Asymptotically almost automorphic solutions of fractional order neutral integro-differential equations

V. Kavitha*, Syed Abbas† and R. Murugesu‡

Abstract

In this paper, we study the existence of asymptotic almost automorphic solution of fractional neutral integro-differential equation. We prove the result by using fixed point theorems. We show the result with Lipschitz condition and without Lipschitz condition on the forcing term. Finally examples are given to illustrate the analytical findings.

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1 Introduction

This work is mainly concerned with the existence of asymptotically almost automorphic mild solutions to fractional order neutral integro-differential equation

$$D_t^{\alpha}[x(t) - k_1(t, x(t))] = A[x(t) - k_1(t, x(t))] + D_t^{\alpha-1} f(t, x(t), Kx(t)), \tag{1.1}$$

$$\mathrm{K}x(t) = \int_{-\infty}^{t} k(t-s)h(s, x(s))ds$$

$$x(0) = x_0 \quad t \in R, \tag{1.2}$$

where $1 < \alpha < 2$ and $A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space $(X, \|.\|)$, k satisfy $|k(t)| \leq c_k e^{-bt}$ for $t \geq 0$ and c_k, b are positive constants, $f : R \times X \times X \to X$, $h : R \times X \to X$ and $k_1 : R \times X \to X$ are asymptotically almost automorphic functions in t for each $x, y \in X$ satisfying suitable conditions. The fractional derivative D_t^{α} is to be understood in Riemann-Liouville sense.

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Neutral differential equations arise in many areas of applied mathematics and for this reason, this type of equation has received much attention in recent years see [14, 15, 23, 24, 26, 31]. Due to their applications in several fields of science [5, 17, 18], fractional differential equations are attracting increasing interest, because of their numerical treatment. Properties of the solutions have been studied in several contexts see [1–4, 7–10, 32] and references therein.

The concept of asymptotically almost automorphy was introduced by N'Guérékata [19]. Since then, these functions have generated lots of developments and applications, we refer the reader to [6, 12, 13, 15, 16, 21, 28, 29, 34] and the references therein.

Recently Kavitha et al [27] studied weighted pseudo almost automorphic solution of the following fractional integro-differential equation

$$D_t^{\alpha}x(t) = Ax(t) + D_t^{\alpha-1}f(t, x(t), Kx(t)) \quad t \in R, \quad \text{where } 1 < \alpha < 2 \text{ and}$$
$$Kx(t) = \int_{-\infty}^{t} k(t-s)h(s, x(s))ds$$

where A is linear densely defined sectorial operator.

Motivated by the above work, in this paper we study the existence of asymptotically almost automorphic solutions to (1.1)-(1.2). The organization of the paper is as follows. In section-2, we give some basic definitions and results. In section-3, we establish the existence of asymptotic almost automorphic solution of equations (1.1)-(1.2). In section 4, examples are given to support the theory.

2 Preliminaries and basic results

In this section, we introduce notations, definitions, lemmas and preliminary facts which are used throughout this work.

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ be two complex Banach spaces. The notation $C(R, X)$, (respectively $C(R \times X, X)$) denote the collection of all continuous functions from R to X. Let $\mathcal{BC}(R, X)$, (respectively $\mathcal{BC}(R \times X, X)$) denote the collection of all X-valued bounded continuous functions (respectively, the class of jointly bounded continuous functions $f : R \times X \to X$). The space $\mathcal{BC}(R, X)$ equipped with the sup norm defined by

$$\|f\|_\infty = \sup_{t \in R} \|f(t)\|$$

is a Banach space. The notation $\mathcal{L}(X, Y)$ stands for the space of bounded linear operators from X into Y endowed with the uniform operator topology and we abbreviate it into $\mathcal{L}(X)$ whenever $X = Y$.

Definition 2.1. *[21, 22]. Let $f : R \to X$ be a bounded continuous function. We say that f is almost automorphic if for every sequence of real numbers $(s_n)_{n\in\mathcal{N}}$, there exists a subsequence $(\tau_n)_{n\in\mathcal{N}}$ such that*

$$g(t) = \lim_{n\to\infty} f(t + \tau_n)$$

is well-defined for each $t \in R$ and

$$\lim_{n\to\infty} g(t-\tau_n) = f(t)$$

for each $t \in R$. Denote by $AA(R, X)$ the set of all such functions.

