Invariants concerning f-domination in graphs

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Abstract

Given a graph G and a function $f : V(G) \to \{0, 1, 2, \ldots\}$, a subset D of $V(G)$ is called an f-dominating set of G if every vertex x outside D is adjacent to at least $f(x)$ vertices in D. In this article we study a few graphical invariants relating to this concept.

Keywords: domination number; f-domination; connected f-domination; total f-domination; bondage number

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1 Introduction

Let $G = (V(G), E(G))$ be a connected graph with order $p(G) := |V(G)|$ and size $q(G) := |E(G)|$, and let $f : V(G) \to \mathbb{Z}_0 := \{0, 1, 2, \ldots\}$ be a function. A subset D of $V(G)$ is called an *f-dominating set* [19, 21] if $|N(x) \cap D| \geq f(x)$ for every $x \in V(G) - D$, where $N(x)$ is the neighbourhood of x in G. If in addition the subgraph $G[D]$ of G induced by D is connected, then D is called a *connected f-dominating set* [22] of G. The *f-domination number* [21] of G, denoted $\gamma_f(G)$, is the minimum cardinality of an f-dominating set of G; and the *connected f-domination number* [22] $\gamma_{c,f}(G)$ is defined similarly. Denote by $d(x) := |N(x)|$ the degree of x in G. A function f is *proper* [22] if $1 \leq f(x) \leq d(x)$ for every $x \in V(G)$. For a proper f, a subset D of $V(G)$ is called a *total f-dominating set* [22] of G if $|N(x) \cap D| \geq f(x)$ for every $x \in V(G)$; the minimum cardinality of such a set is called the *total f-domination number* [22] of G and is denoted by $\gamma_{t,f}(G)$. A subset X of $V(G)$ is an *f-independent set* [21] of G if $|N(x) \cap X| < f(x)$ for every $x \in X$; the maximum cardinality of such a set is called the *f-independence number* [21] of G and is denoted by $\beta_f(G)$. A subset X of $V(G)$ is an *f-transversal* [21] of G if it has non-empty intersection with every non-f-dominating subset of $V(G)$. The *f-transversal number* $\alpha_f(G)$ [21] is the minimum cardinality of an f-transversal of G.

The concepts above are natural generalizations of the corresponding notions involving k-domination and k-independence [6]. In fact, in the special case when $f = k$ (that is, $f(x) = k$ for all $x \in V(G)$) for a fixed integer $k \geq 1$, $\gamma_f(G)$, $\gamma_{c,k}(G) := \gamma_{c,f}(G)$ and $\gamma_{t,k}(G) := \gamma_{t,f}(G)$ are precisely the *k-domination number* $\gamma_k(G)$ [11, 12], the *connected k-domination number* [20] and the *total k-domination number* [6, Section 4] of G, respectively. In particular, $\gamma_1(G)$ $(= \gamma(G))$, $\gamma_{c,1}(G)$ $(= \gamma_c(G))$ and $\gamma_{t,1}(G)$ $(= \gamma_t(G))$ are respectively the ordinary domination, connected domination and total domination numbers [15, 16] of G, and $\alpha_1(G)$ $(= \alpha(G))$ and $\beta_1(G)$ $(= \beta(G))$

are respectively the vertex covering number and the independence number of G in the usual sense.

The concept of f-domination was proposed in [14] for trees, and for general graphs it was evolved from [19], where Stracke and Volkmann studied γ_f for a specific function f. In [21] the author proved among other things that, for any $f : V(G) \to \mathbb{Z}_0$, every f-independent set X of G with $\sum_{x\in X} f(x) - q(G[X])$ as large as possible is an f-dominating set of G. As a consequence we have [21] $\gamma_f(G) \le \beta_f(G)$, which was conjectured in [11, 12] and proved in [9] when $f = k$. In [22] the author established a few inequalities among $\gamma_f(G)$, $\gamma_{c,f}(G)$, $\gamma_{t,f}(G)$ and $i(G)$, where $i(G)$ is the *independence domination number* [16] of G, that is, the minimum cardinality of a subset of $V(G)$ that is both dominating and independent. In [7] it was proved that, if $f(x) < \frac{n}{n+1}\left(d(x) + 1 + \frac{1}{n}\right)$ for every $x \in V(G)$ and a fixed integer $n \ge 1$, then $\gamma_f(G) \le \frac{n}{n+1} \cdot p(G)$. (This is a generalization of Ore's inequality $\gamma(G) \le \frac{1}{2}p(G)$ [16] for any G without isolated vertices.) In [23] several Gallai-type equalities invloving $\gamma_f(G)$ and $\gamma_{c,f}(G)$ were proved. For example, $\gamma_f(G) + \beta_{f^*}(G) = p(G)$ for any proper $f : V(G) \to \mathbb{Z}_0$, where f^* is defined by $f^*(x) = d(x) - f(x) + 1$ for $x \in V(G)$.

