GENERALIZED INTEGRATION OPERATORS FROM MIXED-NORM TO ZYGMUND-TYPE SPACES *

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Abstract: Let φ be an analytic self-map of the unit disk $\mathbb{D}$, $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$ and $g \in H(\mathbb{D})$. The boundedness and compactness of the generalized integration operator

$$I_{g,\varphi}^{(n)} f(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi,\ z \in \mathbb{D},$$

from mixed-norm space to the Zygmund-type space, and the little Zygmund-type space are investigated in this article.

Keywords: Boundedness; Compactness; Generalized integration operators; Mixed-norm space; Zygmund-type space.

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1 Introduction

Let $\mathbb{D}$ be the unit disk in the finite complex plane $\mathbb{C}$, $\partial\mathbb{D}$ boundary of $\mathbb{D}$, $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$, $\mathbb{N}_0$ the set of all nonnegative integers and $\mathbb{N}$ the set of all positive integers.

A positive continuous function ϕ on $[0,1)$ is called a normal if there exist positive numbers a, b, $0 < a < b$ and $t_0 \in [0,1)$, such that

$$\frac{\phi(t)}{(1-t^2)^a} \quad \text{decreases for} \quad t_0 \le t < 1 \quad \text{and} \quad \lim_{t\to 1^-} \frac{\phi(t)}{(1-t^2)^a} = 0,$$
$$\frac{\phi(t)}{(1-t^2)^b} \quad \text{increases for} \quad t_0 \le t < 1 \quad \text{and} \quad \lim_{t\to 1^-} \frac{\phi(t)}{(1-t^2)^b} = \infty$$

(see, for example, [28]).

For $0 < p < \infty$, $0 < q < \infty$ and a normal function ϕ, let $H(p,q,\phi)$ denote the space of all analytic functions f on the unit disk $\mathbb{D}$ such that

$$\|f\|_{p,q,\phi} = \left(\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r} r dr\right)^{1/p},$$

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where the integral means $M_p(f,r)$ are defined by

$$M_p(f,r)=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right)^{1/p},\quad 0\le r<1.$$

For $1\le p<\infty$, $H(p,q,\phi)$ equipped with the norm $\|\cdot\|_{p,q,\phi}$ is a Banach space. When $0<p<1$, $\|\cdot\|_{p,q,\phi}$ is a quasinorm on $H(p,q,\phi)$, $H(p,q,\phi)$ is a Fréchet space but not a Banach space. If $0<p=q<\infty$, then $H(p,p,\phi)$ is the Bergman-type space

$$H(p,p,\phi)=\left\{f\in H(\mathbb{D}):\int_{\mathbb{D}}|f(z)|^p\frac{\phi^p(|z|)}{1-|z|}dA(z)<\infty\right\},$$

where $dA(z)$ denotes the normalized Lebesgue area measure on the unit disk $\mathbb{D}$ such that $A(\mathbb{D})=1$. Note that if $\phi(r)=(1-r)^{(\alpha+1)/p}$, then $H(p,p,\phi)$ is the weighted Bergman space $A^p_\alpha(\mathbb{D})$ defined for $0<p<\infty$ and $\alpha>-1$, as the space of all $f\in H(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\alpha}=\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^\alpha dA(z)<\infty$$

(see, for example, [1, 8]).

Let $\mathcal{Z}_\mu$ denote the space of all $f\in H(\mathbb{D})\cap C(\overline{\mathbb{D}})$ such that

$$\sup_{z\in\mathbb{D}}\mu(|z|)|f''(z)|<\infty.$$

Under the norm

$$\|f\|_{\mathcal{Z}_\mu}=|f(0)|+|f'(0)|+\sup_{z\in\mathbb{D}}\mu(|z|)|f''(z)|, \tag{1}$$

it is easy to see that $\mathcal{Z}_\mu$ is a Banach space. Some information on Zygmund-type spaces on the unit disc and some operators on them, can be found, e.g., in [4, 5, 11, 15, 18, 23, 31], for the case of the upper half-plane see [38], while some information in the setting of the unit ball can be found, e.g., in [6, 17, 18, 20, 41, 54, 56, 57, 59].