Definition 2.2. *[21, 22]. A continuous function $f : R \times X \to X$ is called almost automorphic in t uniformly for x in compact subsets of X if for every compact subset $\mathcal{K}$ of X and every real sequence $(s_n)_{n\in\mathcal{N}}$ there exists a subsequence $(\tau_n)_{n\in\mathcal{N}}$ such that*

$$g(t, x) = \lim_{n\to\infty} f(t+\tau_n, x)$$

is well-defined for each $t \in R$, $x \in \mathcal{K}$ and

$$\lim_{n\to\infty} g(t-\tau_n, x) = f(t, x)$$

for each $t \in R$, $x \in \mathcal{K}$. Denote by $AA(R \times X, X)$ the set of all such functions.

The space of all continuous functions $m : R^+ \to X$ such that $\lim_{t\to\infty} m(t) = 0$ is denoted by $C_0(R^+, X)$. Moreover, we denote $C_0(R^+ \times X, X)$, the space of all continuous functions from $R^+ \times X$ to X satisfying $\lim_{t\to\infty} m(t, x) = 0$ in t and uniformly in $x \in X$.

Definition 2.3. *A continuous function $f : R^+ \to X$ is called asymptotically almost automorphic iff it can be written as $f = g + \phi$, where $g \in AA(R, X)$ and $\phi \in C_0(R^+, X)$. This kind of functions is denoted by $AAA(R^+, X)$.*

Definition 2.4. *A continuous function $f : R^+ \times X \to X$ is called asymptotically almost automorphic iff it can be written as $f = g + \phi$, where $g \in AA(R \times X, X)$ and $\phi \in C_0(R^+ \times X, X)$. This kind of functions is denoted by $AAA(R^+ \times X, X)$.*

We state a Lemma by Liang et.al.[28] about the composition result.

Lemma 2.1. *Let $f(t, x) = g(t, x) + \phi(t, x)$ is an asymptotically almost automorphic function with $g(t, x) \in AA(R \times X, X)$ and $\phi(t, x) \in C_0(R^+ \times X, X)$ and $f(t, x)$ is uniformly continuous on any bounded subset $\Omega \subset X$ uniformly in t. Then for $x(\cdot) \in AAA(R^+, X)$, the function $f(\cdot, x(\cdot)) \in AAA(R^+ \times X, X)$.*

Definition 2.5. *[8]. A closed linear operator $(A, D(A))$ with dense domain $D(A)$ in a Banach space X is said to be sectorial of type ω and angle θ if there are constants $\omega \in R$, $\theta \in (0, \frac{\pi}{2})$, $M > 0$ such that its resolvent exists outside the sector*

$$\omega + \Sigma_\theta := \{\lambda + \omega : \lambda \in \mathcal{C}, |\arg(-\lambda)| < \theta\}, \tag{2.1}$$

$$\|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda - \omega|}, \quad \lambda \notin \omega + \Sigma_\theta. \tag{2.2}$$

Definition 2.6. *Let $1 < \alpha < 2$. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We say that A is the generator of a solution operator if there exist $\omega \in R$ and a strongly continuous functions $S_\alpha : R_+ \to \mathcal{L}(X)$ such that $\{\lambda^\alpha : Re\lambda > \omega\} \subset \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \quad Re\lambda > \omega, \quad x \in X.$$

In [8], Cuesta proves that if A is sectorial of type $\omega \in R$ with $0 \le \theta < \pi(1 - \alpha/2)$, then A is a generator of a solution operator given by

$$S_\alpha(t) = \frac{1}{2\pi i} \int_{\mathbb{G}} e^{\lambda t} \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} d\lambda, \quad t \ge 0$$

with $\mathbb{G}$ a suitable path lying outside the sector $\omega + \Sigma_0$. Furthermore he shows that the following Lemma holds.

Lemma 2.2. *[8][Theorem 1]. Let $A : D(A) \subset X \to X$ be a sectorial operator in a complex Banach space X, satisfying hypothesis (2.1) and (2.2), for some $M > 0, \omega < 0$ and $0 \le \theta < \pi(1 - \alpha/2)$. Then there exists $C(\theta, \alpha) > 0$ depending solely on θ and α, such that*

$$\|S_\alpha(t)\|_{\mathcal{L}(X)} \le \frac{C(\theta, \alpha)M}{1 + |\omega| t^\alpha}, \quad t \ge 0. \tag{2.3}$$

Now, we recall a useful compactness criterion.

