Finite groups with some generalized CAP-subgroups[1]

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Abstract A subgroup H of a finite group G is called to be a CAP*-subgroup of G if H either covers or avoids every non-Frattini chief factor of G. In this paper, we study the influence of the CAP*-subgroups of a finite group G on the structure of G and some recent results were extended.

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1 Introduction

All groups considered in this paper are finite. We use conventional notions and notation, as in Huppert[8]. G always denotes a finite group, $|G|$ the order of G, $\pi(G)$ the set of all primes dividing $|G|$, G_p a Sylow p-subgroup of G for some $p \in \pi(G)$. For clearity, some times we denote the factor group G/N by $\frac{G}{N}$.

Let K and L be normal subgroups of a group G with $K \leq L$. Then K/L is called a *normal factor* of G. A subgroup H of G is said to *cover* K/L if $HK = HL$. On the other hand, if $H \cap K = H \cap L$, then H is said to *avoid* K/L. If K/L is a chief factor of G and $K/L \leq \Phi(G/L)$ (respectively $K/L \not\leq \Phi(G/L)$), then K/L is said to be a *Frattini (respectively non-Frattini) chief factor* of G.

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ *a formation* provided that (i) if $G \in \mathcal{F}$ and $H \lhd G$, then $G/H \in \mathcal{F}$, and (ii) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is saturated formations(ref. [8]).

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The $\mathcal{U}$-hypercenter of G, denote by $Z_{\mathcal{U}}(G)$, is the the product of all normal subgroups H of G such that all G-chief factors of H have prime order. Following [15], the product of all normal subgroups H of G such that all non-Frattini G-chief factors of H have prime order was denoted by $Z_{\mathcal{U}\phi}(G)$.

A subgroup H of a group G is said to have *the cover-avoiding property* in G if H covers or avoids every chief factor of G, in short, H is called to be *a CAP-subgroup* of G ([4]).

In the literature, many people studied the cover-avoidance property of subgroups of finite groups, for example, Gaschütz ([6]), Gillam ([7]), Tomkinson ([16]) and Petrillo ([14]), etc.. By an obvious consequence of the definition of supersolvable group every subgroup of supersolvable group is a CAP-subgroup. In 1993, Ezquerro has proved the converse result ([5, Theorem C and D]): Let G be a group with a normal subgroup H such that G/H is supersolvable. Then G is supersolvable if one of the following holds: (1) all maximal subgroups of the Sylow subgroups of H are CAP-subgroups of G; (2) H is solvable and all maximal subgroups of the Sylow subgroups of $F(H)$ are CAP-subgroups of G. Li and Wang extended Ezquerro's result as follows ([12]): Let G be a group with a normal subgroup H such that G/H is supersolvable. Then G is supersolvable if all maximal subgroups of the Sylow subgroups of $F^*(H)$, the generalized Fitting subgroup of H, are CAP-subgroups of G.

Recently the following definition, as a generalization of the CAP-subgroup, was introduced.

Definition 1.1 *([10]) A subgroup H of a group G is said to be a CAP*-subgroup of G if H either covers or avoids every non-Frattini chief factor of G.*

The authors in [10] and [11] have gotten many structural theorems of groups G under the assumption that some subgroups of G are CAP*-subgroups of G. In this paper, we continue the works in this line to study the influence of CAP*-subgroups on the structure of a groups, many recent results are extended.

2 Preliminaries

Lemma 2.1 *([11, Lemma 2.3]) Let N be a normal subgroup of a group G. If H is a CAP*- subgroup of G, then:*
(1) HN/N is a CAP-subgroup of G/N;*
(2) $H \cap N$ is a CAP-subgroup of G.*

Lemma 2.2 *([4, A, Theorem 9.11]) Let K and N be normal subgroups of a group G with $N \leq K$ and K nilpotent. If $K/N \leq \Phi(G/N)$, then $K \leq \Phi(G)N$.*

Lemma 2.3 *([13, Lemma 2.6]) Let H be a normal subgroup of G. If $H \cap \Phi(G) = 1$, then the Fitting subgroup $F(H)$ of H lies in $Soc(G)$ and therefore $F(H)$ is the direct product of minimal normal subgroups of G which are contained in $F(H)$.*

Lemma 2.4 *([17, Theorem 1.7.19]) Let H be a normal subgroup of G. Then $H \leq Z_{\mathcal{U}}(G)$ if and only if $H/\Phi(H) \leq Z_{\mathcal{U}}(G/\Phi(H))$.*

Lemma 2.5 *([10, Proposition 1.4]) Every non-Frattini chief factor of G is avoided by every subgroup of $\Phi(G)$.*

Lemma 2.6 *([10, Lemma 2.3]) Let p be a prime dividing the order of a group G, H a normal subgroup of G such that $O_{p'}(G) = 1$, and let P be a Sylow p-subgroup of H. If every maximal subgroup of P is a CAP*-subgroup of G and N is a minimal normal subgroup of G contained in H such that $N \cap \Phi(G) = 1$. Then the Sylow p-subgroups of N are of order p. In particular,*

(1) N is of order prime if N is solvable;

(2) N is a non-abelian simple group if N is non-solvable.

