otimes M \to M, \lambda(l, m) = [k, m], \pi(k) = l$ and $\rho : M \otimes L \to M, \rho(m, l) = [m, k], \pi(k) = l$.

**Definition 2.9** Let $(M, \alpha_M)$ and $(L, \alpha_L)$ be Hom-Leibniz algebras together with a Hom-action of $(L, \alpha_L)$ over $(M, \alpha_M)$. Its semi-direct product $(M \rtimes L, \tilde{\alpha})$ is the Hom-Leibniz algebra with underlying $\mathbb{K}$-vector space $M \oplus L$, endomorphism $\tilde{\alpha} : M \rtimes L \to M \rtimes L$ given by $\tilde{\alpha}(m, l) = (\alpha_M(m), \alpha_L(l))$ and bracket

$$[(m_1, l_1), (m_2, l_2)] = ([m_1, m_2] + \alpha_L(l_1) \cdot m_2 + m_1 \cdot \alpha_L(l_2), [l_1, l_2]).$$

Let $(M, \alpha_M)$ and $(L, \alpha_L)$ be Hom-Leibniz algebras with a Hom-action of $(L, \alpha_L)$ over $(M, \alpha_M)$, then we can construct the sequence

$$0 \to (M, \alpha_M) \xrightarrow{i} (M \rtimes L, \tilde{\alpha}) \xrightarrow{\pi} (L, \alpha_L) \to 0 \quad (2)$$

where $i : M \to M \rtimes L, i(m) = (m, 0)$, and $\pi : M \rtimes L \to L, \pi(m, l) = l$. Moreover, this sequence splits by $\sigma : L \to M \rtimes L, \sigma(l) = (0, l)$, that is, $\sigma$ satisfies $\pi \circ \sigma = \text{Id}_L$ and $\tilde{\alpha} \circ \sigma = \sigma \circ \alpha_L$. 

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Definition 2.10 Let \((M, \alpha_M)\) and \((L, \alpha_L)\) be Hom-Leibniz algebras such that there is a Hom-action of \((L, \alpha_L)\) over \((M, \alpha_M)\). Two extensions of \((L, \alpha_L)\) by \((M, \alpha_M), \ 0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0\) and \(0 \to (M, \alpha_M) \xrightarrow{i'} (K', \alpha_{K'}) \xrightarrow{\pi'} (L, \alpha_L) \to 0\), are said to be equivalent if there exists a homomorphism of Hom-Leibniz algebras \(\varphi : (K, \alpha_K) \to (K', \alpha_{K'})\) making the following diagram commutative.

\[
\begin{array}{ccc}
0 & \to & (M, \alpha_M) \\
\downarrow & & \downarrow \varphi \\
0 & \to & (M, \alpha_M) \\
& \xrightarrow{i'} & (K', \alpha_{K'}) \\
\downarrow & & \downarrow \pi' \\
& \to & (L, \alpha_L) \\
\end{array}
\]

Lemma 2.11 Let \((C, \text{Id}_C)\) and \((A, \alpha_A)\) be Hom-Leibniz algebras together with a Hom-action of \((C, \text{Id}_C)\) over \((A, \alpha_A)\).

A sequence of Hom-Leibniz algebras \(0 \to (A, \alpha_A) \xrightarrow{i} (B, \alpha_B) \xrightarrow{\pi} (C, \text{Id}_C) \to 0\) is split if and only if it is equivalent to the semi-direct sequence \(0 \to (A, \alpha_A) \xrightarrow{i} (A \ltimes C, \tilde{\alpha}) \xrightarrow{p} (C, \text{Id}_C) \to 0\).

Proof. If \(0 \to (A, \alpha_A) \xrightarrow{i} (B, \alpha_B) \xrightarrow{\pi} (C, \text{Id}_C) \to 0\) is split by \(s : (C, \text{Id}_C) \to (B, \alpha_B)\), then the Hom-action of \((C, \text{Id}_C)\) over \((A, \alpha_A)\) is given by

\[
c \cdot a = [s(c), i(a)]; \quad a \cdot c = [i(a), s(c)]
\]

With this Hom-action of \((C, \text{Id}_C)\) over \((A, \alpha_A)\) we can construct the following split extension:

\[
\begin{array}{ccc}
0 & \to & (A, \alpha_A) \\
\downarrow & & \downarrow \varphi \\
0 & \to & (A \ltimes C, \tilde{\alpha}) \\
& \xrightarrow{p} & (C, \text{Id}_C) \\
\downarrow & & \downarrow \pi \\
& \to & (C, \text{Id}_C) \\
\end{array}
\]

where \(j : A \to A \ltimes C, \ j(a) = (a, 0), p : A \ltimes C \to C, p(a, c) = c\) and \(\sigma : C \to A \ltimes C, \ \sigma(c) = (0, c)\). Moreover the Hom-action of \((C, \text{Id}_C)\) over \((A, \alpha_A)\) induced by this extension coincides with the initial one:

\[
c \ast a = [\sigma(c), j(a)] = [(0, c), (a, 0)] = ([0, a] + Id_C(c) \cdot a + 0 \cdot 0, [c, 0]) = (c \ast a, 0) \equiv c \ast a
\]

Finally, both extensions are equivalent since the homomorphism of Hom-Leibniz algebras \(\varphi : (A \ltimes C, \tilde{\alpha}) \to (B, \alpha_B), \ \varphi(a, c) = i(a) + s(c),\) makes commutative the following diagram:

\[
\begin{array}{ccc}
0 & \to & (A, \alpha_A) \\
\downarrow & & \downarrow \varphi \\
0 & \to & (B, \alpha_B) \\
& \xrightarrow{\pi} & (C, \text{Id}_C) \\
\end{array}
\]
For the converse, if both extensions are equivalent, i.e. there exists a homomorphism of Hom-Leibniz algebras \( \varphi : (A \rtimes C, \tilde{\alpha}) \to (B, \alpha_B) \) making commutative diagram (3), then \( s : (C, \text{Id}_C) \to (B, \alpha_B) \) given by \( s(c) = \varphi(0, c) \), is a homomorphism that splits the extension. \( \square \)

**Definition 2.12** Let \((M, \alpha_M)\) be a Hom-representation of a Hom-Leibniz algebra \((L, \alpha_L)\). A derivation of \((L, \alpha_L)\) over \((M, \alpha_M)\) is a \(\mathbb{K}\)-linear map \(d : L \to M\) satisfying:

a) \(d[l_1, l_2] = \alpha_L(l_1) \cdot d(l_2) + d(l_1) \cdot \alpha_L(l_2)\)

b) \(d \circ \alpha_L = \alpha_M \circ d\)

for all \(l_1, l_2 \in L\).

**Example 2.13**

a) The \(\mathbb{K}\)-linear map \(\theta : M \rtimes L \to M, \theta(m, l) = m\), is a derivation, where \((M, \alpha_M)\) is a Hom-representation of \((M \rtimes L, \tilde{\alpha})\) via \(\pi\).

b) When \((M, \alpha_M) = (L, \alpha_L)\) is considered as a representation following Example 2.8 b), then a derivation consists of a \(\mathbb{K}\)-linear map \(d : L \to L\) such that \(d[l_1, l_2] = [\alpha_L(l_1), d(l_2)] + [d(l_1), \alpha_L(l_2)]\) and \(d \circ \alpha_L = \alpha_L \circ d\).

**Proposition 2.14** Let \((M, \alpha_M)\) be a Hom-representation of a Hom-Leibniz algebra \((L, \alpha_L)\). For every homomorphism of Hom-Leibniz algebras \(f : (X, \alpha_X) \to (L, \alpha_L)\) and every \(f\)-derivation \(d : (X, \alpha_X) \to (M, \alpha_M)\) there exists a unique homomorphism of Hom-Leibniz algebras \(h : (X, \alpha_X) \to (M \rtimes L, \tilde{\alpha})\), such that the following diagram is commutative

\[
\begin{array}{ccc}
(X, \alpha_X) & \xrightarrow{\delta} & (M \rtimes L, \tilde{\alpha}) \\
\downarrow & & \downarrow \pi \\
(M, \alpha_M) & \xrightarrow{\theta} & (L, \alpha_L) \\
\end{array}
\]

Conversely, every homomorphism of Hom-Leibniz algebras \(h : (X, \alpha_X) \to (M \rtimes L, \tilde{\alpha})\), determines a homomorphism of Hom-Leibniz algebras \(f = \pi \circ h : (X, \alpha_X) \to (L, \alpha_L)\) and any \(f\)-derivation \(d = \theta \circ h : (X, \alpha_X) \to (M, \alpha_M)\).