The little Zygmund-type space $\mathcal{Z}_{\mu,0}$ is defined to be the subspace of $\mathcal{Z}_\mu$ consisting of those $f\in\mathcal{Z}_\mu$ such that [11]

$$\lim_{|z|\to 1}\mu(|z|)|f''(z)|=0.$$

It is easy to see that $\mathcal{Z}_{\mu,0}$ is a closed subspace of $\mathcal{Z}_\mu$ and the set of all polynomials is dense in $\mathcal{Z}_{\mu,0}$.

In this paper, we consider an integration operator $I^{(n)}_{g,\varphi}$ which is defined as

$$I^{(n)}_{g,\varphi}f(z)=\int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi,\ z\in\mathbb{D}.$$

This operator is called the generalized integral operator, which was introduced in [27] and studied in [27, 49]. Also, the operator $I^{(n)}_{g,\varphi}$ is a generalization of the Rimann- Stieltjes operator I_g induced by g, defined as

$$I_gf(z)=\int_0^z f(\xi)g'(\xi)d\xi,\ z\in\mathbb{D}.$$

Y. Yu and Y. Liu in [53] characterized the boundedness and compactness of Riemann-Stieltjes operator I_g from weighted Bloch spaces into Bergman-type spaces. J. Liu, Z. Lou and C. Xiong in [21] investigated the essential norm of the integral operator I_g on some classical Banach spaces (the Bloch space, BMOA and the Dirichlet space). In fact, the operator $I_{g,\varphi}^{(n)}$ can induce many known operators. For example, when $n = 1$, $I_{g,\varphi}^{(n)}$ reduces to an integration operator recently studied by S. Stevi'c, S. Li, X. Zhu and W. Yang in [12, 13, 14, 16, 30]. When $n = 1$ and $g(z) = \varphi'(z)$, we obtain the composition operator C_φ defined as $C_\varphi f = f(\varphi) - f(\varphi(0))$, $f \in H(D)$. $D = D^1$ be the differentiation operator, that is, $Df = f'$. If $n \in \mathbb{N}_0$ then the operator D^n is defined by $D^0 f = f$, $D^n f = f^{(n)}$, $f \in H(\mathbb{D})$, $n = m + 1$ and $g(z) = \varphi'(z)$, then we get the operator $C_\varphi D^m f(z) = f^{(m)}(\varphi(z)) - f^{(m)}(\varphi(0))$ which was studied in [9, 26, 55].

Especially recently years, S. D. Sharma and A. Sharmat in [27] have characterized the boundedness and compactness of generalized integration operators $I_{g,\varphi}^{(n)}$ from Bloch type spaces to weighted BMOA spaces by using logarithmic Carleson measure characterization of the weighted BMOA spaces. Y. Liu and Y. Yu in [24] studied the boundedness and compactness of Riemann-Stieltjes operator from mixed norm spaces to Zygmund-type spaces on the unit ball. S. Stević in [31] studied the boundedness and compactness of the generalized composition operator from mixed-norm space to the Bloch-type space, the little Bloch-type space, the Zygmund space, and the little Zygmund space. X. Zhu in [58] investigated the boundedness and compactness of generalized integration operators from H^∞ to Zygmund-type spaces on the unit disk. Z. He and G. Cao in [7] investigated the boundedness and compactness of generalized integration operators between Bloch-type spaces and $F(p, q, s)$ spaces. For related integral-type operators on the unit disc, see, for example [10, 11, 12, 13, 14, 40, 51, 52]. Some related integral-type operators in $\mathbb{C}^n$ are treated, for example, in [2, 3, 13, 14, 16, 19, 22, 32, 33, 34, 35, 36, 37, 39, 41, 43, 44, 45, 46, 47, 48, 50, 55] (see also the related references therein). Motivated by the results [7, 24, 31, 58], we consider the boundedness and compactness of the operators $I_{g,\varphi}^{(n)}$ from $H(p, q, \phi)$ to the Zygmund space, and the little Zygmund space. For the proof, we need different test functions and some complex calculation skills.

Throughout this paper, the letter C denotes a positive constant which may vary at each occurrence but it is independent of the essential variables. Two quantities a and b are said to be comparable, denoted by $a \asymp b$, if there exists a positive constant C such that $C^{-1}a \le b \le Ca$.