Lemma 2.7 *([10, Lemma 3.1]) Let H be a normal subgroup of a group G and p the smallest prime dividing the order of H, and let P be a Sylow p-subgroup of H. If every maximal subgroup of P is a CAP*-subgroup of G, then H is p-nilpotent.*

3 Main Results

The authors in [10] obtained the following result: Let H ba a normal subgroup of G. If every maximal subgroup of any Sylow subgroup of H is a CAP-subgroup of G, then $H \leq Z_{\mathcal{U}}(G)$ ([10, Theorem 1.6]). There exists examples to illustrates that the conclusion in this result is not true if we replace CAP-subgroup by CAP*-subgroup ([10]). But we have

Theorem 3.1 *Let H ba a normal subgroup of G. If every maximal subgroup of any Sylow subgroup of H is a CAP*-subgroup of G, then $H \leq Z_{\mathcal{U}\phi}(G)$.*

Proof. Suppose that p is the smallest prime dividing the order of H. By Lemma 2.7, we know that H is p-nilpotent. Hence H has the normal Hall p'-subgroup, K say. Obviously, K is normal in G. By induction we have $K \leq Z_{\mathcal{U}\phi}(G)$. By Lemma 2.1, we know that $(G/K, H/K)$ satisfies the hypotheses of the theorem. If $K \neq 1$, then, by induction, we have $H/K \leq Z_{\mathcal{U}\phi}(G/K) = Z_{\mathcal{U}\phi}(G)/K$. Hence $H \leq Z_{\mathcal{U}\phi}(G)$, as desired. Hence we assume that $K = 1$, this means that H is a p-group. Thus we denote $H = P$.

Now we assume that the result is false and G is a counterexample such that $|G|+|P|$ is the smallest number. We will conduct a contradiction in several steps.

(1) P is not a minimal normal subgroup of G.

Assume that P is a minimal normal subgroup of G. If $P \leq \Phi(G)$, then $P \leq Z_{\mathcal{U}\phi}(G)$, a contradiction. Hence $P \not\leq \Phi(G)$, i.e., $P/1$ is a non-Frattini G-chief factor of G. Pick a maximal subgroup P_1 of P. Then P_1 is a CAP^*-subgroup of G, i.e., P_1 either covers or avoids $P/1$. We only have $P_1 = P \cap P_1 = 1$. Therefore, P is of order p. So $P \leq Z_{\mathcal{U}\phi}(G)$, a contradiction.

(2) Suppose that N is a minimal normal subgroup of G contained in P. Then $P/N \leq Z_{\mathcal{U}\phi}(G/N)$.

Let M/N be a maximal subgroup of PN/N. It is easy to see $M = P_1N$ for some maximal subgroup P_1 of P. By the hypotheses, P_1 is a CAP*-subgroup of G. Hence $M/N = P_1N/N$ is a CAP*-subgroup of G/N by Lemma 2.1. Therefore G/N satisfies the hypotheses of the theorem. Hence (2) holds.

(3) $P \cap \Phi(G) = 1$.

Otherwise, we can pick a minimal normal subgroup N of G contained in $P \cap \Phi(G) \neq 1$. Then, by (2), $P/N \leq Z_{\mathcal{U}\phi}(G/N) = Z_{\mathcal{U}\phi}(G)/N$. Hence $P \leq Z_{\mathcal{U}\phi}(G)$, a contradiction.

(4) The final contradiction.

By Lemma 2.3, we can denote $P = R_1 \times \cdots \times R_s$, where all $R_j (j = 1, 2, \cdots, s)$ are minimal normal subgroups of G. Pick a maximal subgroup R_1^* of R_1. Set $P^* = R_1^* R_2 \cdots R_s$. Then P^* is a maximal subgroup of P. By hypotheses, P^* is a CAP*-subgroup of G. Hence $R_1^* = R_1 \cap P^*$ is a CAP*-subgroup of G by Lemma 2.1. Obviously $R_1/1$ is a non-Frattini factor of G. Then R_1^* either covers or avoids $R_1/1$. This implies that $R_1^* = R_1^* \cap R_1 = 1$. So R_1 is of order p. Similarly, we can prove that all R_i are of prime order. Hence $P \leq Z_{\mathcal{U}}(G) \leq Z_{\mathcal{U}\phi}(G)$, the final contradiction.