**Proof.** The homomorphism \(h : X \to M \rtimes L, h(x) = (d(x), f(x))\) satisfies all the conditions. \( \square \)

**Corollary 2.15** The set of all derivations from \((L, \alpha_L)\) to \((M, \alpha_M)\) is in one-to-one correspondence with the set of Hom-Leibniz algebra homomorphisms \(h : (L, \alpha_L) \to (M \rtimes L, \tilde{\alpha})\) such that \(\pi \circ h = \text{Id}_L\).
3 Functorial properties

In this section we analyze functorial properties of the universal $(\alpha)$-central extensions of $(\alpha)$-perfect Hom-Leibniz algebras. For detailed motivation, constructions and characterizations we refer to [7].

Definition 3.1 A short exact sequence of Hom-Leibniz algebras $(K) : 0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is said to be central if $[M, K] = 0 = [K, M]$. Equivalently, $M \subseteq Z(K)$.

We say that $(K)$ is $\alpha$-central if $[\alpha_M(M), K] = 0 = [K, \alpha_M(M)]$. Equivalently, $\alpha_M(M) \subseteq Z(K)$.

A central extension $(K) : 0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is said to be universal if for every central extension $(K') : 0 \to (M', \alpha_M') \xrightarrow{i'} (K', \alpha_K') \xrightarrow{\pi'} (L, \alpha_L) \to 0$ there exists a unique homomorphism of Hom-Leibniz algebras $h : (K, \alpha_K) \to (K', \alpha_K')$ such that $\pi' \circ h = \pi$.

We say that the central extension $(K) : 0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ is universal $\alpha$-central if for every central $\alpha$-extension $(K) : 0 \to (M', \alpha_M') \xrightarrow{i'} (K', \alpha_K') \xrightarrow{\pi'} (L, \alpha_L) \to 0$ there exists a unique homomorphism of Hom-Leibniz algebras $h : (K, \alpha_K) \to (K', \alpha_K')$ such that $\pi' \circ h = \pi$.

Remark 3.2 Obviously, every universal $\alpha$-central extension is a universal central extension. Note that in the case $\alpha_M = \text{Id}_M$, both notions coincide.

A perfect $(L = [L, L])$ Hom-Leibniz algebra $(L, \alpha_L)$ admits universal central extension, which is $(\text{uce}(L), \tilde{\alpha})$, where $\text{uce}(L) = \frac{L \otimes L}{I_L}$ and $I_L$ is the subspace of $L \otimes L$ spanned by the elements of the form $-[x_1, x_2] \otimes \alpha_L(x_3) + [x_1, x_3] \otimes \alpha_L(x_2) + \alpha_L(x_1) \otimes [x_2, x_3], x_1, x_2, x_3 \in L$; every class $x_1 \otimes x_2 + I_L$ is denoted by $\{x_1, x_2\}$, for all $x_1, x_2 \in L$. $\text{uce}(L)$ is endowed with a structure of Hom-Leibniz algebra with respect to the bracket $\{x_1, x_2\} = \{y_1, y_2\}$ and the endomorphism $\tilde{\alpha} : \text{uce}(L) \to \text{uce}(L)$ defined by $\tilde{\alpha}(\{x_1, x_2\}) = \{\alpha_L(x_1), \alpha_L(x_2)\}$. By construction, $u_L : (\text{uce}(L), \tilde{\alpha}) \to (L, \alpha_L)$, given by $u_L\{x_1, x_2\} = \{x_1, x_2\}$, gives rise to the universal central extension $0 \to (\text{HLeib}_2(L, \tilde{\alpha})) \to (\text{uce}(L), \tilde{\alpha}) \xrightarrow{\tilde{\text{uce}}(L)} (L, \alpha_L) \to 0$.

A Hom-Leibniz algebra $(L, \alpha_L)$ is said to be $\alpha$-perfect if $L = [\alpha_L(L), \alpha_L(L)]$. Theorem 5.5 in [7] shows that a Hom-Leibniz algebra $(L, \alpha_L)$ is $\alpha$-perfect if and only if it admits a universal $\alpha$-central extension, which is $(\text{uce}_{\text{Leib}}^\alpha(L), \overline{\alpha})$, where $\text{uce}_{\text{Leib}}^\alpha(L) = \frac{\alpha_L(L) \otimes \alpha_L(L)}{I_L}$ and $I_L$ is the vector subspace spanned by the elements of the form $-[x_1, x_2] \otimes \alpha_L(x_3) + [x_1, x_3] \otimes \alpha_L(x_2), x_1, x_2, x_3 \in L$. We denote by $\{\alpha_L(x_1), \alpha_L(x_2)\}$ the equivalence class of $\alpha_L(x_1) \otimes \alpha_L(x_2) + I_L$. $\text{uce}_{\text{Leib}}^\alpha(L)$ is endowed with a structure of Hom-Leibniz algebra with respect to the bracket $\{\alpha_L(x_1), \alpha_L(x_2)\} = \{\alpha_L(y_1), \alpha_L(y_2)\}$ and the endomorphism $\overline{\alpha} : \text{uce}_{\text{Leib}}^\alpha(L) \to \text{uce}_{\text{Leib}}^\alpha(L)$ defined by $\overline{\alpha}(\{\alpha_L(x_1), \alpha_L(x_2)\})$.
\[ \{\alpha_2^L(x_1), \alpha_2^L(x_2)\}. \] The homomorphism of Hom-Leibniz algebras \( U_\alpha : ucL_{\text{Leib}}(L) \to L \) given by \( U_\alpha(\{\alpha_L(x_1), \alpha_L(x_2)\}) = [\alpha_L(x_1), \alpha_L(x_2)] \) gives rise to the universal \( \alpha \)-central extension \( 0 \to (\text{Ker}(U_\alpha), \alpha_1) \to (ucL_{\text{Leib}}(L), \alpha) \to (L, \alpha_L) \to 0. \) See [7] for details.

**Definition 3.3** A perfect Hom-Leibniz algebra \((L, \alpha_L)\) is said to be centrally closed if its universal central extension is

\[ 0 \to 0 \to (L, \alpha_L) \tilde{\to} (L, \alpha_L) \to 0 \]

i.e. \( HL_2^0(L) = 0 \) and \((ucL_{\text{Leib}}(L), \tilde{\alpha}) \cong (L, \alpha_L).\)

A Hom-Leibniz algebra \((L, \alpha_L)\) is said to be superperfect if \( HL_1^0(L) = HL_2^0(L) = 0.\)

**Corollary 3.4** If \( 0 \to (\text{Ker}(U_\alpha), \alpha_K) \to (K, \alpha_K) \overset{U_\alpha}{\to} (L, \alpha_L) \to 0 \) is the universal \( \alpha \)-central extension of an \( \alpha \)-perfect Hom-Leibniz algebra \((L, \alpha_L)\), then \((K, \alpha_K)\) is centrally closed.

**Proof.** By Corollary 4.12 a) in [7], \( HL_2^0(K) = 0. \)

\( HL_1^0(K) = 0 \) if and only if \((K, \alpha_K)\) is perfect. By Theorem 4.11 c) in [7] it admits a universal central extension \( 0 \to (HL_2^0(K), \tilde{\alpha}_1) \to (ucL(K), \tilde{\alpha}) \overset{\sim}{\to} (K, \alpha_K) \to 0. \) Since \( HL_2^0(K) = 0, \) then \( u \) is an isomorphism.

**Lemma 3.5** Let \( \pi : (K, \alpha_K) \to (L, \alpha_L) \) be a central extension where \((L, \alpha_L)\) is a perfect Hom-Leibniz algebra. Then the following statements hold:

a) \( K = [K, K] + \text{Ker}(\pi) \) and \( \bar{\pi} : ([K, K], \alpha_{[K,K]}) \to (L, \alpha_L) \) is an epimorphism where \(([K, K], \alpha_{[K,K]}\) is a perfect Hom-Leibniz algebra.

b) \( \pi(Z(K)) \subseteq Z(L) \) and \( \alpha_L(Z(L)) \subseteq \pi(Z(K)). \)

**Proof.**
a) It suffices to consider the following commutative diagram:

\[
\begin{array}{ccc}
(Ker(\pi) \cap [K, K], \alpha_{Ker(\pi)\cap[K,K]}) & \xrightarrow{\pi} & ([K, K], \alpha_{[K,K]}) \\
\downarrow & & \downarrow \\
(Ker(\pi), \alpha_{K}) & \xrightarrow{\bar{\pi}} & (K, \alpha_{K}) \\
\downarrow & & \downarrow \\
* & \xrightarrow{\pi} & (L, \alpha_{L}) \\
\downarrow & & \downarrow \\
(K/[K, K], \alpha_{K}) & \xrightarrow{U_\alpha} & (L/[L, L], \alpha_{L})
\end{array}
\]

b) Direct checking \( \square \)
Definition 3.6 A Hom-Leibniz algebra \((L, \alpha_L)\) is said to be simply connected if every central extension \(\tau : (F, \alpha_F) \to (L, \alpha_L)\) splits uniquely as the product of Hom-Leibniz algebras \((F, \alpha_F) = (\text{Ker}(\tau), \alpha_F) \times (L, \alpha_L)\).