2 The boundedness and compactness of $I_{g,\varphi}^{(n)}$ from $H(p, q, \phi)$ to Zygmund space

In this section, we study the boundedness and compactness of $I_{g,\varphi}^{(n)} : H(p, q, \phi) \to \mathcal{Z}_\mu$. To do so we need the following lemmas.

Lemma 2.1 *([31]) Assume that $p, q \in (0, \infty)$, ϕ is normal and $f \in H(p, q, \phi)$. Then for each $n \in \mathbb{N}_0$, there is a positive constant C independent of f such that*

$$|f^{(n)}(z)| \le C \frac{\|f\|_{p,q,\phi}}{\phi(|z|)(1 - |z|^2)^{1/q+n}}, z \in \mathbb{D}.$$

By standard arguments (see, for example, [4] or Lemma 3 in [29]) the following lemma follows.

Lemma 2.2 *Assume that φ is an analytic self-map of $\mathbb{D}$. Then $I_{g,\varphi}^{(n)} : H(p, q, \phi) \to \mathcal{Z}_\mu$ is compact if and only if $I_{g,\varphi}^{(n)} : H(p, q, \phi) \to \mathcal{Z}_\mu$ is bounded and for any bounded sequence $\{f_k\}$ in $H(p, q, \phi)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, we have $\|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{Z}} \to 0$ as $k \to \infty$.*

The proof of the following lemma is similar to that of [25, Lemma 1], and the details are omitted here.

Lemma 2.3 *A closed set K in $\mathcal{Z}_{\mu,0}$ is compact if and only if K is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(|z|)|f''(z)| = 0.$$

Lemma 2.4 *([8]) For any real β, let*

$$J_\beta(z) = \int_0^{2\pi} \frac{d\theta}{|1 - ze^{-i\theta}|^{1+\beta}},\ z \in \mathbb{D}.$$

Then we have

$$J_\beta(z) \asymp \begin{cases} 1 & ,\ if\ \beta < 0, \\ \log \frac{1}{1-|z|^2} & ,\ if\ \beta = 0, \\ \frac{1}{(1-|z|^2)^\beta} & ,\ if\ \beta > 0, \end{cases} \quad as\ \ |z| \to 1^-.$$

Lemma 2.5 *([28]) For $\beta > -1$ and $\gamma > 1 + \beta$ we have*

$$\int_0^1 \frac{(1-r)^\beta}{(1-r\rho)^\gamma} dr \le C(1-\rho)^{1+\beta-\gamma},\ 0 < \rho < 1.$$

Now we are in a position to characterize the boundedness of $I_{g,\varphi}^{(n)} : H(p, q, \phi) \to \mathcal{Z}_\mu$.

Theorem 2.6 *Assume that φ is an analytic self-map of $\mathbb{D}$. Then $I_{g,\varphi}^{(n)} : H(p, q, \phi) \to \mathcal{Z}_\mu$ is bounded if and only if the following conditions are satisfied,*

$$M_1 = \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{1/q+n+1}} < \infty, \tag{2}$$

and

$$M_2 = \sup_{z\in\mathbb{D}} \frac{\mu(|z|)|g'(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{1/q+n}} < \infty. \tag{3}$$

Proof. Assume that conditions (2) and (3) hold. Then, for every $z \in \mathbb{D}$ and $f \in H(p, q, \phi)$, by Lemma 2.1, we have

$$\begin{aligned}
&\left|\mu(|z|)(I_{g,\varphi}^{(n)} f)''(z)\right| \\
&= \mu(|z|)\left|\left(f^{(n)}(\varphi(z))g(z)\right)'\right| \\
&= \mu(|z|)\left|f^{(n+1)}(\varphi(z))\varphi'(z)g(z) + f^{(n)}(\varphi(z))g'(z)\right| \\
&\le C\left(\frac{\mu(|z|)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{1/q+n+1}} + \frac{\mu(|z|)|g'(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{1/q+n}}\right)\|f\|_{p,q,\phi} \\
&\le C(M_1 + M_2)\|f\|_{p,q,\phi}. \qquad (4)
\end{aligned}$$

On the other hand, we have

$$|(I_{g,\varphi}^{(n)}f)(0)| = 0, \tag{5}$$

and

$$\begin{aligned} |(I_{g,\varphi}^{(n)}f)'(0)| &= |f^{(n)}(\varphi(0))g(0)| \\ &\le C\frac{|g(0)|}{\phi(|\varphi(0)|)(1-|\varphi(0)|^2)^{1/q+n}}\|f\|_{p,q,\phi}. \end{aligned} \tag{6}$$

Applying conditions (2), (3), (4), (5) and (6), we deduce that the operator $I_{g,\varphi}^{(n)} : H(p,q,\phi) \to \mathcal{Z}_\mu$ is bounded.