This completes the proof of the theorem. □

From our Theorem 3.1, we can extend Ezuquerro's results ([5, Theorem C and D]) as follows.

Corollary 3.2 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup H such that $G/H \in \mathcal{F}$. Then:*

(1) If all maximal subgroups of the Sylow subgroups of H are CAP*-subgroups of G, then $G \in \mathcal{F}$;

(2) If H is solvable and all maximal subgroups of the Sylow subgroups of $F(H)$ are CAP*-subgroups of G, then $G \in \mathcal{F}$ ([10, Theorem 4.4]).

Proof. (1) By Theorem 3.1 we know that $H \leq Z_{\mathcal{U}\phi}(G)$. Pick a minimal normal subgroup N of G contained in H. Then $N \leq \Phi(G)$ or N is of prime order. In particular, N is a p-group for some prime $p \in \pi(H)$. By Lemma 2.1, we know that $(G/N, H/N)$ satisfies the hypotheses of the corollary. Hence $G/N \in \mathcal{F}$ by induction.

If $N \leq \Phi(G)$, then $G \in \mathcal{F}$ as $\mathcal{F}$ is saturated. If N is of prime order, then $G \in \mathcal{F}$ as $\mathcal{F}$ contains $\mathcal{U}$.

Hence (1) holds.

(2) Denote $H_1 = H \cap \Phi(G)$. Then $H_1 = F(H) \cap \Phi(G)$. By Theorem 3.1, we have that $F(H) \leq Z_{\mathcal{U}\phi}(G)$.

Pick an arbitrary G-chief factor R/S over $F(H) \cap \Phi(G)$ and below $F(H)$.

If R/S is Frattini, i.e., $R/S \leq \Phi(G/S)$, then $R \leq \Phi(G)S$ by Lemma 2.2. Then

$$R \leq \Phi(G)S \cap F(H) = S(\Phi(G) \cap F(H)) = S,$$

a contradiction. Hence R/S is non-Frattini, then R/S is of order prime as R/S is a G-chief factor below $Z_{\mathcal{U}\phi}(G)$. Therefore,

$$\frac{F(H)}{F(H) \cap \Phi(G)} = \frac{F(H)}{H_1} \leq Z_{\mathcal{U}}(G/H_1).$$

Then

$$\frac{G/H_1}{C_{G/H_1}(F(H)/H_1)}$$

is supersolvable by [4, IV, Theorem 6.10]. Since $G/H \in \mathcal{F}$, we have

$$\frac{G/H_1}{H/H_1} \in \mathcal{F}.$$

Therefore,

$$\frac{G/H_1}{C_{H/H_1}(F(H/H_1))}$$

$$= \frac{G/H_1}{(H/H_1) \cap C_{G/H_1}(F(H/H_1))} \in \mathcal{F}.$$

Since H is solvable,

$$C_{H/H_1}(F(H/H_1)) \leq F(H/H_1).$$

So

$$\frac{G/H_1}{F(H/H_1)} \in \mathcal{F}.$$

Since

$$F(H/H_1) = F(H)/H_1 \leq Z_{\mathcal{U}}(G/H_1),$$

we conclude that

$$G/H_1 \in \mathcal{F}.$$

Since $\mathcal{F}$ is saturated, we have $G \in \mathcal{F}$.

This completes the proof of this corollary. □

Remark 3.1 *The following example indicates that we can not delete the hypothesis that H is solvable and replace $F(H)$ by $F^*(H)$, the generalized Fitting subgroup of H (ref. [9, X, 13]), in Corollary 3.2(2).*

Example 3.1 *Suppose that G is a non-split extension $(Z_2)^3 L_3(2)$ of an elementary abelian subgroup $(Z_2)^3$ of order 2^3 by $L_3(2)$. G is a maximal subgroup of $G_2(3)$ (ref. [3, page 61]). Then $F^*(G) = F(G) = \Phi(G) = (Z_2)^3$. It is easy to see that every maximal subgroup of the Sylow subgroup of $F^*(G)$ is a CAP*-subgroup of G by Lemma 2.5. But G is not solvable.*

The following result is a uniform generalization of [10, Theorem 4.1 and 4.2].