Proposition 3.7 For a perfect Hom-Leibniz algebra \((L, \alpha_L)\), the following statements are equivalent:

\[\begin{align*}
    a) \quad (L, \alpha_L) & \text{ is simply connected.} \\
    b) \quad (L, \alpha_L) & \text{ is centrally closed.}
\end{align*}\]

If \(u : (L, \alpha_L) \to (M, \alpha_M)\) is a central extension, then:

\[\begin{align*}
    c) \quad \text{Statement } a) \text{ (respectively, statement } b)\text{) implies that } u : (L, \alpha_L) \to (M, \alpha_M)\text{ is a universal central extension.}
    \\
    d) \quad \text{If in addition } u : (L, \alpha_L) \to (M, \alpha_M)\text{ is a universal } \alpha\text{-central extension, then statements } a) \text{ and } b) \text{ hold.}
\end{align*}\]

Proof. \(a) \Rightarrow b)\) Let \(0 \to (\text{Ker}(u_\alpha) = H^2_\alpha(L), \tilde{\alpha}) \to (\text{uce}_\alpha(L), \tilde{\alpha}) \xrightarrow{u_\alpha} (L, \alpha_L) \to 0\) be the universal central extension of \((L, \alpha_L)\), then it is split. Consequently there exists an isomorphism \(\text{uce}_\alpha(L) \cong L\) and \(H^2_\alpha(L) = 0\).

\(b) \Rightarrow a)\) The universal central extension of \((L, \alpha_L)\) is \(0 \to 0 \to (L, \alpha_L) \xrightarrow{\cong} (L, \alpha_L) \to 0\). Consequently every central extension splits uniquely thanks to the universal property.

\(c)\) Let \(u : (L, \alpha_L) \to (M, \alpha_M)\) be a central extension. By Theorem 4.11 \(b)\) in [7], it is universal if \((L, \alpha_L)\) is perfect and every central extension of \((L, \alpha_L)\) splits.

\((L, \alpha_L)\) is perfect by hypothesis and by statement \(a)\), it is simply connected, which means that every central extension splits.

\(d)\) If \(u : (L, \alpha_L) \to (M, \alpha_M)\) is a universal \(\alpha\)-central extension, then by Theorem 4.1. \(a)\) in [7] every central extension \((L, \alpha_L)\) splits. Consequently \((L, \alpha_L)\) is simply connected, equivalently, it is centrally closed. \(\square\)

Now we are going to study functorial properties of the universal central extensions.

Consider a homomorphism of perfect Hom-Leibniz algebras \(f : (L', \alpha_{L'}) \to (L, \alpha_L)\). This homomorphism induces a \(K\)-linear map \(f \otimes f : L' \otimes L' \to L \otimes L\) given by \((f \otimes f)(x_1 \otimes x_2) = f(x_1) \otimes f(x_2)\), that maps the submodule \(I_L\) to the submodule \(I_{L'}\), hence \(f \otimes f\) induces a \(K\)-linear map \(\text{uce}(f) : \text{uce}(L') \to \text{uce}(L)\), given by \(\text{uce}(f) \{x_1, x_2\} = \{f(x_1), f(x_2)\}\), which is a homomorphism of Hom-Leibniz algebras as well.
Moreover, the following diagram is commutative:

\[
\begin{array}{ccc}
HL_2^3(L') & \rightarrow & HL_2^3(L) \\
\downarrow & & \downarrow \\
(uec(L'), \tilde{\alpha}') & \xrightarrow{uec(f)} & (uec(L), \tilde{\alpha}) \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
algebras category. Consequently, an automorphism $f$ of $(L, \alpha_L)$ gives rise to an automorphism $\text{uce}_\alpha(f)$ of $(\text{uce}_\alpha(L), \overline{\pi})$. Commutativity of diagram (5) implies that $\text{uce}_\alpha(f)$ leaves $\text{Ker}(U_\alpha)$ invariant. So the homomorphism of Hom-groups

$$\text{Aut}(L, \alpha_L) \rightarrow \{ g \in \text{Aut}(\text{uce}_\alpha(L), \overline{\pi}) : f(\text{Ker}(U_\alpha)) = \text{Ker}(U_\alpha) \}$$

is obtained.

Now we consider a derivation $d$ of the $\alpha$-perfect Hom-Leibniz algebra $(L, \alpha_L)$. The linear map $\varphi : \alpha_L(L) \otimes \alpha_L(L) \rightarrow \alpha_L(L) \otimes \alpha_L(L)$ given by $\varphi(\alpha_L(x_1) \otimes \alpha_L(x_2)) = d(\alpha_L(x_1)) \otimes \alpha_L^2(x_2) + \alpha_L^2(x_1) \otimes d(\alpha_L(x_2))$, leaves invariant the vector subspace $I_L$ of $\alpha_L(L) \otimes \alpha_L(L)$ spanned by the elements of the form $-x_1 \otimes \alpha_L(x_3) + \alpha_L(x_1) \otimes [x_2, x_3]$, $x_1, x_2, x_3 \in L$. Hence it induces a linear map $\text{uce}_\alpha(d) : (\text{uce}_\alpha(L), \overline{\pi}) \rightarrow (\text{uce}_\alpha(L), \overline{\alpha})$, given by $\text{uce}_\alpha(d)(\{(\alpha_L(x_1), \alpha_L(x_2))\}) = \{d(\alpha_L(x_1)), \alpha_L^2(x_2)\} + \{\alpha_L^2(x_1), d(\alpha_L(x_2))\}$, that makes commutative the following diagram:

$$\frac{(\text{uce}_\alpha(L), \overline{\alpha})}{(L, \alpha_L)} \xrightarrow{\text{uce}_\alpha(d)} \frac{(\text{uce}_\alpha(L), \overline{\alpha})}{(L, \alpha_L)}$$

Consequently, a derivation $d$ of $(L, \alpha_L)$ gives rise to a derivation $\text{uce}_\alpha(d)$ of $(\text{uce}_\alpha(L), \overline{\pi})$. The commutativity of diagram (6) implies that $\text{uce}_\alpha(d)$ maps $\text{Ker}(U_\alpha)$ on itself.

Hence, it is obtained the homomorphism of Hom-$K$-vector spaces

$$\text{uce}_\alpha : \text{Der}(L, \alpha_L) \rightarrow \{ \delta \in \text{Der}(\text{uce}_\alpha(L), \overline{\pi}) : \delta(\text{Ker}(U_\alpha)) \subseteq \text{Ker}(U_\alpha) \}$$

whose kernel belongs to the subalgebra of derivations of $(L, \alpha_L)$ such that vanish on $[\alpha_L(L), \alpha_L(L)]$.

The functorial properties of $\text{uce}_\alpha(-)$ relative to the derivations are described by the following result.

**Lemma 3.8** Let $f : (L', \alpha_{L'}) \rightarrow (L, \alpha_L)$ be a homomorphism of $\alpha$-perfect Hom-Leibniz algebras. Consider $d \in \text{Der}(L)$ and $d' \in \text{Der}(L')$ such that $f \circ d' = d \circ f$, then $\text{uce}_\alpha(f) \circ \text{uce}_\alpha(d') = \text{uce}_\alpha(d) \circ \text{uce}_\alpha(f)$.

**Proof.** Routine checking. \qed

## 4 Lifting automorphisms and derivations

In this section we analyze under what conditions an automorphism or a derivation can be lifted to an $\alpha$-cover. We restrict the study to $\alpha$-covers since we must
compose central extensions in the constructions. This fact does not allow to obtain more general results, mainly due to Lemma 4.10 in [7].

**Definition 4.1** A central extension of Hom-Leibniz algebras \( f : (L', \alpha_{L'}) \to (L, \alpha_L) \), where \((L', \alpha_{L'})\) is an \(\alpha\)-perfect Hom-Leibniz algebra, is said to be an \(\alpha\)-cover.

**Lemma 4.2** If \( f : (L', \alpha_{L'}) \to (L, \alpha_L) \) is a surjective homomorphism of Hom-Leibniz algebras and \( (L', \alpha_{L'}) \) is \(\alpha\)-perfect, then \((L, \alpha_L)\) is \(\alpha\)-perfect as well.