Let $h : R^+ \to [1, \infty)$ be a continuous function such that $h(t) \to \infty$ as $t \to \infty$. We consider the space

$$C_h(X) = \left\{ u \in C(R^+, X) : \lim_{t\to\infty} \frac{u(t)}{h(t)} = 0 \right\}.$$

The space $C_h(X)$ is a Banach space equipped with the norm
$\|u\|_h = \sup_{t \in R^+} \frac{\|u(t)\|}{h(t)}$. (see[11]).

Lemma 2.3. *[11]. A subset $K^{'} \subset C_h(X)$ is a relatively compact set if it verifies the following conditions:*

(c-1) The set $K^{'}_b = \{u_{[0,b]} : u \in K^{'}\}$ is relatively compact in $C([0, b], X)$ for all $b \ge 0$.

(c-2) $\lim_{t\to\infty} \frac{\|u(t)\|}{h(t)} = 0$ *uniformly for all $u \in K^{'}$.*

3 Asymptotically almost automorphic mild solutions

Before starting our main results in this section, we recall the definition of the mild solution to (1.1)-(1.2).

Definition 3.1. *[3]. A continuous function $x : R^+ \to X$ satisfying the integral equation*

$$x(t) = S_\alpha(t)[x_0 - k_1(0, x_0)] + k_1(t, x(t)) + \int_0^t S_\alpha(t-s) f\big(s, x(s), Kx(s)\big) ds,$$

is called the mild solution of the problem (1.1)-(1.2).

We only need integrability of function f so that the right hand expression is well defined and therefore it is called mild solution. If we put the condition $f \in \mathfrak{C}_\mu$, $1 < \mu < 2$, where $\mathfrak{C}_\mu$ is the space of all functions such that $t^\mu f$ is continuous, then the solution is called classical solution.

We make the following assumptions:

(H1) A is a sectorial operator of type $\omega < 0$.

(H2) $k_1 \in AAA(R^+ \times X, X)$ and $f \in AAA(R^+ \times X \times X, X)$ and there exist positive constants L_1, L_2, L_3 such that

$$(i) \quad \|k_1(t,x) - k_1(t,y)\| \le L_1 \|x - y\|, \quad x, y \in X$$

$$(ii) \quad \|f(t, x_1, y_1) - f(t, x_2, y_2)\| \le L_2 \|x_1 - x_2\| + L_3 \|y_1 - y_2\|,$$

where $x_i, y_i \in X$, $i = 1, 2$ and $t \in R^+$.

(H3) The function $h : R^+ \times X \to X$ is an asymptotically almost automorphic in t uniformly in $x \in X$ and satisfies

$$\|h(t,x) - h(t,y)\| \le L_4 \|x - y\| \quad \text{for each } x, y \in X.$$

The following lemmas are from [16].

Lemma 3.1. *Let $f = g + \phi \in AAA(R^+ \times X, X)$ with $g \in AA(R \times X, X)$, $\phi \in C_0(R^+ \times X, X)$ satisfying the Hypothesis (H2)(ii). If $x(t) \in AAA(R^+, X)$ then $f(\cdot, x(\cdot)) \in AAA(R^+ \times X, X)$.*

Lemma 3.2. *Let $f = g + \phi \in AAA(R^+ \times X \times X \to X)$ with $g \in AA(R, X)$, $\phi \in C_0(R^+, X)$. Then $Q(t) := \int_0^t S_\alpha(t-s) f(s) ds \in AAA(R^+, X)$.*

Proof. We observe that

$$\begin{aligned} Q(t) &= \int_0^t S_\alpha(t-s) g(s) ds + \int_0^t S_\alpha(t-s)\phi(s) ds \\ &= \int_{-\infty}^t S_\alpha(t-s) g(s) ds - \int_{-\infty}^0 S_\alpha(t-s) g(s) ds + \int_0^t S_\alpha(t-s)\phi(s) ds \end{aligned}$$

Let $Q(t) = R(t) + S(t)$, where

$$R(t) := \int_{-\infty}^t S_\alpha(t-s) g(s) ds$$

$$S(t) := \int_0^t S_\alpha(t-s)\phi(s)ds - \int_{-\infty}^0 S_\alpha(t-s)g(s)ds.$$

Now, let $(s_n^{'})$ be an arbitrary sequence of real numbers. Since $g \in AA(R, X)$ there exists a subsequence s_n of $(s_n^{'})$ such that

$$\lim_{n\to\infty} g(t+s_n) = \overline{g}(t), \text{ for all } t \in R$$

and

$$\lim_{n\to\infty} \overline{g}(t-s_n) = g(t), \text{ for all } t \in R.$$

We define $\overline{R}(t) := \int_{-\infty}^t S_\alpha(t-s)\overline{g}(s)ds$.