Conversely, assume that $I_{g,\varphi}^{(n)} : H(p,q,\phi) \to \mathcal{Z}_\mu$ is bounded, that is there exists a constant C such that

$$\|I_{g,\varphi}^{(n)}f\|_{\mathcal{Z}_\mu} \le C\|f\|_{p,q,\phi}$$

for all $f \in H(p,q,\phi)$. For a fixed $w \in \mathbb{D}$, set

$$f_w(z) = \frac{(1-|w|^2)^{b+1}}{\phi(|w|)}\left(\frac{1}{(1-\overline{w}z)^\alpha} - \frac{\alpha(1-|w|^2)}{(\alpha+n)(1-\overline{w}z)^{\alpha+1}}\right), \tag{7}$$

where the constant b is from the definition of the normality of the function ϕ and $\alpha = 1/q + b + 1$.

A straightforward calculation shows that

$$\begin{aligned} f_w^{(n)}(z) &= \frac{(1-|w|^2)^{b+1}}{\phi(|w|)}\frac{\alpha(\alpha+1)\dots(\alpha+n-1)(\overline{w})^n}{(1-\overline{w}z)^{\alpha+n}} \\ &- \frac{(1-|w|^2)^{b+1}}{\phi(|w|)}\frac{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)(\alpha+n)(1-|w|^2)(\overline{w})^n}{(\alpha+n)(1-\overline{w}z)^{\alpha+n+1}}, \end{aligned} \tag{8}$$

$$\begin{aligned} f_w^{(n+1)}(z) &= \frac{(1-|w|^2)^{b+1}}{\phi(|w|)}\frac{\alpha(\alpha+1)\dots(\alpha+n)(\overline{w})^{n+1}}{(1-\overline{w}z)^{\alpha+n+1}} \\ &- \frac{(1-|w|^2)^{b+1}}{\phi(|w|)}\frac{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n+1)(1-|w|^2)(\overline{w})^{n+1}}{(\alpha+n)(1-\overline{w}z)^{\alpha+n+2}}. \end{aligned} \tag{9}$$

By Lemma 2.4, we have

$$M_q(f_w, r) \le C\frac{(1-|w|^2)^{b+1}}{\phi(|w|)(1-r|w|)^{b+1}}.$$

As ϕ is normal and by applying Lemma 2.5, we obtain (see [31, 42])

$$\sup_{w\in\mathbb{D}} \|f_w\|_{p,q,\phi} \le C. \tag{10}$$

From (8) and (9), we have

$$f_w^{(n)}(w) = 0,\ f_w^{(n+1)}(w) = -\frac{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)(\overline{w})^{n+1}}{\phi(|w|)(1-|w|^2)^{1/q+n+1}}. \tag{11}$$

Hence

$$C \geq \|I_{g,\varphi}^{(n)} f_{\varphi(w)}\|_{\mathcal{Z}_\mu} \geq \mu(|w|)|\varphi'(w)g(w)| \left| f_{\varphi(w)}^{(n)}(\varphi(w)) \right|$$
$$= \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1) \frac{\mu(|w|)|\varphi'(w)g(w)||\varphi(w)|^{n+1}}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/q+n+1}}. \tag{12}$$

From (12) we have

$$\sup_{|\varphi(w)|>\frac{1}{2}} \frac{\mu(|w|)|\varphi'(w)g(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/q+n+1}}$$
$$\leq \sup_{|\varphi(w)|>\frac{1}{2}} 2^{n+1} \frac{\mu(|w|)|\varphi'(w)g(w)||\varphi(w)|^{n+1}}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/q+n+1}}$$
$$\leq C < \infty. \tag{13}$$

Since $f(w) = \frac{w^n}{n!} \in H(p,q,\phi)$ it follows that

$$\mu(|w|)|g'(w)| \leq \|I_{g,\varphi}^{(n)} f\|_{\mathcal{Z}_\mu} \leq \|I_{g,\varphi}^{(n)}\| \cdot \|f\|_{p,q,\phi} \leq C. \tag{14}$$