So far a number of results on $\gamma_k, \gamma_{c,k}, \gamma_{t,k}$ and several other invariants involving k-domination and k-independence have been obtained by various researchers, as shown in a recent survey [6]. In contrast, our understanding to more general invariants involving f-domination and f-independence is very limited. In this article we give a lower bound for each of $\gamma_{c,f}$ and $\gamma_{t,f}$ (Theorem 2.1), answer a question about relations between $\gamma_{c,f}$ and $\gamma_{t,f}$ (Theorem 2.4), and prove a few upper bounds on the f-bondage number of G (Section 3).

Throughout the paper G is a connected graph with $p(G) \ge 2$ and $f : V(G) \to \mathbb{Z}_0$ is a function. As usual, $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of G, respectively. For $X \subseteq V(G)$, denote

$$M(f) = \max_{x\in V(G)} f(x), \quad f(X) = \sum_{x\in X} f(x), \quad d(X) = \sum_{x\in X} d(x).$$

The reader is referred to [4] for undefined notation and terminology on graphs. For some recent progresses on a few generalized domination concepts, the reader is referred to [1, 2, 10, 18].

2 Connected and total f-domination numbers

We first prove the following results, where the known bound on $\gamma_f(G)$ is stated for the purpose of comparison.

Theorem 2.1. *The following inequalities hold:*

(a) $\gamma_f(G) \ge \left\lceil \frac{f(V(G))-q(G)}{M(f)} \right\rceil$ *([21, Proposition 3]);*

(b) *if* $M(f) \ge 2$, *then* $\gamma_{c,f}(G) \ge \max\left\{ \left\lceil \frac{f(V(G))-q(G)-1}{M(f)-1} \right\rceil, \left\lceil \frac{f(V(G))-2}{M(f)+\Delta(G)-2} \right\rceil \right\}$*;*

(c) *if* f *is proper, then* $\gamma_{t,f}(G) \ge \max\left\{ \left\lceil \frac{2(f(V(G))-q(G))}{M(f)} \right\rceil, \left\lceil \frac{f(V(G))}{\Delta(G)} \right\rceil \right\}$.

Proof We prove (b) first. Since $M(f) \geq 2$, we have $M(f)+\Delta(G)-2 \geq 1$. Let D be a connected f-dominating set of G with $|D| = \gamma_{c,f}(G)$. Then $q(G[D]) \geq |D|-1$ as $G[D]$ is connected. Since D is f-dominating, each $x \in V(G)-D$ contributes at least $f(x)$ edges with one end-vertex x and the other end-vertex in D. Thus, $q(G) \geq f(V(G)-D)+|D|-1 \geq (f(V(G)) - M(f)|D|)+|D|-1$, yielding

$$\gamma_{c,f}(G) = |D| \geq (f(V(G)) - q(G) - 1)/(M(f) - 1).$$

Since $G[D]$ is connected, summing up the degrees of the vertices in D we obtain $f(V(G) - D) + 2(|D| - 1) \leq d(D)$, which implies $f(V(G)) - M(f)|D| + 2(|D| - 1) \leq \Delta(G)|D|$. Therefore,

$$\gamma_{c,f}(G) \geq (f(V(G)) - 2)/(M(f) + \Delta(G) - 2)$$

and the proof of (b) is complete.