Now, consider

$$\begin{aligned} R(t+s_n) &= \int_{-\infty}^{t+s_n} S_\alpha(t+s_n-s)g(s)ds \\ &= \int_{-\infty}^{t} S_\alpha(t-\sigma)g(\sigma+s_n)d\sigma \\ &= \int_{-\infty}^{t} S_\alpha(t-\sigma)g_n(\sigma)d\sigma \end{aligned}$$

where $g_n(\sigma) = g(\sigma+s_n),\ n = 1, 2, \cdots$

$$R(t+s_n) = \int_0^\infty S_\alpha(\sigma)g_n(t-\sigma)d\sigma$$

Now, by inequality (2.3)

$$\begin{aligned} \|R(t+s_n)\| &\le \int_0^\infty \frac{C(\theta,\alpha)M}{1+|\omega|\sigma^\alpha}\|g_n(t-\sigma)\|d\sigma \\ &\le C(\theta,\alpha)M\frac{|w|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}\|g\|_\infty \end{aligned}$$

and by continuity of $S_\alpha(\cdot)x$ we have $S_\alpha(t-\sigma)g_n(\sigma) \to S_\alpha(t-\sigma)\overline{g}(\sigma)$ as $n \to \infty$ for each $\sigma \in R$ fixed and any $t \ge \sigma$. Then by the Lebesgue dominated convergence theorem,

$$R(t+s_n) \to \overline{R}(t) \quad \text{as} \quad n \to \infty \quad \text{for all} \quad t \in R.$$

In similar way we can show that

$$\overline{R}(t-s_n) \to R(t) \quad \text{as} \quad n \to \infty \quad \text{for all} \quad t \in R.$$

This shows that $R(t) \in AA(R, X)$.

Now let us show that $S(t) \in C_0(R^+, X)$. Since $\phi \in C_0(R^+, X)$, for each $\epsilon > 0$ there exists a constant $T > 0$ such that $\|\phi(s)\| \le \epsilon$ for all $s \ge T$. Then for all $t \ge T$, we deduce,

$$\|S(t)\| \le C(\theta,\alpha)M\|\phi\|_\infty \int_0^{t/2} \frac{1}{1+|\omega|(t-s)^\alpha}ds + \epsilon C(\theta,\alpha)M\int_{t/2}^t \frac{1}{1+|\omega|(t-s)^\alpha}ds$$

$$
\begin{aligned}
&+ C(\theta,\alpha)M\|g\|_\infty \int_{-\infty}^{0} \frac{1}{1+|\omega|(t-s)^\alpha} ds \\
&\le C(\theta,\alpha)M[\|\phi\|_\infty + \|g\|_\infty] \int_t^\infty \frac{1}{1+|\omega|s^\alpha} ds \\
&+ \epsilon C(\theta,\alpha)M\|g\|_\infty \int_0^\infty \frac{1}{1+|\omega|s^\alpha} ds \\
&\le C(\theta,\alpha)M[\|\phi\|_\infty + \|g\|_\infty] \int_t^\infty \frac{1}{1+|\omega|s^\alpha} ds + \frac{\epsilon C(\theta,\alpha)M|\omega|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)}.
\end{aligned}
$$

Therefore, $\lim_{t\to\infty} S(t) = 0$, that is, $S(t) \in C_0(R^+, X)$. This completes the proof. □

The first existence and uniqueness result is based on Banach's contraction principle.

Theorem 3.1. *Let $f = g + \phi \in AAA(R^+ \times X \times X, X)$ with $g \in AA(R \times X \times X, X)$ and $\phi \in C_0(R^+ \times X \times X, X)$. Assume that (H1)-(H3) hold. Then (1.1)-(1.2) has a unique asymptotically almost automorphic mild solution provided*

$$
L_1 + \Big(L_2 + L_3 L_4 \frac{c_k}{b}\Big) C(\theta,\alpha) M \frac{|w|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)} < 1. \tag{3.1}
$$

Proof. Consider the operator $\Gamma : AAA(R^+, X) \to AAA(R^+, X)$ such that

$$
(\Gamma x)(t) = S_\alpha(t)[x_0 - k_1(0, x_0)] + k_1(t, x(t)) + \int_0^t S_\alpha(t-s) f(s, x(s), Kx(s)) ds.
$$