Since $h(w) = \frac{w^{n+1}}{(n+1)!} \in H(p,q,\phi)$, from (14) and the boundedness of φ it follows that

$$\mu(|w|)|\varphi'(w)g(w)| \leq \|I_{g,\varphi}^{(n)} h\|_{\mathcal{Z}_\mu} + \mu(|w|)|\varphi(w)g'(w)|$$
$$\leq \|I_{g,\varphi}^{(n)}\| \cdot \|h\|_{p,q,\phi} + C \leq 2C. \tag{15}$$

From this and the fact ϕ is normal we obtain

$$\sup_{|\varphi(w)|\leq\frac{1}{2}} \frac{\mu(|w|)|\varphi'(w)g(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/q+n+1}}$$
$$\leq C \sup_{|\varphi(w)|\leq\frac{1}{2}} \mu(|w|)|\varphi'(w)g(w)| \leq C < \infty. \tag{16}$$

From (13) and (16) it follows that (2) holds.

For a fixed $w \in \mathbb{D}$, set

$$g_w(z) = \frac{(1-|w|^2)^{b+1}}{\phi(|w|)} \left(\frac{\alpha+n+1}{(1-\overline{w}z)^\alpha} - \frac{\alpha(1-|w|^2)}{(1-\overline{w}z)^{\alpha+1}} \right), \tag{17}$$

It is easy to see that

$$g_w^{(n)}(z) = \frac{(1-|w|^2)^{b+1}}{\phi(|w|)} \frac{\alpha(\alpha+1)\dots(\alpha+n-1)(\alpha+n+1)(\overline{w})^n}{(1-\overline{w}z)^{\alpha+n}}$$
$$- \frac{(1-|w|^2)^{b+1}}{\phi(|w|)} \frac{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n)(1-|w|^2)(\overline{w})^n}{(1-\overline{w}z)^{\alpha+n+1}}, \tag{18}$$

$$g_w^{(n+1)}(z) = \frac{(1-|w|^2)^{b+1}}{\phi(|w|)} \frac{\alpha(\alpha+1)\dots(\alpha+n+1)(\overline{w})^{n+1}}{(1-\overline{w}z)^{\alpha+n+1}}$$
$$- \frac{(1-|w|^2)^{b+1}}{\phi(|w|)} \frac{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n+1)(1-|w|^2)(\overline{w})^{n+1}}{(1-\overline{w}z)^{\alpha+n+2}}. \tag{19}$$

By Lemmas 2.4 and 2.5, we get (see [31, 42])

$$\sup_{w\in\mathbb{D}} \|g_w\|_{p,q,\phi} \le C. \tag{20}$$

From (18) and (19), we have

$$g_w^{(n)}(w) = \frac{\alpha(\alpha+1)\dots(\alpha+n-1)(\overline{w})^n}{\phi(|w|)(1-|w|^2)^{1/q+n}},\ g_w^{(n+1)}(w) = 0. \tag{21}$$

Hence

$$\begin{aligned} C &\ge \|I_{g,\varphi}^{(n)} g_{\varphi(w)}\|_{\mathcal{Z}_\mu} \ge |\mu(|w|)g'(w)|g_{\varphi(w)}^{(n)}(\varphi(w))| \\ &= \frac{\alpha(\alpha+1)\dots(\alpha+n-1)|\mu(|w|)g'(w)||\varphi(w)|^n}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/q+n}}. \end{aligned} \tag{22}$$

From (22) we have that

$$\begin{aligned} &\sup_{|\varphi(w)|>\frac{1}{2}} \frac{\mu(|w|)|g'(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/q+n}} \\ &\le \alpha(\alpha+1)\dots(\alpha+n-1) \sup_{|\varphi(w)|>\frac{1}{2}} 2^n \frac{\mu(|w|)|g'(w)|\varphi(w)|^n}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/q+n}} \\ &\le C < \infty. \end{aligned} \tag{23}$$

Using (14) and the fact ϕ is normal we obtain

$$\begin{aligned} &\sup_{|\varphi(z)|\le\frac{1}{2}} \frac{\mu(|w|)|g'(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/q+n}} \\ &\le C \sup_{|\varphi(w)|\le\frac{1}{2}} \mu(|w|)|g'(w)| \le C < \infty. \end{aligned} \tag{24}$$

Combining (23) with (24) we get (3), finishing the proof of the theorem.