Now we prove (c). Let D be a minimum total f-dominating set of G. Then $|N(x)\cap D| \geq f(x)$ for $x \in D$, and hence $q(G[D]) \geq f(D)/2$. Since D is f-dominating, we then have $q(G) \geq f(V(G) - D) + q(G[D]) \geq f(V(G)) - (f(D)/2) \geq f(V(G)) - (M(f)|D|/2)$. Hence

$$\gamma_{t,f}(G) = |D| \geq \frac{2(f(V(G)) - q(G))}{M(f)}.$$

Computing the sum of the degrees of the vertices in D, we obtain

$$\begin{aligned}
\Delta(G)\gamma_{t,f}(G) &\geq d(D) \\
&= \sum_{x\in D} (|N(x) \cap D| + |N(x) \cap (V(G) - D)|) \\
&\geq f(D) + \sum_{x\in V(G)-D} |N(x) \cap D| \\
&\geq f(D) + f(V(G) - D) \\
&= f(V(G)).
\end{aligned}$$

Thus $\gamma_{t,f}(G) \geq f(V(G))/\Delta(G)$ and (c) is proved. □

Theorem 2.1 implies the following new lower bounds on $\gamma_{c,k}$ and $\gamma_{t,k}$, where $\rho(G) := q(G) - p(G) + 1$ is the cyclomatic number of G.

Corollary 2.2. *The following inequalities hold:*

(a) $\gamma_k(G) \geq p(G) - \left\lfloor \frac{q(G)}{k} \right\rfloor$ *([11, Theorem 4]);*

(b) *if* $k \geq 2$, *then* $\gamma_{c,k}(G) \geq \max\left\{ \left\lceil p(G) - \frac{\rho(G)}{k-1} \right\rceil, \left\lceil \frac{kp(G)-2}{\Delta(G)+k-2} \right\rceil \right\}$ *([20]);*

(c) *if* $1 \leq k \leq \delta(G)$, *then* $\gamma_{t,k}(G) \geq \max\left\{ \left\lceil 2\left(p(G) - \frac{q(G)}{k}\right) \right\rceil, \left\lceil \frac{kp(G)}{\Delta(G)} \right\rceil \right\}$.

The following example shows that the lower bounds in Theorem 2.1 and Corollary 2.2 are all attainable.

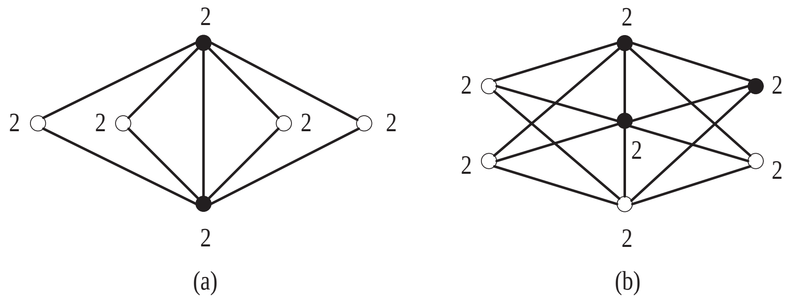

Figure 1: (a) A graph with connected 2-domination number 2; (b) a graph with total 2-domination number 3.

Example 2.3. Let $K_{1,n}$ be the star with $n+1 \geq 3$ vertices. Let $f(x) = n$ if x is the center of $K_{1,n}$ and $f(x) = 1$ otherwise. Then f is proper and $\gamma_f(K_{1,n}) = 1$. The lower bound in (a) of Theorem 2.1 for $(K_{1,n}, f)$ is 1 and hence is sharp.

Let G be the graph in Figure 1(a), and let $f(x) = 2$ for every $x \in V(G)$. Then the two filled vertices form a minimum connected f-dominating set. Hence $\gamma_{c,f}(G) = 2$, which agrees with the lower bound in (b) of Theorem 2.1. For the graph in Figure 1(b) with $f(x) = 2$ for all x, both sides of the inequality (c) in Theorem 2.1 are equal to 3, and the three filled vertices form a minimum total f-dominating set. $\square$

It is well-known [17] that, for any connected graph G with $\Delta(G) < p(G) - 1$,

$$\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G). \tag{1}$$

Clearly, we have $\gamma_f(G) \leq \gamma_{c,f}(G)$ and $\gamma_f(G) \leq \gamma_{t,f}(G)$ when f is proper. In view of (1), it is natural to ask whether we always have $\gamma_{t,f}(G) \leq \gamma_{c,f}(G)$ for any proper function f. The next result shows that this is not the case in general, and the two invariants are incomparable. Moreover, even for trees one of these invariants can exceed the other by an arbitrary integer.