Applying Lemma 3.1, we infer that $k_1(\cdot, x(\cdot))$ and $f(\cdot, x(\cdot))$ belong to $AAA(R^+, X)$. By Lemma 3.2, we obtain that Γ is $AAA(R^+, X)$-valued. Furthermore, we have the estimate

$$
\begin{aligned}
\|(\Gamma x)(t) - (\Gamma y)(t)\| &= \Big\| [k_1(t, x(t)) - k_1(t, y(t))] \\
&\quad + \int_0^t S_\alpha(t-s)\big[f(s, x(s), Kx(s)) - f(s, y(s), Ky(s))\big] ds \Big\| \\
&\le \|k_1(t, x(t)) - k_1(t, y(t))\| \\
&\quad + \int_0^t \|S_\alpha(t-s)\|_{L(X)} \big\| f(s, x(s), Kx(s)) - f(s, y(s), Ky(s)) \big\| ds \\
&\le L_1 \|x(t) - y(t)\| + \int_0^t \frac{C(\theta,\alpha)M}{1+|\omega|(t-s)^\alpha} \big[L_2 \|x(s) - y(s)\| \\
&\quad + L_3 \|Kx(s) - Ky(s)\| \big] ds.
\end{aligned} \tag{3.2}
$$

Consider

$$
\begin{aligned}
\|Kx(s) - Ky(s)\| &\le \int_0^t |k(t-s)| \|h(s, x(s)) - h(s, y(s))\| ds \\
&\le \int_0^t |k(t-s)| L_4 \|x(s) - y(s)\| ds \\
&\le \sup_{t \in R^+} \|x(t) - y(t)\| L_4 \Big(\int_0^t |k(t-s)| ds \Big)
\end{aligned}
$$

$$\leq \sup_{t\in R^+}\|x(t)-y(t)\|L_4\int_0^t |k(s)|ds$$
$$\leq \sup_{t\in R^+}\|x(t)-y(t)\|L_4\int_0^t c_k e^{-bs}ds$$
$$\leq c_k\frac{(1-e^{-bt})}{b}L_4\sup_{t\in R^+}\|x(t)-y(t)\|.$$

Using the above estimate, inequality (3.2) becomes,

$$\|(\Gamma x)(t)-(\Gamma y)(t)\|$$
$$\leq L_1\sup_{t\in R^+}\|x(t)-y(t)\|+\Big[L_2+L_3L_4c_k\Big(\frac{1-e^{-bt}}{b}\Big)\Big]\sup_{t\in R^+}\|x(t)-y(t)\|\int_0^t\frac{C(\theta,\alpha)M}{1+|\omega|s^\alpha}ds$$
$$\leq\Big[L_1+\Big[L_2+L_3L_4c_k\Big(\frac{1-e^{-bt}}{b}\Big)\Big]C(\theta,\alpha)M\frac{|w|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}\Big]\|x-y\|_\infty.$$

This implies

$$\|\Gamma x-\Gamma y\|_\infty\leq\Big[L_1+\Big[L_2+L_3L_4c_k\Big(\frac{1-e^{-bt}}{b}\Big)\Big]C(\theta,\alpha)M\frac{|w|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}\Big]\|x-y\|_\infty.$$

which proves that Γ is a contraction we conclude that Γ has a unique fixed point in $AAA(R^+,X)$. This completes the proof. □

We next study the existence of asymptotically almost automorphic mild solutions of (1.1)-(1.2) when the perturbation f is not necessarily Lipschitz continuous. For that, we require the following assumptions:

(H4) There exists a continuous nondecreasing function $W:[0,\infty)\to(0,\infty)$ such that

$$\|f(t,x,y)\|\leq W(\|x\|+\|y\|)\quad\text{for all}\quad t\geq 0\text{ and }\quad x\in X.$$

(H5) The functions $f:R^+\times X\times X\to X$, $h:R^+\times X\to X$ and $k_1:R^+\times X\to X$ are asymptotically almost automorphic in t and uniformly for x in compact subsets of X and uniformly continuous on bounded sets of X uniformly in $t\geq 0$.

Theorem 3.2. *Assume that the conditions (H1) and (H4)-(H5) hold. Let inequality (2.3) be satisfied. In addition, suppose the following properties hold:*

(i) *For each $C\geq 0$*

$$\lim_{t\to\infty}\frac{1}{h(t)}\int_0^t\frac{W\big((1+K)Ch(s)\big)}{1+|\omega|(t-s)^\alpha}ds=0,$$

where h is the function given in Lemma 2.3.