Theorem 2.7 *Assume that φ is an analytic self-map of $\mathbb{D}$. Then $I_{g,\varphi}^{(n)} : H(p,q,\phi) \to \mathcal{Z}_\mu$ is compact if and only if $I_{g,\varphi}^{(n)} : H(p,q,\phi) \to \mathcal{Z}_\mu$ is bounded, and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+n+1}} = 0 \tag{25}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+n}} = 0. \tag{26}$$

Proof. Assume that $I_{g,\varphi}^{(n)} : H(p,q,\phi) \to \mathcal{Z}_\mu$ is bounded and that conditions (25) and (26) hold. For any bounded sequence $\{f_k\}$ in $H(p,q,\phi)$ with $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. To establish the assertion, it suffices, in view of Lemma 2.2, to show that

$$\|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{Z}_\mu} \to 0 \ \text{ as } \ k \to \infty.$$

We assume that $\|f_k\|_{p,q,\phi} \leq 1$. From (25) and (26), given $\epsilon > 0$, there exists a $\delta \in (0,1)$, when $\delta < |\varphi(z)| < 1$, we have

$$\frac{\mu(|z|)}{\phi(|\varphi(z)|)} \left(\frac{|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^{1/q+n+1}} + \frac{|g'(z)|}{(1-|\varphi(z)|^2)^{1/q+n}} \right) < \epsilon. \tag{27}$$

From the boundedness of $I_{g,\varphi}^{(n)} : H(p,q,\phi) \to \mathcal{Z}_\mu$, we see that (14) and (15) hold by Theorem 2.6. Since $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimate gives that $f_k^{(n)}$ and $f_k^{(n+1)}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, there exists a $K_0 \in \mathbb{N}$ such that $k > K_0$ implies that

$$\begin{aligned} &|f_k^{(n)}(\varphi(0))g(0)| + \sup_{|\varphi(z)|\leq\delta} \mu(|z|)|\varphi'(z)g(z)f_k^{(n+1)}(\varphi(z))| + \sup_{|\varphi(z)|\leq\delta} \mu(|z|)\left|g'(z)f_k^{(n)}(\varphi(z))\right| \\ &\leq |f_k^{(n)}(\varphi(0))g(0)| + C \sup_{|\varphi(z)|\leq\delta} |f_k^{(n+1)}(\varphi(z))| + C \sup_{|\varphi(z)|\leq\delta} \left|f_k^{(n)}(\varphi(z))\right| < C\epsilon. \end{aligned} \tag{28}$$

From (27) and (28) we have

$$\begin{aligned} &\|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{Z}_\mu} = |(I_{g,\varphi}^{(n)} f_k)(0)| + |(I_{g,\varphi}^{(n)} f_k)'(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|(I_{g,\varphi}^{(n)} f_k)''(z)| \\ &\leq |f_k^{(n)}(\varphi(0))g(0)| + \sup_{|\varphi(z)|\leq\delta} \mu(|z|)\left|\varphi'(z)g(z)f_k^{(n+1)}(\varphi(z))\right| + \sup_{|\varphi(z)|\leq\delta} \mu(|z|)\left|g'(z)f_k^{(n)}(\varphi(z))\right| \\ &+ \sup_{\delta<|\varphi(z)|<1} \left(\mu(|z|)\left|\varphi'(z)g(z)f_k^{(n+1)}(\varphi(z))\right| + \mu(|z|)\left|g'(z)f_k^{(n)}(\varphi(z))\right|\right) \\ &\leq C\epsilon + C \sup_{\delta<|\varphi(z)|<1} \frac{\mu(|z|)}{\phi(|\varphi(z)|)} \left(\frac{|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^{1/q+n+1}} + \frac{|g'(z)|}{(1-|\varphi(z)|^2)^{1/q+n}} \right) \\ &< 2C\epsilon, \end{aligned}$$

when $k > K_0$. It follows that the operator $I_{g,\varphi}^{(n)} : H(p,q,\phi) \to \mathcal{Z}_\mu$ is compact.