Let \( f : (L', \alpha_{L'}) \to (L, \alpha_L) \) be an \(\alpha\)-cover. Thanks to Lemma 4.2 \((L, \alpha_L)\) is an \(\alpha\)-perfect Hom-Leibniz algebra as well. By Theorem 5.5 in [7], everyone admits universal \(\alpha\)-central extension. Having in mind the functorial properties given in diagram (5), we can construct the following diagram:

\[
\begin{array}{ccc}
Ker(U_{\alpha'}) & \rightarrow & Ker(U_{\alpha}) \\
\downarrow \quad uce_{\alpha}(U_{\alpha'}) & & \downarrow uce_{\alpha}(U_{\alpha}) \\
\quad (uce_{\alpha}(L'), \alpha') & \rightarrow & (uce_{\alpha}(L), \alpha) \\
\quad \downarrow U_{\alpha'} & & \downarrow U_{\alpha} \\
\quad (L', \alpha_{L'}) & \rightarrow & (L, \alpha_L) \\
f & & \\
\end{array}
\]

Since \( U_{\alpha'} : (uce_{\alpha'}(L'), \alpha') \to (L', \alpha_{L'}) \) is a universal \(\alpha\)-central extension, then by Remark 3.2, it is a universal central extension as well. Since \( f : (L', \alpha_{L'}) \to (L, \alpha_L) \) is a central extension and \( U_{\alpha'} : (uce_{\alpha'}(L'), \alpha') \to (L', \alpha_{L'}) \) is a universal central extension, then by Proposition 4.15 in [7] the extension \( f \circ U_{\alpha'} : (uce_{\alpha'}(L'), \alpha') \to (L, \alpha_L) \) is \(\alpha\)-central which is universal in the sense of Definition 4.13 in [7].

On the other hand, since \( U_{\alpha} : (uce_{\alpha}(L), \alpha) \to (L, \alpha_L) \) is a universal \(\alpha\)-central extension, then there exists a unique homomorphism \( \varphi : (uce_{\alpha}(L), \alpha) \to (uce_{\alpha'}(L'), \alpha') \) such that \( f \circ U_{\alpha'} \circ \varphi = U_{\alpha} \).

Moreover \( \varphi \circ uce_{\alpha}(f) = Id \) since the following diagram is commutative

\[
\begin{array}{ccc}
0 & \rightarrow & (Ker(f \circ U_{\alpha'}), \alpha') \\
\downarrow \varphi \circ uce_{\alpha}(f) & & \downarrow Id \\
0 & \rightarrow & (Ker(f \circ U_{\alpha'}), \alpha') \\
f \circ U_{\alpha'} & & \\
\end{array}
\]

and \( f \circ U_{\alpha'} \) is an \(\alpha\)-central extension which is universal in the sense of Definition 4.13 in [7].
Conversely, $\text{uce}_\alpha(f) \circ \varphi = Id$ since the following diagram is commutative

\[
\begin{array}{c}
0 \rightarrow (\text{Ker}(U_\alpha), \alpha_\parallel) \xrightarrow{\varphi} (\text{uce}_\alpha(L), \overline{\alpha}) \xrightarrow{U_\alpha} (L, \alpha_L) \rightarrow 0 \\
\downarrow \text{uce}_\alpha(f) \circ \varphi \quad \downarrow \text{Id} \\
0 \rightarrow (\text{Ker}(U_\alpha), \alpha_\parallel) \xrightarrow{\varphi} (\text{uce}_\alpha(L), \overline{\alpha}) \xrightarrow{U_\alpha} (L, \alpha_L) \rightarrow 0
\end{array}
\]

whose horizontal rows are central extensions and $(\text{uce}_\alpha(L), \overline{\alpha})$ is $\alpha$-perfect, then Lemma 5.4 in [7] guarantees the uniqueness of the vertical homomorphism.

Consequently $\text{uce}_\alpha(f)$ is an isomorphism and from now on we will use the notation $\text{uce}_\alpha(f)^{-1}$ instead of $\varphi$.

On the other hand, $U_{\alpha'} \circ \text{uce}_\alpha(f)^{-1} : (\text{uce}_\alpha(L), \overline{\alpha}) \rightarrow (L', \alpha_{L'})$ is an $\alpha-$cover. In the sequel, we will denote its kernel by

\[
C := \text{Ker}(U_{\alpha'} \circ \text{uce}_\alpha(f)^{-1}) = \text{uce}_\alpha(f)(\text{Ker}(U_{\alpha'})).
\]

**Theorem 4.3** Let $f : (L', \alpha_{L'}) \rightarrow (L, \alpha_L)$ be an $\alpha-$cover.

For any $h \in \text{Aut}(L, \alpha_L)$, there exists a unique $\theta_h \in \text{Aut}(L', \alpha_{L'})$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
(L', \alpha_{L'}) & \xrightarrow{f} & (L, \alpha_L) \\
\downarrow \theta_h & & \downarrow h \\
(L', \alpha_{L'}) & \xrightarrow{f} & (L, \alpha_L)
\end{array}
\tag{7}
\]

if and only if the automorphism $\text{uce}_\alpha(h)$ of $(\text{uce}_\alpha(L), \overline{\alpha})$ satisfies $\text{uce}_\alpha(h)(C) = C$.

In this case, it is uniquely determined by diagram (7) and $\theta_h(\text{Ker}(f)) = \text{Ker}(f)$.

Moreover, the map

\[
\Theta : \{h \in \text{Aut}(L, \alpha_L) : \text{uce}_\alpha(h)(C) = C\} \rightarrow \{g \in \text{Aut}(L', \alpha_{L'}) : g(\text{Ker}(f)) = \text{Ker}(f)\} \\
h \mapsto \theta_h
\]

is a group isomorphism.

**Proof.** Let $h \in \text{Aut}(L, \alpha_L)$ and assume that there exists a $\theta_h \in \text{Aut}(L', \alpha_{L'})$ such that diagram (7) is commutative.

Then $h \circ f : (L', \alpha_{L'}) \rightarrow (L, \alpha_L)$ is an $\alpha-$cover, hence $\theta_h$ is a homomorphism from the $\alpha$-cover $h \circ f$ to the $\alpha$-cover $f$ which is unique by Remark 5.3 b) and Lemma 4.7 in [7].

By application of the functor $\text{uce}_\alpha(-)$ to diagram (7), one obtains the following commutative diagram:

\[
\begin{array}{c}
(\text{uce}_\alpha(L'), \overline{\alpha_{L'}}) \xrightarrow{\text{uce}_\alpha(f)} (\text{uce}_\alpha(L), \overline{\alpha_L}) \\
\downarrow \text{uce}_\alpha(\theta_h) & & \downarrow \text{uce}_\alpha(h) \\
(\text{uce}_\alpha(L'), \overline{\alpha_{L'}}) \xrightarrow{\text{uce}_\alpha(f)} (\text{uce}_\alpha(L), \overline{\alpha_L})
\end{array}
\]
Theorem 4.5

Let \( f : \alpha \rightarrow (L, \alpha L) \) be an \( \alpha \)-cover. Denote by \( C = \text{uce}_\alpha(f)(\text{Ker}(U_\alpha)) \subseteq \text{Ker}(U_\alpha) \). Then the following statements hold:

a) For any \( d \in \text{Der}(L, \alpha L) \) there exists a \( \delta_d \in \text{Der}(L', \alpha L') \) such that the following diagram is commutative

\[
\begin{array}{c}
(L, \alpha L) \\
\downarrow^{\delta_d} \\
(L', \alpha L')
\end{array} \xrightarrow{f} \begin{array}{c}
(L, \alpha L) \\
\downarrow^{d} \\
(L', \alpha L')
\end{array}
\]

\[
(8)
\]

Hence \( \text{uce}_\alpha(h)(C) = \text{uce}_\alpha(h) \circ \text{uce}_\alpha(f)(\text{Ker}(U_\alpha)) = \text{uce}_\alpha(f) \circ \text{uce}_\alpha(\theta_h)(\text{Ker}(U_\alpha)) = \text{uce}_\alpha(f)(\text{Ker}(U_\alpha)) = C. \)

Conversely, from diagram (5), we have that \( U_\alpha = f \circ U_\alpha' \circ \text{uce}_\alpha(f)^{-1} \), hence we obtain the following diagram:

\[
\begin{array}{c}
(C, \alpha) \xrightarrow{\text{uce}_\alpha(L)} (L', \alpha L') \xrightarrow{f} (L, \alpha L) \\
\downarrow^{\text{uce}_\alpha(h)} \downarrow^{\text{uce}_\alpha(f)^{-1}} \downarrow^{h}
\end{array}
\]

If \( \text{uce}_\alpha(h)(C) = C \), then \( U_\alpha' \circ \text{uce}_\alpha(h)^{-1} \circ \text{uce}_\alpha(h)(C) = U_\alpha' \circ \text{uce}_\alpha(f)^{-1}(C) = 0 \), then there exists a unique \( \theta_h : (L', \alpha L') \rightarrow (L, \alpha L) \) such that \( \theta_h \circ U_\alpha' \circ \text{uce}_\alpha(f)^{-1} = U_\alpha' \circ \text{uce}_\alpha(f)^{-1} \circ \text{uce}_\alpha(h). \)