We set

$$\beta(C):=\frac{1}{h(t)}\Big(\|S_\alpha(t)(x_0-k_1(0,x_0)\|+\|k_1(t,x(t))\|$$
$$+C(\theta,\alpha)M\int_0^t\frac{W\big((1+K)Ch(s)\big)}{1+|\omega|(t-s)^\alpha}ds\Big),$$

where $C(\theta,\alpha)$ and M are constants given in (2.3).

(ii) *There is a constant* $L_1 > 0$ *such that* $\|k_1(t, h(t)x) - k_1(t, h(t)y)\| \le L_1\|x - y\|$ *for all* $t \ge 0$ *and* $x, y \in X$. *We set*

$$\Omega(C) := \frac{C(\theta,\alpha)M}{h(t)} \int_0^t \frac{W\big((1+K)Ch(s)\big)}{1+|\omega|(t-s)^\alpha} ds,$$

where $C(\theta, \alpha)$ *and* M *are the constants given in (2.3) and* h *is given in Lemma 2.3.*

(iii) *For each* $\epsilon > 0$ *there is* $\delta > 0$ *such that for every* $u, v \in C_h(X)$, $\|u - v\|_h \le \delta$ *implies that*

$$C(\theta,\alpha)M \int_0^t \frac{\|f(s,u(s),Ku(s)) - f(s,v(s),Kv(s))\|}{1+|\omega|(t-s)^\alpha} ds \le \epsilon,$$

for all $t \in R$.

(iv) $L_1 + \liminf_{r} \to \infty \dfrac{\Omega(r)}{r} < 1$.

(v) $\lim \inf_{\xi \to \infty} \dfrac{\xi}{\beta(\xi)} > 1$.

(vi) *For all* $a, b \in R$, $a < b$ *and* $r > 0$, *the set* $\{f(s, h(s)x, K(h(s)x)) : a \le s \le b,\ x \in C_h(X),\ \|x\|_h \le r\}$ *is relatively compact in* X.

Then equation (1.1)-(1.2) *has an asymptotically almost automorphic mild solution.*

Proof. We define the operator $\Gamma : C_h(X) \to C_h(X)$ by

$$(\Gamma x)(t) = S_\alpha(t)[x_0 - k_1(0, x_0)] + k_1(t, x(t)) + \int_0^t S_\alpha(t-s) f(s, x(s), Kx(s)) ds,\ t \ge 0.$$

Now, we decompose Γ as $\Gamma = \Gamma_1 + \Gamma_2$, where

$$(\Gamma_1 x)(t) = S_\alpha(t)[x_0 - k_1(0, x_0)] + k_1(t, x(t))$$
$$(\Gamma_2 x)(t) = \int_0^t S_\alpha(t-s) f(s, x(s), Kx(s)) ds.$$

Now, we will show that the operator Γ_1 is contraction and Γ_2 is completely continuous. For better readability, we break the proof into sequence of steps.

Step 1: We show that Γ_1 is contraction on $C_h(X)$.

Let $x \in C_h(X)$, we have that,

$$\begin{aligned}\frac{\|(\Gamma_1 x)(t)\|}{h(t)} &\le \frac{1}{h(t)}\Big[\|S_\alpha(t)\|[\|x_0\| + \|k_1(0,x_0)\|] + \|k_1(t,x(t)) - k_1(t,0)\| + \|k_1(t,0)\|\Big] \\ &\le \frac{1}{h(t)}\Big[C(\theta,\alpha)M[\|x_0\| + \|k_1(0,x_0)\|] + L_1\|x\|_h + \|k_1(\cdot,0)\|_\infty\Big].\end{aligned}$$

Hence, Γ_1 is $C_h(X)$-valued. On the other hand, Γ_1 is an L_1-contraction.

Next we show that, Γ_2 is completely continuous.

Step 2: The operator Γ_2 is continuous.

In fact, for any $\epsilon > 0$, we take $\delta > 0$ involved in condition (iii). If $x, y \in C_h(X)$ and $\|x-y\|_h \leq \delta$ then

$$\begin{aligned}\|(\Gamma_2 x)(t) - (\Gamma_2 y)(t)\| &\leq C(\theta,\alpha)M\int_0^t \frac{\|f(s,x(s),Kx(s)) - f(s,y(s),Ky(s))\|}{1+|\omega|(t-s)^\alpha}ds\\ &\leq \epsilon,\end{aligned}$$

which shows the assertion.

Step 3: We next show that Γ_2 is completely continuous.