Conversely, assume that $I_{g,\varphi}^{(n)} : H(p,q,\phi) \to \mathcal{Z}_\mu$ is compact. Then it is clear that $I_{g,\varphi}^{(n)} : H(p,q,\phi) \to \mathcal{Z}_\mu$ is bounded. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. We can use the test functions

$$f_k(z) = f_{\varphi(z_k)}(z), \tag{29}$$

f_w here is defined in (7). From (10) and (11) we have

$$\sup_{k\in\mathbb{N}} \|f_k\|_{p,q,\phi} \leq C$$

and

$$f_k^{(n)}(\varphi(z_k)) = 0,\ f_k^{(n+1)}(\varphi(z_k)) = -\frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)(\overline{\varphi(z_k)})^{n+1}}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{1/q+n+1}}.$$

For $|z| = r < 1$, using the fact that ϕ is normal, we have

$$|f_k(z)| \leq \frac{C}{(1-r)^{1/q+1}}\,(1-|\varphi(z_k)|) \to 0 \ (\text{as}\, k \to \infty),$$

that is, f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$, using the compactness of $I_{g,\varphi}^{(n)}$: $H(p,q,\phi) \to \mathcal{Z}_\mu$ we obtain

$$\frac{\mu(|z_k|)|\varphi'(z_k)g(z_k)||\varphi(z_k)|^{n+1}}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{1/q+n+1}} \le \|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{Z}_\mu} \to 0 \text{ as } k \to \infty.$$

From this, and $|\varphi(z_k)| \to 1$, it follows that

$$\lim_{k\to\infty} \frac{\mu(|z_k|)|\varphi'(z_k)g(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{1/q+n+1}} = 0,$$

and consequently (25) holds.

In order to prove (26), choose

$$g_k(z) = g_{\varphi(z_k)}(z), \tag{30}$$

g_w here is defined in (17). It follows from (20) and (21) that

$$\sup_{k\in\mathbb{N}} \|g_k\|_{p,q,\phi} \le C$$

and

$$g_k^{(n)}(\varphi(z_k)) = \frac{\alpha(\alpha+1)\dots(\alpha+n-1)(\overline{\varphi(z_k)})^n}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{1/q+n}}, \;\; g_k^{(n+1)}(\varphi(z_k)) = 0,$$

and g_k converges to 0 uniformly on compact subsets of $\mathbb{D}$. The compactness of $I_{g,\varphi}^{(n)} : H(p,q,\phi) \to \mathcal{Z}_\mu$ implies that

$$\lim_{k\to\infty} \|I_{g,\varphi}^{(n)} g_k\|_{\mathcal{Z}_\mu} = 0.$$

It follows that

$$\frac{\mu(|z_k|)|g'(z_k)|\left|(\overline{\varphi(z_k)})^n\right|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{1/q+n}} \le C\|I_{g,\varphi}^{(n)} g_k\|_{\mathcal{Z}_\mu} \to 0 \text{ as } k \to \infty. \tag{31}$$

$|\varphi(z_k)| \to 1$ implies that

$$\lim_{k\to\infty} \frac{\mu(|z_k|)|g'(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{1/q+n}} = 0,$$

(26) holds.

3 The boundedness and compactness of $I_{g,\varphi}^{(n)}$ from $H(p,q,\phi)$ to the little Zygmund-type space

In this section, we study the boundedness and compactness of $I_{g,\varphi}^{(n)} : H(p,q,\phi) \to \mathcal{Z}_{\mu,0}$.

The following result is proved similar to Theorem in [31], hence we omit it.

Theorem 3.1 *Assume that φ is an analytic self-map of $\mathbb{D}$. Then $I_{g,\varphi}^{(n)} : H(p,q,\phi) \to \mathcal{Z}_{\mu,0}$ is bounded if and only if $I_{g,\varphi}^{(n)} : H(p,q,\phi) \to \mathcal{Z}_\mu$ is bounded,*

$$\lim_{|z|\to 1} \mu(|z|)\,|\varphi'(z)g(z)| = 0,$$

and

$$\lim_{|z|\to 1}\mu(|z|)|g'(z)| = 0.$$

In the next theorem, we characterize the compactness of $I_{g,\varphi}^{(n)} : H(p,q,\phi)\to \mathcal{Z}_{\mu,0}$.