On the other hand, \( h \circ f \circ U_\alpha' \circ \text{uce}_\alpha(f)^{-1} = f \circ U_\alpha' \circ \text{uce}_\alpha(f)^{-1} \circ \text{uce}_\alpha(h) = f \circ U_\alpha' \circ \text{uce}_\alpha(h) \circ \text{uce}_\alpha(f)^{-1} = f \circ \theta_h \circ U_\alpha' \circ \text{uce}_\alpha(f)^{-1} \), then \( h \circ f = f \circ \theta_h. \)

In conclusion, \( \theta_h \) is uniquely determined by diagram (7) and moreover \( \theta_h(\text{Ker}(f)) = \text{Ker}(f). \)

By the previous arguments, it is easy to check that \( \Theta \) is a well-defined map, it is a monomorphism thanks to the uniqueness of \( \theta_h \) and it is an epimorphism, since every \( g \in \text{Aut}(L', \alpha L') \) with \( g(\text{Ker}(f)) = \text{Ker}(f) \), induces a unique homomorphism \( h : (L, \alpha L) \rightarrow (L, \alpha L) \) such that \( h \circ f = f \circ g. \) Then \( g = \theta_h \) and \( \text{uce}_\alpha(h)(C) = C. \)

\[\square\]

Corollary 4.4 If \( (L, \alpha L) \) is an \( \alpha \)-perfect Hom-Leibniz algebra, then the map

\[
\text{Aut}(L, \alpha L) \rightarrow \{ g \in \text{Aut}(\text{uce}_\alpha(L), \overline{\alpha}) : g(\text{Ker}(U_\alpha)) = \text{Ker}(U_\alpha) \}
\]

\[
h \mapsto \text{uce}_\alpha(h)
\]

is a group isomorphism.

Proof. By application of Theorem 4.3 to the \( \alpha \)-cover \( U : (\text{uce}_\alpha(L), \overline{\alpha}) \rightarrow (L, \alpha L) \), it is enough to have in mind that under these conditions \( C = 0 \) and \( \text{uce}_\alpha(f)(0) = 0. \)

Now we analyze under what conditions a derivation of an \( \alpha \)-perfect Hom-Leibniz algebra can be lifted to an \( \alpha \)-cover.

Theorem 4.5 Let \( f : (L', \alpha L') \rightarrow (L, \alpha L) \) be an \( \alpha \)-cover. Denote by \( C = \text{uce}_\alpha(f)(\text{Ker}(U_\alpha)) \subseteq \text{Ker}(U_\alpha) \). Then the following statements hold:

a) For any \( d \in \text{Der}(L, \alpha L) \) there exists a \( \delta_d \in \text{Der}(L', \alpha L') \) such that the following diagram is commutative
if and only if the derivation $\text{uce}_a(d)$ of $(\text{uce}_a(L), \overline{\alpha}_L)$ satisfies $\text{uce}_a(d)(C) \subseteq C$.

In this case, $\delta_d$ is uniquely determined by (8) and $\delta_d(Ker(f)) \subseteq Ker(f)$.

b) The map

$$\Delta: \{d \in \text{Der}(L, \alpha_L) : \text{uce}_a(d)(C) \subseteq C\} \rightarrow \{\rho \in \text{Der}(L', \alpha_{L'}) : \rho(Ker(f)) \subseteq Ker(f)\}$$

$$d \mapsto \delta_d$$

is an isomorphism of Hom-vector spaces (see [19, 2.2]).

c) For the $\alpha$-cover $U_\alpha : (\text{uce}_a(L), \overline{\alpha}_L) \rightarrow (L, \alpha_L)$, the map

$$\text{uce}_a : \text{Der}(L, \alpha_L) \rightarrow \{\delta \in \text{Der}(\text{uce}_a(L), \overline{\alpha}_L) : \delta(Ker(U_\alpha)) \subseteq Ker(U_\alpha)\}$$

is an isomorphism of Hom-vector spaces.

Proof. a) Let $d \in \text{Der}(L, \alpha_L)$ and assume the existence of a $\delta_d \in \text{Der}(L', \alpha_{L'})$ such that diagram (8) is commutative. Then, by Lemma 3.8, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
\text{uce}_a(L') \times \overline{\alpha}_{L'} & \xrightarrow{\text{uce}_a(f)} & \text{uce}_a(L) \times \overline{\alpha}_L \\
\downarrow \text{uce}_a(\delta_d) & & \downarrow \text{uce}_a(d) \\
\text{uce}_a(L') \times \overline{\alpha}_{L'} & \xrightarrow{\text{uce}_a(f)} & \text{uce}_a(L) \times \overline{\alpha}_L
\end{array}
$$

Hence, having in mind the properties derived from diagram (6), we obtain:

$$\text{uce}_a(d)(C) = \text{uce}_a(d) \circ \text{uce}_a(f)(Ker(U_\alpha')) = \text{uce}_a(f) \circ \text{uce}_a(\delta_d)(Ker(U_\alpha')) \subseteq \text{uce}_a(f)(Ker(U_\alpha')) = C.$$ 

Conversely, from diagram (5) we have that $U_\alpha = f \circ U_{\alpha'} \circ \text{uce}_a(f)^{-1}$ and consider the following diagram:

$$
\begin{array}{ccc}
C \xrightarrow{U_{\alpha'} \circ \text{uce}_a(f)^{-1}} (L', \alpha_{L'}) \xrightarrow{f} (L, \alpha_L) \\
\downarrow \text{uce}_a(d) \quad \downarrow d \quad \downarrow \delta_d \\
C \xrightarrow{U_{\alpha'} \circ \text{uce}_a(f)^{-1}} (L', \alpha_{L'}) \xrightarrow{f} (L, \alpha_L)
\end{array}
$$

Since $\text{uce}_a(d)(C) \subseteq C$, then $U_{\alpha'} \circ \text{uce}_a(f)^{-1} \circ \text{uce}_a(d)(C) \subseteq U_{\alpha'} \circ \text{uce}_a(f)^{-1}(C) = U_{\alpha'}(Ker(U_\alpha')) = \emptyset$. Hence there exists a unique $K$-linear map $\delta_d : (L', \alpha_{L'}) \rightarrow (L', \alpha_{L'})$ such that $\delta_d \circ U_{\alpha'} \circ \text{uce}_a(f)^{-1} = U_{\alpha'} \circ \text{uce}_a(f)^{-1} \circ \text{uce}_a(d)$.

On the other hand $d \circ f \circ U_{\alpha'} \circ \text{uce}_a(f)^{-1} = d \circ U_{\alpha'} \circ \text{uce}_a(f) \circ \text{uce}_a(f)^{-1} = U_{\alpha} \circ \text{uce}_a(d) = f \circ U_{\alpha'} \circ \text{uce}_a(f)^{-1} \circ \text{uce}_a(d)$, since $\delta_d \circ U_{\alpha'} \circ \text{uce}_a(f)^{-1} = U_{\alpha'} \circ \text{uce}_a(f)^{-1} \circ \text{uce}_a(d)$, then $d \circ f = f \circ \delta_d$.

Finally, a direct checking shows that $\delta_d$ is a derivation of $L'$, which is uniquely determined by diagram (8) and $\delta_d(Ker(f)) \subseteq Ker(f)$. 

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b) The map $\Delta$ is a homomorphism of Hom-vector spaces by construction, which is injective by the uniqueness of $\delta_d$, and surjective, since for every $\rho \in \text{Der} (L', \alpha_L)$ such that $\rho (\text{Ker}(f)) \subseteq \text{Ker}(f)$ there exists the following diagram commutative:

\[
\begin{array}{ccc}
\text{Ker}(f) & \xrightarrow{\sim} & (L', \alpha_L') \\
\downarrow & & \downarrow \rho \\
\text{Ker}(f) & \xrightarrow{\sim} & (L', \alpha_L')
\end{array}
\]

where $d : (L, \alpha_L) \rightarrow (L, \alpha_L)$ is a derivation satisfying $\text{uce}_\alpha (d) (C) = \text{uce}_\alpha (d) \circ \text{uce}_\alpha (f) (\text{Ker}(U_{\alpha'})) = \text{uce}_\alpha (f) (\text{Ker}(U_{\alpha'})) = C$

Finally, the uniqueness of $\delta_d$ implies that $\Delta (d) = \delta_d = \rho$.

c) It is enough to write the statement b) for the $\alpha$—cover $U_{\alpha} : (\text{uce}_\alpha (L), \overline{\alpha L}) \rightarrow (L, \alpha_L)$. Now $C = \text{uce}_\alpha (U_{\alpha}) (\text{Ker}(U_{\alpha})) = 0$, and $\Delta$ is the map $\text{uce}_\alpha$ derived from diagram (6).