Let $V^{'}(t) = \Gamma_2(B_r(C_h(X)))$ and $v^{'} = \Gamma_2(x)$ for $x \in B_r(C_h(X))$. Initially, we can infer that $V_b^{'}(t)$ is a relatively compact subset of X for each $t \in [0,b]$. Infact, using condition (vi) we get that $N = \{S_\alpha(s)f(\xi, h(\xi)x, K(h(\xi)x)) : 0 \leq s \leq t, 0 \leq \xi \leq t, \|x\| \leq r\}$ is relatively compact. It is each to see that $V_b^{'}(t) \subset S_\alpha(t)[x_0 - k_1(0,x_0)] + k_1(t,x(t)) + t\overline{C(N)}$, which establishes our assertion.

From the decomposition of

$$\begin{aligned}v^{'}(t+s) - v^{'}(t) = {}& [S_\alpha(t+s) - S_\alpha(t)][x_0 - k_1(0,x_0)] + k_1(t+s, x(t+s)) - k_1(t,x(t))\\ &+ \int_t^{t+s} S_\alpha(t+s-\xi)f(\xi, x(\xi), Kx(\xi))d\xi\\ &+ \int_0^t [S_\alpha(\xi+s) - S_\alpha(\xi)]f(t-\xi, x(t-\xi), Kx(t-\xi))d\xi,\end{aligned}$$

it follows that the set $V_b^{'}$ is equicontinuous.

From the condition (i), we have,

$$\begin{aligned}\frac{\|v^{'}(t)\|}{h(t)} &\leq \frac{1}{h(t)}\Big[S_\alpha(t)[x_0 - k_1(0,x_0)] + k_1(t,x(t))\Big] + \frac{C(\theta,\alpha)M}{h(t)}\int_0^t \frac{W((1+\|K\|)rh(s)}{1+|\omega|(t-s)^\alpha}ds\\ &\to 0 \qquad \text{as} \qquad t \to \infty.\end{aligned}$$

From Lemma 2.3, we deduce that, $V^{'}$ is relatively compact set in $C_h(X)$.

Let us denote $x^\lambda(\cdot)$ be a solution of equation $x^\lambda = \lambda\Gamma(x^\lambda)$ for some $\lambda \in (0,1)$. Now using the estimate,

$$\begin{aligned}\|x^\lambda\|_h &\leq \|S_\alpha(t)[x_0 - k_1(0,x_0)\| + \|k_1(t, \|x^\lambda(t)\|_h)\|\\ &\quad + C(\theta,\alpha)M\int_0^t \frac{W((1+\|K\|)r\|x^\lambda\|_h h(s)}{1+|\omega|(t-s)^\alpha}ds\\ &\leq \beta(\|x^\lambda\|_h),\end{aligned}$$

we get, $\frac{\|x^\lambda\|_h}{\beta(\|x^\lambda\|_h)} \leq 1$. Using the condition (v) of Theorem 3.2, we have $\{x^\lambda : x^\lambda = \lambda\Gamma(x^\lambda)\}, \lambda \in (0,1)$ is bounded. From Lemmas 3.1 and 3.2, we have that

$$\Gamma_i(AAA(R^+ \times X, X)) \subset AAA((R^+ \times X, X)), i = 1, 2.$$

Hence, $\Gamma(AAA(R^+ \times X, X)) \subset AAA((R^+ \times X, X))$ and $\Gamma_2 : (AAA(R^+ \times X, X)) \to AAA((R^+ \times X, X))$ is completely continuous.

Putting $B_r := B_r(AAA(R^+ \times X, X))$, we claim that there is $r > 0$ such that $\Gamma(B_r) \subset B_r$. In fact, if we assume that this assertion is false, then for all $r > 0$ we can choose $x^r \in B_r$ and $t^r \geq 0$ such that $\|\Gamma x^r(t^r)\|/h(t^r) > r$. We observe that

$$\|\Gamma x^r(t^r)\| \leq C(\theta, \alpha)M(\|x_0\| + \|k_1(0, x_0)\|) + L_1 r + \|k_1(\cdot, 0)\|_\infty + C(\theta, \alpha)M \int_0^{t^r} \frac{W((1 + \|K\|)rh(s)}{1 + |\omega|(t^r - s)^\alpha} ds.$$

Thus, $1 \leq L_1 + \liminf_{r\to\infty} \frac{\Omega(r)}{r}$, which is contrary to assumption (iv). We have that Γ_1 is a contraction on B_r and $\Gamma_2(B_r)$ is a compact set. It follows from [30] [Corollary 4.3.2] that Γ has a fixed point $x \in AAA(R^+ \times X, X)$. More precisely, $x \in AAA(R^+ \times X, X)$ and this finishes the proof. □