Theorem 2.4. *For any non-negative integer n,*

(a) *there exist a tree T_1 and a proper $f : V(T_1) \to \mathbb{Z}_0$ such that $\gamma_{c,f}(T_1) - \gamma_{t,f}(T_1) = n$;*

(b) *there exist a tree T_2 and a proper $f : V(T_2) \to \mathbb{Z}_0$ such that $\gamma_{t,f}(T_2) - \gamma_{c,f}(T_2) = n$.*

Proof (a) Let P be the path P_{2n+6} on $2n+6$ vertices if n is even, and the path P_{2n+7} on $2n+7$ vertices if n is odd. Denote the vertices of P by $x_1, x_2, \ldots, x_m$, where $m = |V(P)|$. Define T_1 to be the tree obtained from P by attaching three pendant edges to each of x_1 and x_m (see Figure 2). Let x_0 and x_{m+1} be two pendant vertices adjacent in T_1 to x_1 and x_m, respectively. Let $f(x_1) = f(x_m) = 2$, and $f(x) = 1$ for all other vertices x of T_1. Then the vertices on P form a minimum connected f-dominating set of T_1. Hence

$$\gamma_{c,f}(T_1) = \begin{cases} 2n+6, & \text{if } n \text{ is even} \\ 2n+7, & \text{if } n \text{ is odd.} \end{cases}$$

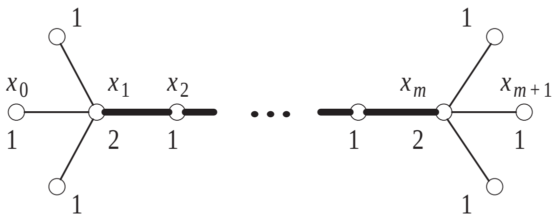

Figure 2: Proof of Theorem 2.4(a).

We claim that both x_1 and x_m must be in any total f-dominating set D of T. In fact, if $x_1 \notin D$, then $x_0 \in D$, but x_0 has no neighbour in D, a contradiction. Hence $x_1 \in D$. Similarly, $x_m \in D$. Based on this one can show that, if n is even, then

$$\{x_i, x_{i+1} : 1 \leq i \leq 2n+6,\ i \equiv 1 \pmod 4\} \cup \{x_0, x_{m+1}\}$$

is a minimum total f-dominating set of T_1. Similarly, if n is odd, then

$$\{x_i, x_{i+1} : 1 \leq i \leq 2n+6,\ i \equiv 1 \pmod 4\} \cup \{x_0, x_{m-1}, x_m, x_{m+1}\}$$

is a minimum total f-dominating set of T_1. Thus

$$\gamma_{t,f}(T_1) = \begin{cases} n+6, & \text{if } n \text{ is even} \\ n+7, & \text{if } n \text{ is odd.} \end{cases}$$

It then follows that $\gamma_{c,f}(T_1) - \gamma_{t,f}(T_1) = n$ regardless of the parity of n.

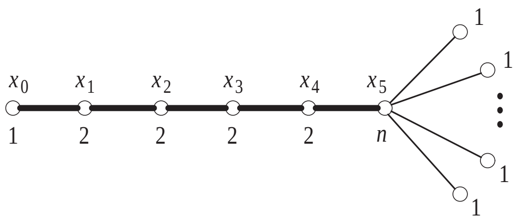

Figure 3: Proof of Theorem 2.4(b).

(b) Let $P_6 = x_0x_1x_2x_3x_4x_5$ be a path of six vertices. Let T_2 be the tree obtained from P_6 by attaching n pendant edges to x_5 (see Figure 3). Let $f(x) = 1$ if x is a pendant vertex of T_2, and let $f(x_5) = n$ and $f(x_i) = 2$ for $1 \leq i \leq 4$. Then $\{x_1, x_2, \ldots, x_5\}$ is a minimum connected f-dominating set, and so $\gamma_{c,f}(T_2) = 5$. For any total f-dominating set D of T_2, since $f(x_i) = d(x_i) = 2$ for $1 \leq i \leq 4$, we have $\{x_0, x_1, \ldots, x_4, x_5\} \subseteq D$. Since $f(x_5) = d(x_5) - 1 = n$, at most one of the pendant vertices adjacent to x_5 is not in D. It follows that $|D| \geq n+5$. On the other hand, for any pendant vertex x adjacent to x_5, $V(T_2) - \{x\}$ is a total f-dominating set of T_2. Hence $\gamma_{t,f}(T_2) = n+5$ and $\gamma_{t,f}(T_2) - \gamma_{c,f}(T_2) = n$. □

3 Bounding the f-bondage number

In this section we investigate perturbation of $\gamma_f(G)$ under deletion of edges of G. This leads to a new invariant, namely the f-bondage number of a graph. As usual, denote by $G - E$

the spanning subgraph obtained from G by deleting all edges in a given subset $E \subseteq E(G)$; in particular, we use $G - e$ in place of $G - \{e\}$ for $e \in E(G)$. The notion of f-bondage number is based on the following observation.