\section{Universal $\alpha$-central extension of the semi-direct product}

Consider a split extension of $\alpha$-perfect Hom-Leibniz algebras

\[
0 \longrightarrow (M, \alpha_M) \xrightarrow{t} (G, \alpha_G) \xrightarrow{p} (Q, Id_Q) \longrightarrow 0
\]

where, by Lemma 2.11, $(G, \alpha_G) \cong (M, \alpha_M) \times (Q, Id_Q)$, whose Hom-action of $(Q, Id_Q)$ on $(M, \alpha_M)$ is given by $q \cdot m = [s(q), t(m)]$ and $m \cdot q = [t(m), s(q)], q \in Q, m \in M$. Moreover we will assume, when it is needed, that the previous action is symmetric, i.e. $q \cdot m + m \cdot q = 0, q \in Q, m \in M$.

An example of the above situation is the direct product $(G, \alpha_G) = (M, \alpha_M) \times (Q, Id_Q) = (M \times Q, \alpha_M \times Id_Q)$, where $(M, \alpha_M)$ is an $\alpha$-perfect Hom-Leibniz algebra and $(Q, Id_Q)$ is a perfect Hom-Leibniz algebra.

Applying the functorial properties of $\text{uce}_\alpha (-)$ given in diagram (5) and having in mind that $(Q, Id_Q)$ is perfect is equivalent to $Q$ is perfect, we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Ker}(U_{\alpha}^M) & \xrightarrow{\tau} & \text{Ker}(U_{\alpha}^G) \\
\downarrow & & \downarrow \pi \\
\text{Ker}(U_{\alpha}^M) & \xrightarrow{\pi} & (Q, Id_{\text{uce}(Q)})
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{t} & (M, \alpha_M) \\
\downarrow \text{uce}_\alpha (M) & & \downarrow \text{uce}_\alpha (G) \\
0 & \xrightarrow{u_{\alpha}^M} & (G, \alpha_G)
\end{array}
\]

\[
\begin{array}{ccc}
& & \text{HL}_2(Q) \\
\uparrow \text{uce}_\alpha (M) & & \uparrow \text{uce}_\alpha (G) \\
0 & \xrightarrow{u_{\alpha}^G} & (Q, Id_Q)
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{t} & (G, \alpha_G) \\
\downarrow \text{uce}_\alpha (M) & & \downarrow \text{uce}_\alpha (G) \\
0 & \xrightarrow{u_{\alpha}^G} & (Q, Id_Q)
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{t} & (G, \alpha_G) \\
\downarrow \text{uce}_\alpha (M) & & \downarrow \text{uce}_\alpha (G) \\
0 & \xrightarrow{u_{\alpha}^G} & (Q, Id_Q)
\end{array}
\]

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Here $\tau = \text{ucc}_\alpha(t)$, $\pi = \text{ucc}_\alpha(p)$, $\sigma = \text{ucc}_\alpha(s)$.

The sequence

$$(\text{ucc}_\alpha(M), \alpha_M) \overset{\tau}{\longrightarrow} (\text{ucc}_\alpha(G), \alpha_G) \overset{\pi}{\longrightarrow} (\text{ucc}(Q), Id_{\text{ucc}(Q)})$$

is split, since $p \circ s = Id_Q$, then $\text{ucc}_\alpha(p) \circ \text{ucc}_\alpha(s) = \text{ucc}_\alpha(Id_Q)$, i.e. $\pi \circ \sigma = Id_{\text{ucc}(Q)}$.

Obviously $\pi$ is surjective and there exists a Hom-action of $(\text{ucc}(Q), Id_{\text{ucc}(Q)})$ on $(\ker(\pi), \alpha_G)$ induced by the section $\sigma$, which is given by:

$\lambda : \text{ucc}(Q) \otimes \ker(\pi) \to \ker(\pi),$

$\lambda([q_1, q_2] \otimes \{\alpha_G(g_1), \alpha_G(g_2)\}) = \{q_1, q_2\} \cdot \{\alpha_G(g_1), \alpha_G(g_2)\} =

\{\{\alpha_G(g_1), \alpha_G(g_2)\} \cdot \{q_1, q_2\}\}$

$\rho : \ker(\pi) \otimes \text{ucc}(Q) \to \ker(\pi),$

$\rho([\alpha_G(g_1), \alpha_G(g_2)] \otimes \{q_1, q_2\}) = \{\alpha_G(g_1), \alpha_G(g_2)\} \cdot \{q_1, q_2\} =

\{\{\alpha_G(g_1), \alpha_G(g_2)\} \cdot \{q_1, q_2\}\}$

By Lemma 2.11, the split exact sequence

$0 \longrightarrow (\ker(\pi), \alpha_G) \overset{i}{\longrightarrow} (\text{ucc}_\alpha(G), \alpha_G) \overset{\pi}{\longrightarrow} (\text{ucc}(Q), Id_{\text{ucc}(Q)}) \longrightarrow 0$

is equivalent to the semi-direct product sequence, i.e.

$$(\text{ucc}_\alpha(G), \alpha_G) \cong (\ker(\pi), \alpha_G) \ltimes (\text{ucc}_\alpha(Q), Id_{\text{ucc}(Q)})$$

Let $q \in Q$ and $\alpha_M(m_1), \alpha_M(m_2) \in \alpha_M(M)$, then in $(\text{ucc}_\alpha(G), \alpha_G)$ the following identities hold:

$\{\alpha_G(s(q)), [t(\alpha_M(m_1)), t(\alpha_M(m_2))]\} = \{[s(q), t(\alpha_M(m_1))], \alpha_G(t(\alpha_M(m_2)))\}$

and

$\{[t(\alpha_M(m_1)), t(\alpha_M(m_2))], \alpha_G(s(q))\} = \{\alpha_G(t(\alpha_M(m_1))), t(\alpha_M(m_2)), s(q)\}$

These equalities together with the $\alpha$-perfection of $(M, \alpha_M)$ imply:

$\{s(Q), M\} = \{\alpha_G(s(Q)), [\alpha_M(M), \alpha_M(M)]\} \subseteq \{\alpha_M(M), \alpha^2_M(M)\} \subseteq \{\alpha_M(M), \alpha_M(M)\}$

and

$\{M, s(Q)\} = \{[\alpha_M(M), \alpha_M(M)], \alpha_G(s(Q))\} \subseteq \{\alpha^2_M(M), \alpha_M(M)\} + \{\alpha_M(M), \alpha^2_M(M)\} \subseteq \{\alpha_M(M), \alpha_M(M)\}$.

Moreover

$$\tau(\text{ucc}_\alpha(M), \alpha_M) \equiv \{\{\alpha_M(M), \alpha_M(M)\}, \alpha_G\}$$

since we have the following identification: $\tau\{\alpha_M(m_1), \alpha_M(m_2)\} = \{t(\alpha_M(m_1)), t(\alpha_M(m_2))\} \equiv \{\alpha_M(m_1), \alpha_M(m_2)\}$, and

$$\sigma(\text{ucc}(Q)) = \{s(Q), s(Q)\} = \{\alpha_G(s(Q)), \alpha_G(s(Q))\}$$

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since \( \sigma(\{q_1, q_2\}) = \{s(q_1), s(q_2)\} = \{\alpha_G(s(q_1)), \alpha_G(s(q_2))\} \).

On the other hand, for every \( \alpha_G(g) \in G \), there exists an \( \alpha_M(m) \in \alpha_M(M) \) such that \( \alpha_G(g) = s(p(\alpha_G(g))) + \alpha_M(m) \). Hence

\[
(\text{uce}_a (G), \overline{\alpha_G}) = (\{s (Q), s (Q)\} + \{\alpha_M(M), \alpha_M(M)\}, \overline{\alpha_G})
\] (10)

**Proposition 5.1**

\[
(Ker(\pi), \overline{\alpha_G}) = (\{\alpha_M(M), \alpha_M(M)\}, \overline{\alpha_G}) = \tau(\text{uce}_a (M), \overline{\alpha_M}) .
\]

**Proof.** Let \( \{g_1, g_2\} \in Ker(\pi) \). From (10), \( \{g_1, g_2\} = \{s(q_1), s(q_2)\} + \{\alpha_M(g_1), \alpha_M(g_2)\} \in \text{uce}_a (G) \). Then \( \overline{\pi} = \pi \{g_1, g_2\} = \{p(s(q_1)), p(s(q_2))\} + \{p(\alpha_M(g_1)), p(\alpha_M(g_2))\} = \{q_1, q_2\} \), i.e. \( q_1 \otimes q_2 \in I_Q \). Consequently, \( \sigma(\{q_1, q_2\}) = \{s(q_1), s(q_2)\} = 0 \) since \( s(q_1) \otimes s(q_2) \in \sigma(I_Q) \subseteq I_G \). So any element in the kernel has the form \( \{\alpha_M(g_1), \alpha_M(g_2)\} \). The reverse inclusion is obvious.

Second equality was proved in (9). \( \square \)

On the other hand \( \sigma(\text{uce}(Q), Id_{\text{uce}(Q)}) = (\{s(Q), s(Q)\}, \overline{\alpha_G}) \).