4 Example

Example 1. *Consider the following example for the Theorem 3.1.*

$$\begin{aligned} \partial_t^\alpha[w(t, x) - k_1(t, w(t, x))] &= \partial_x^2[w(t, x) - k_1(t, w(t, x))] - \mu w(t, x) \\ &+ \partial_t^{\alpha-1}\Big[\beta w(t, x)(\cos t + \cos \sqrt{2}t) + \beta e^{-t} \sin(w(t, x)) \\ &+ \sin\Big(\int_0^t e^{t-s} h(s, w(t, s))ds\Big)\Big], \quad t \geq 0, \, x \in [0, \pi], \\ w(t, 0) = w(t, \pi) = 0, &\quad t \geq 0, \, \mu > 0, \end{aligned}$$

where $1 < \alpha < 2$ and $w_0 \in L^2[0, \pi]$. Define the linear operator A on $X = (L^2([0, \pi]), |\cdot|_2)$ by $Aw = w'' - \mu w$ with the domain

$$D(A) = \{w \in X : w'' \in X, w(0) = w(\pi) = 0\}.$$

It is known that $\Delta w = w''$ is the infinitesimal generator of analytic semigroup on $L^2[0, \pi]$ and thus A is sectorial of type $w = -\mu < 0$. Denote $w(t)x = w(t, x)$ and

$$f(t, w, Kw)(x) = \beta w(x)(\cos t + \cos \sqrt{2}t) + \beta e^{-|t|} \sin(w(x)) + \sin(Kw(x))$$

for each $w \in X$. One can easily see that the function $f(t, x, Kx)$ is asymptotically almost automorphic in t for each $x \in X$. Now under the condition

$$\beta + 1 < \frac{\alpha \sin(\pi/\alpha)}{3C(\theta, \alpha)M\mu^{-1/\alpha}\pi} - L_1,$$

there exists an unique asymptotically almost automorphic mild solution.

Example 2. *One can also consider the following fractional order delay relaxation oscillation equation for $\alpha \in (1, 2)$,*

$$\frac{\partial^\alpha}{\partial t^\alpha}(u(t, x) - k_1(t, u(t, x))) = \frac{\partial^2}{\partial x^2}((u(t, x) - k_1(t, u(t, x)))) - pu(t, x)$$

$$+\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}(f(t,u(t,x),u(t-\tau,x))),\ \tau>0,$$
$$t\in R,\ x\in(0,\pi)$$
$$\begin{aligned} u(t,0) &= u(t,\pi)=0, \quad t\in R,\\ u(t,x) &= \phi(t,x) \quad t\in[-\tau,0], \end{aligned} \tag{4.1}$$

where $p>0$ and f is a asymptotic almost automorphic function in t. Also assume that f satisfies Lipschitz condition in both variable with Lipschitz constants L_2, L_3. Note that $\int_{-\infty}^{t} k(t-s)h(s,u(s))ds = \int_{-\infty}^{t} k(-s)h(s,u_t(s))ds = J(u_t)$, which can be thought like function of u_t and hence can be considered as functional differential equations. Using the transformation $u(t)x=u(t,x)$ and define $Au=\frac{\partial^2 u}{\partial x^2}-pu,\ u\in D(A)$, where

$$D(A)=\Big\{u\in L^2((0,\pi),R), u^{'}\in L^2((0,\pi),R), u^{''}\in L^2((0,\pi),R), u(0)=u(\pi)=0\Big\},$$

the above equation can be transform into

$$\frac{d^{\alpha}}{dt^{\alpha}}(u(t)-k_1(t,u(t)))=A(u(t)-k_1(t,u(t)))+\frac{d^{\alpha-1}}{dt^{\alpha-1}}g(t,u(t),u_t(-\tau)), \tag{4.2}$$

$t\in R$ and $u(t)=\phi(t)$ $t\in[-\tau,0]$. It is to note that A generates an analytic semigroup $\{T(t):t\geq 0$ on X, where $X=L^2((0,\pi),R)$. Hence $pI-A$ is sectorial of type $\omega=-p<0$. Further A has discrete spectrum with eigenvalues of the form $-k^2; k\in N$, and corresponding normalized eigenfunctions given by $z_k(x)=(\frac{2}{\pi})^{\frac{1}{2}}\sin(kx)$. As A is analytic. Thus under all the required assumption on f, the existence of asymptotic almost automorphic solutions is ensured accordingly.

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