Lemma 3.1. *For any $e \in E(G)$, we have*

$$\gamma_f(G) \leq \gamma_f(G - e) \leq \gamma_f(G) + 1.$$

Proof Since any f-dominating set of $G - e$ is an f-dominating set of G, we have $\gamma_f(G) \leq \gamma_f(G - e)$ immediately. (Note that f may not be a proper function for $G - e$ even if f is proper for G.) Let D be a minimum f-dominating set of G, and x and y the end-vertices of e. If both x and y are in D, or both of them are in $V(G) - D$, then D is also an f-dominating set of $G - e$. In this case we have $\gamma_f(G - e) \leq \gamma_f(G)$ and hence $\gamma_f(G) = \gamma_f(G - e)$. Assume then that exactly one of x, y is in D, say, $y \in D$ and $x \in V(G) - D$. Then $D \cup \{x\}$ is an f-dominating set of $G - e$ and so $\gamma_f(G - e) \leq \gamma_f(G) + 1$. □

By Lemma 3.1 the deletion of any edge from G either keeps γ_f or increases γ_f by one. We may ask at least how many edges one must delete from G so that the f-domination number increases. For the ordinary domination number, this problem was studied by Bauer, Harary, Nieminen and Suffel [3], who called the minimum number of edges required the *bondage number* of G and denoted it by $b(G)$. In general, we define the *f-bondage number* of G, $b_f(G)$, to be the minimum cardinality of a subset E of $E(G)$ such that $\gamma_f(G - E) > \gamma_f(G)$. By Lemma 3.1, for a minimum cardinality E with $\gamma_f(G - E) > \gamma_f(G)$, we must have $\gamma_f(G - E) = \gamma_f(G) + 1$.

Theorem 3.2. *Let $f : V(G) \to \mathbb{Z}_0$ be a proper function. Then*

$$b_f(G) \leq \min\{M(f)\gamma_f(G) + q(G) - f(V(G)) + 1,\ q(G)\}. \tag{2}$$

Proof We may assume that $M(f)\gamma_f(G) + q(G) - f(V(G)) + 1 < q(G)$ for otherwise the bound is trivial. Let E be a set of edges of G with $|E| = M(f)\gamma_f(G) + q(G) - f(V(G)) + 1$. By Theorem 2.1(a) we have

$$\begin{aligned}
\gamma_f(G - E) &\geq \left\lceil \frac{f(V(G)) - q(G - E)}{M(f)} \right\rceil \\
&= \left\lceil \frac{f(V(G)) - q(G) + |E|}{M(f)} \right\rceil \\
&\geq \left\lceil \gamma_f(G) + \frac{1}{M(f)} \right\rceil \\
&> \gamma_f(G).
\end{aligned}$$

Thus $b_f(G) \leq |E|$ as required. □

By (2), any upper bound on $\gamma_f(G)$ gives rise to an upper bound on $b_f(G)$. In particular, Theorem 3.2 and two upper bounds in [21] on $\gamma_f(G)$ imply the following bounds.

Corollary 3.3. *Let $f : V(G) \to \mathbb{Z}_0$ be a proper function. Then*

(a) $b_f(G) \leq \min\{M(f)\alpha(G) + q(G) - f(V(G)) + 1,\ q(G)\}$;

(b) $b_f(G) \le \min\{M(f)\beta_f(G)+q(G)-f(V(G))+1,\ q(G)\}$;

(c) *if* $f(x) \le \frac{d(x)+1}{2}$ *for all* $x \in V(G)$, *then* $b_f(G) \le \min\left\{\frac{M(f)p(G)}{2}+q(G)-f(V(G))+1,\ q(G)\right\}$.

Proof (a) For a maximum independence set X of G, we have $N(x) \subseteq V(G)-X$ for every $x \in X$. Since $f(x) \le d(x)$ for all $x \in V(G)$, it follows that $V(G)-X$ is an f-dominating set of G. Hence $\gamma_f(G) \le p(G)-|X| = p(G)-\beta(G) = \alpha(G)$ and the result follows from (2).