Since \( \pi \circ \sigma = Id_{\text{uce}(Q)} \), then \( (\text{uce}_a (G), \overline{\alpha_G}) = (Ker(\pi), \overline{\alpha_G}) \times \sigma(\text{uce}(Q), Id_{\text{uce}(Q)}) \).

Moreover \( \sigma \) is an isomorphism between \( (\text{uce}(Q), Id_{\text{uce}(Q)}) \) and \( \sigma(\text{uce}(Q), Id_{\text{uce}(Q)}) \).

These facts imply:

1. \( (\text{uce}_a (G), \overline{\alpha_G}) = \tau(\text{uce}_a (M), \overline{\alpha_M}) \times \sigma(\text{uce}(Q), Id_{\text{uce}(Q)}) \).

2. \( \sigma(\text{uce}(Q), Id_{\text{uce}(Q)}) \cong (\text{uce}(Q), Id_{\text{uce}(Q)}) \).

From 1., an element of \( (\text{uce}_a (G), \overline{\alpha_G}) \) can be written as \( (\tau(m), \sigma(q)) \), for \( m \in (\text{uce}_a (M), \overline{\alpha_M}) \) and \( q \in (\text{uce}(Q), Id_{\text{uce}(Q)}) \) with a suitable choice. Such an element belongs to \( Ker(U^G) \) if and only if \( U^G(\tau(m), \sigma(q)) = 0 \), i.e. \( m \in Ker(U^G) \) and \( q \in H L_2(Q) \).

From these facts we can derive that

3. \( (Ker(U^G), \overline{\alpha_G}) \cong \tau(Ker(U^M), \overline{\alpha_M}) \times \sigma(H L_2(Q), Id_{\text{uce}(Q)}) \).

Since there exists a symmetric Hom-action of \( (Q, Id_Q) \) on \( (M, \alpha_M) \), then there is a Hom-action of \( (\text{uce}(Q), Id_{\text{uce}(Q)}) \) on \( (\text{uce}_a (M), \overline{\alpha_M}) \) given by:

\[
\lambda : \text{uce}(Q) \otimes \text{uce}_a (M) \rightarrow \text{uce}_a (M)
\]

\[
\{q_1, q_2\} \otimes \{\alpha_M(m_1), \alpha_M(m_2)\} \mapsto \{q_1, q_2\} \cdot \{\alpha_M(m_1), \alpha_M(m_2)\} =
\]

\[
\{[q_1, q_2] \cdot \alpha_M(m_1), \alpha_M^2(m_2)\} -
\]

\[
\{[q_1, q_2] \cdot \alpha_M(m_2), \alpha_M^2(m_1)\}
\]

and

\[
\rho : \text{uce}_a (M) \otimes \text{uce}(Q) \rightarrow \text{uce}_a (M)
\]

\[
\{\alpha_M(m_1), \alpha_M(m_2)\} \otimes \{q_1, q_2\} \mapsto \{\alpha_M(m_1), \alpha_M(m_2)\} \cdot \{q_1, q_2\} =
\]

\[
\{\alpha_M(m_1), [q_1, q_2], \alpha_M^2(m_2)\} -
\]

\[
\{\alpha_M^2(m_1), [q_1, q_2] \cdot \alpha_M(m_2)\}
\]

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Then we can define the following homomorphism of Hom-Leibniz algebras:

$$\tau \times \sigma : (uce_\alpha(M), \overline{\alpha_M}) \times (uce(Q), Id_{use(Q)}) \to (uce_\alpha(G), \overline{\alpha_G}) \cong$$

$$\tau (uce_\alpha(M), \overline{\alpha_M}) \times \sigma (uce(Q), Id_{use(Q)})$$

$$\{\alpha_M(m_1), \alpha_M(m_2)\}, \{q_1, q_2\} \mapsto \{t(\alpha_M(m_1)), t(\alpha_M(m_2))\}, \{s(q_1), s(q_2)\}$$

Moreover $$\tau \times \sigma$$ is an epimorphisms since

$$(uce_\alpha(G), \overline{\alpha_G}) \cong (uce_\alpha(M), \overline{\alpha_M}) \times (uce(Q), Id_{use(Q)})$$.

By the relations coming from the action induced by the split extension

$$\tau (\{q_1, q_2\} \cdot \{\alpha_M(m_1), \alpha_M(m_2)\}) = \{s(q_1), s(q_2)\}, \{t(\alpha_M(m_1)), t(\alpha_M(m_2))\}$$

and

$$\tau (\{\alpha_M(m_1), \alpha_M(m_2)\} \cdot \{q_1, q_2\}) = \{t(\alpha_M(m_1)), t(\alpha_M(m_2))\}, \{s(q_1), s(q_2)\}$$

one derives that:

$$t \circ U^M_\alpha (\{q_1, q_2\}, \{\alpha_M(m_1), \alpha_M(m_2)\}) = [q_1, q_2], [\alpha_M(m_1), \alpha_M(m_2)],$$

and

$$t \circ U^M_\alpha (\{\alpha_M(m_1), \alpha_M(m_2)\} \cdot \{q_1, q_2\}) = [\alpha_M(m_1), \alpha_M(m_2)] \cdot [q_1, q_2].$$

4. Now we define the surjective homomorphism of Hom-Leibniz algebras

$$\Phi := (t \circ U^M_\alpha) \times (s \circ u_Q) : (uce_\alpha(M) \times uce(Q), \overline{\alpha_M} \times Id_{use(Q)}) \to (G, \alpha_G)$$

$$\{\alpha_M(m_1), \alpha_M(m_2)\}, \{q_1, q_2\} \mapsto \{t[\alpha_M(m_1), \alpha_M(m_2)], s[q_1, q_2]\}$$

that makes commutative the following diagram:

$$\begin{array}{c}
(uce_\alpha(M) \times uce(Q), \overline{\alpha_M} \times Id_{use(Q)}) \\
\Phi \\
\downarrow \tau \times \sigma \\
(G, \alpha_G)
\end{array}$$

(11)

Now we prove that

$$uce(Q) \cdot Ker(U^M_\alpha \oplus Ker(U^M_\alpha) \cdot uce(Q) \subseteq Ker(\tau) \subseteq Ker(U^M_\alpha)$$

Second inclusion is obvious since $$t \circ U^M_\alpha = U^G_\alpha \circ \tau$$ and $$t$$ is injective.

From the commutativity of the following diagram

$$\begin{array}{c}
Ker(\tau) \\
\downarrow \\
(uce_\alpha(M), \overline{\alpha_M}) \quad \tau \\
\downarrow U^M_\alpha \\
(M, \alpha_M) \quad t \\
\downarrow U^G_\alpha \\
(G, \alpha_G)
\end{array}$$

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we have that \( U^G \circ \tau (Ker(U^M_a)) = t \circ U^M_a (Ker(U^M_a)) = 0 \), then \( \tau (Ker(U^M_a)) \subseteq Ker(U^G_a) \subseteq Z(uce_a(G)) \), so,
\[
\tau (uce(Q) \cdot Ker(U^M_a)) = [\sigma (uce(Q)), \tau (Ker(U^M_a))] = 0
\]
and
\[
\tau (Ker(U^M_a) \cdot uce(Q)) = [\tau (Ker(U^M_a)), \sigma (uce(Q))] = 0
\]
Consequently, \( uce(Q) \cdot Ker(U^M_a) \oplus Ker(U^M_a) \cdot uce(Q) \subseteq Ker(\tau) \).

On the other hand, we observe that \( (uce(Q) \cdot Ker(U^M_a) \oplus Ker(U^M_a) \cdot uce(Q), \overline{\alpha_M}) \) is a two-sided ideal of \( (uce_a(M), \overline{\alpha_M}) \). Then the Hom-action of \( (uce(Q), Id_Q) \) on \( (uce_a(M), \overline{\alpha_M}) \) induces a Hom-action of \( (uce(Q), Id_Q) \) on
\[
\left( uce_a(M), \overline{\alpha_M} \right) = \left( \frac{uce_a(M)}{uce(Q) \cdot Ker(U^M_a) \oplus Ker(U^M_a) \cdot uce(Q)}, \overline{\alpha_M} \right).
\]

Since \( \tau \) vanishes on \( uce(Q) \cdot Ker(U^M_a) \oplus Ker(U^M_a) \cdot uce(Q) \), then it induces \( \tau : uce_a(M) \rightarrow (uce_a(M), \overline{\alpha_M}) \). This fact is illustrated in the following diagram where the notation \( I = uce(Q) \cdot Ker(U^M_a) \oplus Ker(U^M_a) \cdot uce(Q) \) is employed:

\[
\begin{array}{ccc}
(I, \overline{\alpha_M}) & \xrightarrow{0} & (uce_a(G), \overline{\alpha_G}) \\
(uce_a(M), \overline{\alpha_M}) & \xrightarrow{\tau} & (uce_a(M), \overline{\alpha_M}) \\
(uce_a(M), \overline{\alpha_M}) & \xrightarrow{\tau} & (uce_a(G), \overline{\alpha_G})
\end{array}
\]