Theorem 3.2 *Assume that φ is an analytic self-map of $\mathbb{D}$. Then $I_{g,\varphi}^{(n)} : H(p,q,\phi)\to \mathcal{Z}_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1}\frac{\mu(|z|)\,|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+n+1}} = 0, \tag{32}$$

and

$$\lim_{|z|\to 1}\frac{\mu(|z|)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+n}} = 0. \tag{33}$$

Proof. Assume that conditions (32) and (33) hold. Then it is clear that (2) and (3) hold. Hence $I_{g,\varphi}^{(n)} : H(p,q,\phi)\to \mathcal{Z}_{\mu}$ is bounded by Theorem 2.6. From inequality (4) we see that $I_{g,\varphi}^{(n)}f \in \mathcal{Z}_{\mu,0}$ for each $f\in H(p,q,\phi)$, it follows that $I_{g,\varphi}^{(n)} : H(p,q,\phi)\to \mathcal{Z}_{\mu,0}$ is bounded. Taking the supremum in inequality (4) over all $f\in H(p,q,\phi)$ such that $\|f\|_{p,q,\phi}\le 1$ and letting $|z|\to 1$, yields

$$\lim_{|z|\to 1}\sup_{\|f\|_{p,q,\phi}\le 1}\mu(|z|)|(I_{g,\varphi}^{(n)}f)''(z)| = 0.$$

Hence, by Lemma 2.3 we see that the operator $I_{g,\varphi}^{(n)} : H(p,q,\phi)\to \mathcal{Z}_{\mu,0}$ is compact.

Now assume that $I_{g,\varphi}^{(n)} : H(p,q,\phi)\to \mathcal{Z}_{\mu,0}$ is compact. Then $I_{g,\varphi}^{(n)} : H(p,q,\phi)\to \mathcal{Z}_{\mu,0}$ is bounded, and by taking the function $f(z)=\frac{z^n}{n!}$, it follows that

$$\lim_{|z|\to 1}\mu(|z|)|g'(z)| = 0. \tag{34}$$

By taking the function $f(z)=\frac{z^{n+1}}{(n+1)}$, we have

$$\lim_{|z|\to 1}\mu(|z|)|\varphi'(z)g(z)+\varphi(z)g'(z)| = 0, \tag{35}$$

from (34), (35), we get

$$\lim_{|z|\to 1}\mu(|z|)|\varphi'(z)g(z)| = 0. \tag{36}$$

Since $f_{\varphi(z)},\ g_{\varphi(z)}\in H(p,q,\phi)$, we have $I_{g,\varphi}^{(n)}f_{\varphi(z)}, I_{g,\varphi}^{(n)}g_{\varphi(z)}\in \mathcal{Z}_{\mu,0}$. Because $|\varphi(z)|\to 1$ implies $|z|\to 1$, we obtain

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(|z|)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+n+1}} = 0, \tag{37}$$

and

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(|z|)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+n}} = 0. \tag{38}$$

We only prove that (36) and (37) imply (32). The proof of (33) is similar, hence it will be omitted.

From (37), it follows that for every $\epsilon>0$, there exists $\delta\in(0,1)$ such that

$$\frac{\mu(|z|)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+n+1}} < \epsilon, \tag{39}$$

when $\delta < |\varphi(z)| < 1$. Using (36) we see that there exists $\tau \in (0,1)$ such that

$$\mu(|z|)|\varphi'(z)g(z)| < \epsilon \inf_{t\in[0,\delta]} \phi(t)(1-t^2)^{1/q+n+1}, \tag{40}$$

when $\tau < |z| < 1$.

Therefore, when $\tau < |z| < 1$ and $\delta < |\varphi(z)| < 1$, by (39) we have

$$\frac{\mu(|z|)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+n+1}} < \epsilon. \tag{41}$$

On the other hand, when $\tau < |z| < 1$ and $|\varphi(z)| \le \delta$, by (40) we obtain

$$\frac{\mu(|z|)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+n+1}} \le \frac{\mu(|z|)|\varphi'(z)g(z)|}{\inf\limits_{t\in[0,\delta]} \phi(t)(1-t^2)^{1/q+n+1}} < \epsilon. \tag{42}$$

From (41) and (42), we obtain (32), as desired. The proof is completed.

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