(b) This follows from (2) and $\gamma_f(G) \le \beta_f(G)$ [21].

(c) Since $f(x) \le (d(x)+1)/2$, $x \in V(G)$, we have $\gamma_f(G) \le p(G)/2$ by [21, Corollary 4]. From this and (2) the bound in (c) follows. □

It is well-known that $\alpha(G)+\beta(G) = p(G)$ [13] for any graph G. In general, we have $\alpha_f(G)+\beta_f(G) = p(G)$ [21, Theorem 3]. Using these two equalities and Corollary 3.3, we obtain the following upper bounds on the k-bondage number of a graph.

Corollary 3.4. *Let k be an integer with $1 \le k \le \delta(G)$. Then*

$$b_k(G) \le \min\{q(G)-k\beta(G)+1, q(G)-k\alpha_k(G)+1\}. \tag{3}$$

In particular,

$$b(G) \le \rho(G)+\min\{\alpha(G),\beta(G)\}. \tag{4}$$

Note that $q(G)-k\alpha_k(G)+1 \ge 1$, for otherwise we would have $b_f(G) \le 0$, which is impossible. Thus, as a by-product of Corollary 3.4, we obtain the following upper bound on the k-transversal number $\alpha_k(G)$.

Corollary 3.5. *Let k be an integer with $1 \le k \le \delta(G)$. Then*

$$\alpha_k(G) \le \left\lfloor \frac{q(G)}{k} \right\rfloor.$$

All upper bounds above for b_f, b_k and b are attainable. For example, if $G = K_{2,4}$ and $f(x) = 2$ for each vertex x, then $\gamma_f(G) = 2$, $b_f(G) = 1$, and the right-hand sides of (2), (3) and Corollary 3.3(a) are all equal to 1. Thus these three bounds are all attainable. If $G = C_4$ (cycle of length 4) and $f(x) = 1$ for each vertex x, then $\gamma_f(G) = 2$, $b_f(G) = b(G) = 3$, and the right-hand sides of (4) and (b)-(c) in Corollary 3.3 are all equal to 3. Hence these three bounds are attainable as well.

The bounds on b_f above are valid for any connected graph G with $p(G) \ge 2$ and any proper function $f : V(G) \to \mathbb{Z}_0$, and the bounds on b_k are valid for all positive integers k up to $\delta(G)$. Due to this generality it is unrealistic to expect that these bounds are tight in all occasions, though they can be sharp in some cases as seen above. In fact, all the bounds above might be poor for some pairs (G, f) and (G, k). It seems challenging to obtain good estimates of $b_f(G)$ for general (G, f). However, for trees we have the following result, which is a generalization of [3, Proposition 13] asserting that $b(T) \le 2$ for any tree T.

Theorem 3.6. *Let T be a tree with $p(T) \ge 2$. Then, for any proper $f : V(T) \to \mathbb{Z}_0$, we have*

$$1 \le b_f(T) \le 2.$$

Proof Since f is proper, we have $1 \le f(x) \le d(x)$ for each $x \in V(T)$. In particular, if x is a pendant vertex of T, then $f(x) = 1$. Since the result is trivial when $p(T) \le 3$, we assume $p(T) \ge 4$ in the following.

Case 1: T contains a vertex y which is adjacent to at least two pendant vertices of T, say, x_1, x_2. In this case y must be in every minimum f-dominating set of T. Setting $e = x_1 y$, we claim that $\gamma_f(T - e) = \gamma_f(T) + 1$. Suppose otherwise, that is, $\gamma_f(T - e) = \gamma_f(T)$, and let D be a minimum f-dominating set of $T - e$. Then $x_1 \in D$, and either x_2 or y is in D. If $x_2 \in D$ but $y \notin D$, then $(D - \{x_2\}) \cup \{y\}$ is also an f-dominating set of $T - e$. Thus, without loss of generality we may suppose $y \in D$ and $x_2 \notin D$. But then $D - \{x_1\}$ is an f-dominating set of T with $\gamma_f(T) - 1$ vertices, a contradiction. Thus $\gamma_f(T - e) = \gamma_f(T) + 1$ and $b_f(T) = 1$.