Now we can construct the following commutative diagram:

\[
\begin{array}{ccc}
I & \xrightarrow{\kappa} & I \\
Ker(\tau \times \sigma) & \xrightarrow{\psi} & (uce_a(M) \times uce(Q), \overline{\alpha_M} \times Id_{uce(Q)}) \\
Ker(\tau \times \sigma) & \xrightarrow{\psi} & (uce_a(G), \overline{\alpha_G})
\end{array}
\]

whose bottom row is a central extension. Moreover \( (uce_a(G), \overline{\alpha_G}) \) is an \( \alpha \)-perfect Hom-Leibniz algebra, then by Theorem 5.5 in [7], it admits a universal \( \alpha \)-central extension and, by Corollary 3.4, \( uce_a(G) \) is centrally closed, i.e. \( uce(uce_a(G)) \cong uce_a(G) \).
Having in mind the following diagram,

\[
\begin{array}{ccc}
\left( uce_a(M) \times uce(Q), \alpha_M \times I_{díu(Q)} \right) & \xrightarrow{\Psi} & \left( uce_a(G), \alpha_G \right) \\
\downarrow{\psi} & & \downarrow{\Id} \\
\left( uce_a(G), \alpha_G \right) & \xrightarrow{\Id} & \left( uce_a(G), \alpha_G \right) \\
\downarrow{\mu} & & \downarrow{\Id} \\
\left( uce_a(M) \times uce(Q), \alpha_M \times I_{díu(Q)} \right) & \xrightarrow{\Psi} & \left( uce_a(G), \alpha_G \right)
\end{array}
\]

where \( \Id : (uce_a(G), \alpha_G) \to (uce_a(G), \alpha_G) \) is a universal central extension since \( (uce_a(G), \alpha_G) \) is centrally closed and \( \Psi \) is a central extension, then there exists a unique homomorphism of Hom-Leibniz algebras \( \mu : (uce_a(G), \alpha_G) \to \left( uce_a(M) \times uce(Q), \alpha_M \times I_{díu(Q)} \right) \) such that \( \Psi \circ \mu = \Id \).

Since \( \Psi \circ \mu \circ \Psi = \Id \circ \Psi = \Psi = \Psi \circ \Id \) and \( uce_a(M) \times uce(Q) \) is \( \alpha \)-perfect, then Lemma 5.4 in [7] implies that \( \mu \circ \Psi = \Id \). Consequently, \( \Psi \) is an isomorphism, then \( \Ker(\Psi) = Ker(\tau \times \sigma) = 0 \), so \( Ker(\tau \times \sigma) \subseteq I \).

The above discussion can be summarized in:

5. \( Ker(\tau \times \sigma) \cong uce(Q) \cdot Ker(U^M_\alpha) \oplus Ker(U^M_\alpha) \cdot uce(Q) \)

We summarize the above results in the following

**Theorem 5.2** Consider a split extension of \( \alpha \)-perfect Hom-Leibniz algebras

\[
0 \to (M, \alpha_M) \xrightarrow{t} (G, \alpha_G) \xrightarrow{p} (Q, I_{díu(Q)}) \to 0
\]

where the induced Hom-action of \( (Q, I_{díu(Q)}) \) on \( (M, \alpha_M) \) is symmetric. Then the following statements hold:

1. \( (uce_a(G), \alpha_G) = \tau (uce_a(M), \alpha_M) \times \sigma (uce(Q), I_{díu(Q)}) \).

2. \( \sigma (uce(Q), I_{díu(Q)}) \cong (uce(Q), I_{díu(Q)}) \).

3. \( (Ker(U^G_\alpha), \alpha_G) = \tau (Ker(U^M_\alpha), \alpha_M) \oplus \sigma \left( HL_2(Q), I_{díu(Q)} \right) \).

4. The homomorphism of Hom-Leibniz algebras

\[
\Phi : (uce_a(M) \times uce(Q), \alpha_M \times I_{díu(Q)}) \to (G, \alpha_G)
\]

given by \( \Phi (\{\alpha_M(m_1), \alpha_M(m_2)\}, \{q_1, q_2\}) = (t[\alpha_M(m_1), \alpha_M(m_2)], s[q_1, q_2]) \) is an epimorphism that makes commutative diagram (11) and its kernel is \( Ker(U^M_\alpha) \oplus HL_2(Q) \).
5. $\text{Ker} (\tau \times \sigma) \cong \text{uce} (Q) \cdot \text{Ker}(U^M) \oplus \text{Ker}(U^M) \cdot \text{uce} (Q)$

Remark 5.3 Let us observe that the hypothesis of symmetric Hom-action is not needed in statements 1., 2. and 3. in Theorem 5.2, so they are valid in general.

Theorem 5.4 The following statements are equivalent:

a) $\Phi = (t \circ U^M_\alpha) \times (s \circ u_Q) : (\text{uce}_\alpha (M) \times \text{uce} (Q), \overline{\alpha M} \times \text{Id}_{\text{uce}_\alpha (Q)}) \rightarrow (G, \alpha_G)$ is a central extension, hence is an $\alpha$-cover.

b) The Hom-action of $(\text{uce}(Q), \text{Id}_Q)$ on $(\text{Ker}(U^M_\alpha), \overline{\alpha M})$ is trivial.

c) $\tau \times \sigma$ is an isomorphism. Consequently $\text{uce}_\alpha (M) \times \text{uce} (Q)$ is the universal $\alpha$-central extension of $(G, \alpha_G)$.

d) $\tau$ is injective.

In particular, for the direct product $(G, \alpha_G) = (M, \alpha_M) \times (Q, \text{Id}_Q)$ the following isomorphism holds:

$$(\text{uce}_\alpha (M \times Q), \overline{\alpha M} \times \text{Id}_Q) \cong (\text{uce}_\alpha (M) \times \text{uce} (Q), \overline{\alpha M} \times \text{Id}_{\text{uce}(Q)})$$

Proof. a) $\iff$ b)

If $\Phi : (\text{uce}_\alpha (M) \times \text{uce}_\alpha (Q), \overline{\alpha M} \times \text{Id}_{\text{uce}_\alpha (Q)}) \rightarrow (G, \alpha_G)$ is a central extension and having in mind that $\text{Ker}(\Phi) = \text{Ker}(U^M_\alpha) \oplus HL_2(Q)$, then the Hom-action of $(\text{uce}(Q), \text{Id}_Q)$ on $(\text{Ker}(U^M_\alpha), \overline{\alpha M})$ is trivial and vice versa.

Moreover $(\text{uce}_\alpha (M) \times \text{uce} (Q), \overline{\alpha M} \times \text{Id}_{\text{uce}(Q)})$ is $\alpha$-perfect since the Hom-action is trivial.

b) $\iff$ c)

By statement 5. in Theorem 5.2 we know that $\text{Ker} (\tau \times \sigma) \cong \text{uce} (Q) \cdot \text{Ker}(U^M_\alpha) \oplus \text{Ker}(U^M_\alpha) \cdot \text{uce} (Q)$, then $\tau \times \sigma$ is injective if and only if the Hom-action is trivial.

From this fact and having in mind diagram (11), immediately follows that $\text{uce}_\alpha (M) \times \text{uce} (Q)$ is the universal $\alpha$-central extension of $(G, \alpha_G)$.

c) $\iff$ d)

It suffices to have in mind the identification of $\tau$ with $\tau \times \sigma$ given by $\tau \{\alpha_M (m_1), \alpha_M (m_2)\} = (\tau \times \sigma) \{\alpha_M (m_1), \alpha_M (m_2)\}, 0)$, since $\text{Ker}(\tau) \cong \text{Ker} (\tau \times \sigma)$, then the equivalence is obvious.

Finally, in case of the direct product $(G, \alpha_G) = (M, \alpha_M) \times (Q, \text{Id}_Q)$ the Hom-action of $(Q, \text{Id}_Q)$ on $(M, \alpha_M)$ is trivial, then the Hom-action of $(\text{uce}(Q), \text{Id}_{\text{uce}(Q)})$ on $(\text{uce}_\alpha (M), \overline{\alpha M})$ is trivial as well and, consequently, $(\text{uce}_\alpha (M) \times \text{uce} (Q), \overline{\alpha M} \times \text{Id}_{\text{uce}(Q)}) = (\text{uce}_\alpha (M) \times \text{uce} (Q), \overline{\alpha M} \times \text{Id}_{\text{uce}(Q)})$.

The proof is finished by application of statement c) to this particular case. $\square$

Remark 5.5 Note that when the Hom-Leibniz algebras are considered as Leibniz algebras, i.e. the endomorphisms $\alpha$ are identities, then the results in this section recover the corresponding results for Leibniz algebras given in [6].
References