Case 2: Each vertex of T is adjacent to at most one pendant vertex. In this case T must contain a pendant vertex x which is adjacent to a vertex y of degree two. Let z be the vertex other than x which is adjacent to y, and let $e = yz$. If $\gamma_f(T - e) > \gamma_f(T)$, then $b_f(G) \le 1$. Otherwise, we have $\gamma_f(T - \{e, e'\}) > \gamma_f(T)$ where $e' = xy$, and hence $b_f(G) \le 2$. □

We have the following immediate consequence of Theorem 3.6.

Corollary 3.7. $b(G) \le \rho(G) + 2$.

Theorem 3.6 suggests that graphs with small cyclomatic number may have small f-bondage number. This is supported by Corollary 3.7 for the ordinary bondage number b. The following result confirms this for *unicyclic graphs*, namely graphs having a unique cycle.

Theorem 3.8. *Let G be a connected unicyclic graph. Then for any proper function $f : V(G) \to \mathbb{Z}_0$ we have*

$$1 \le b_f(G) \le 3.$$

Proof Let C denote the unique cycle of G. If G contains a vertex which is adjacent to at least two pendant vertices, or if G contains a pendant vertex which is adjacent to a degree-two vertex, then by a similar argument as in the proof of Theorem 3.6 we can prove that $b_f(G) \le 2$. In the remaining case, C must be a dominating cycle, that is, each vertex of G outside C is adjacent to at least one vertex on C. Moreover, every vertex on C is adjacent to at most one vertex outside C.

Case 1: $G = C$. In this case, if $f(x) = 1$ for all $x \in V(G)$, then it is not difficult to verify that $b_f(G) \le 3$. Suppose then $f(x) \ge 2$ for some vertex x, and let x, y, z, w be consecutive vertices on C. Let H be the graph obtained from G by deleting the edges xy, yz and zw. Let D be a smallest f-dominating set of H. Since x is pendant and y, z are isolated in H, and since $f(x) = 2$, $f(y) \ge 1$ and $f(z) \ge 1$, we must have $x, y, z \in D$. So $D - \{y\}$ is an f-dominating set of G. Hence $\gamma_f(G) < \gamma_f(H)$ and consequently $b_f(G) \le 3$.

Case 2: $G \ne C$. In this case, since C is the unique cycle of G, any two non-consecutive vertices on C are not adjacent. Hence there is a pendant vertex x which is adjacent to a vertex y in C. Let z and w be the two neighbours of y on C. Let H be the graph obtained from G by deleting the edges xy, yz and zw. Then $\gamma_f(G) < \gamma_f(H)$ and hence $b_f(G) \le 3$. □

4 Remarks

In view of [6] a number of problems concerning f-domination and f-independence may be studied. In [21, Theorem 4] it was proved that, for any $f : V(G) \to \mathbb{Z}_0$, there exists a subset of $V(G)$ that is both f-dominating and f-independent. Thus we may define $i_f(G)$ to be the minimum cardinality of such a subset and call it the *independence f-domination number* of G. A subset of $V(G)$ that is both f-dominating and f-independent must be a maximal f-independent set (with respect to the set-theoretic inclusion), but the converse is not true [22]. Thus $i_f(G)$ may not be the same as the minimum cardinality $i'_f(G)$ of a maximal f-dominating set of G. This is different from the case of ordinary domination and independence for which $i'_1(G) = i_1(G) = i(G)$. It is natural to ask under what conditions we have $i'_f(G) = i_f(G)$.

By [21, Theorem 4],

$$\gamma_f(G) \le i'_f(G) \le i_f(G) \le \beta_f(G).$$

This is the analogy of part of the following well-known domination chain [16]:

$$ir(G) \le \gamma(G) \le i(G) \le \beta(G) \le \Gamma(G) \le Ir(G). \tag{5}$$

In [21] the *upper f-domination number* $\Gamma_f(G)$ was defined to be the maximum cardinality of a minimal f-dominating set of G. In the case where $f = 1$, this is exactly the upper domination number $\Gamma(G)$ of G in the usual sense [16]. The inequality $\beta(G) \le \Gamma(G)$ in (5) is based on the fact [16] that any maximal independent set is a minimal dominating set. Since similar statement is not true for f-domination and f-independence, we do not know whether $\beta_f(G) \le \Gamma_f(G)$ holds in general. In fact, as far as we know, there is no any result in the literature about $\Gamma_f(G)$, $i_f(G)$ and $i'_f(G)$ for general (G, f). Finally, one may study invariants involving f-domination and f-independence for various special functions f.

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