Theorem 3.3 *Let p be a prime dividing the order of G and P a Sylow p-subgroup of G. Then the following statements are equivalent:*

(1) Every maximal subgroup of P is a CAP-subgroup of G.*

(2) $|P| = p$ or G is p-supersolvable.

Proof. Since (2) ⇒ (1) is obvious, we consider (1) ⇒ (2).

Assume that the result is false and G is a counterexample with minimal order. We will conduct a contradiction in several steps.

Step 1. $O_{p'}(G) = 1$.

It follows from Lemma 2.1, $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. The minimal choice implies that $O_{p'}(G) = 1$.

Step 2. Let N be a minimal normal subgroup. Then either G/N is p-supersolvable or G/N is a non-p-solvable group whose Sylow p-subgroups are of order p.

Consider G/N. Let M/N be a maximal subgroup of PN/N. It is easy to see $M = P_1N$ for some maximal subgroup P_1 of P. By the hypotheses, P_1 is a CAP*-subgroup of G. Hence $M/N = P_1N/N$ is a CAP*-subgroup of G/N by Lemma 2.1. Therefore G/N satisfies the hypotheses of the theorem. The choice of G yields that either G/N is p-supersolvable or G/N is a non-p-solvable group whose Sylow p-subgroups are of order p.

Step 3. Let N be a minimal normal subgroup of G. Then $1 \neq N_p = N \cap P < P$. By Step 1, the prime p divides the order of N and then $N_p = N \cap P \neq 1$.

Notice that if $P \cap N = P$, then $P \leq N$. If $N \leq \Phi(G)$, then $P \leq \Phi(G)$. This is impossible. Hence $N \nleq \Phi(G)$. By Lemma 2.6, we have $|N|_p = p$, i.e., $|P| = p$, a contradiction.

Step 4. The minimal normal subgroup of G is unique, N say.

Suppose that there exist two distinct minimal normal subgroups M and N of G. By Step 3, we know that both G/N and G/M satisfy the hypotheses of the theorem. If G/N and G/M are p-supersolvable, the G is p-supersolvable, a contradiction. Suppose that G/N is a non-p-solvable group whose Sylow p-subgroups are of order p and G/M is p-supersolvable. Since NM/M is a minimal normal subgroup of G/M, we know, by Step 1, that NM/M is of order p. Hence $|N| = p$ and P is of order p^2. On the other hand, MN/N is a minimal normal subgroup of G/N and p divides the order of M. We have M is a non-abelian simple group. As a maximal subgroup of P, the Sylow p-subgroup M_p of M is a CAP*-subgroup of G by the hypotheses. Hence M_p covers the G-chief factor $M/1$. So $M = M_p$ is solvable, a contradiction. Finally, we assume that both G/N and G/M are non-p-solvable groups whose Sylow p-subgroups are of order p. With the same arguments as above, we have $|P| = p^2$ and M is a non-abelian simple group whose Sylow p-subgroups are of order p, and M_p is a CAP*-subgroup of G. Again, this implies that M_p covers the G-chief factor $M/1$ and $M = M_p$ is solvable, a contradiction.

This completes the proof of this step.

Step 5. $N \nleq \Phi(P)$.

Suppose that $N \leq \Phi(P)$. Then, by [4, A, 9.2.d], we have $N \leq \Phi(G)$.

If G/N is p-supersolvable, then G is p-supersolvable as the class of all p-supersolvable groups is a saturated formation, a contradiction. Hence G/N is a non-p-solvable group whose Sylow p-subgroups are of order p by Step 2. Since $N \leq \Phi(P)$, we have P is a cyclic group. Therefore N is a cyclic group of order p and $|P| = p^2$. By [2, Theorem 7], this is impossible.

Step 6. $O_p(G) = 1$.

Suppose that $O_p(G) \neq 1$. Then $N \leq O_p(G)$ by Step 4.

If $N \nleq \Phi(G)$, then N is of order p by Lemma 2.7. Applying Step 2, we have G/N is a non-p-solvable group whose Sylow p-subgroups are of order p. So $|P| = p^2$. By [2, Theorem 7], this is impossible. So we have $N \leq \Phi(G)$. Again, applying Step 2, we have G/N is a non-p-solvable group whose Sylow p-subgroups are of order p. In this case, $O^{p'}(G/N)$ is a non-abelain simple group by [1, Lemma 3.1]. Denote $K/N = O^{p'}(G/N)$. Then K/N is a non-Frattini G-chief factor.

By Step 5, we can pick a maximal subgroup P_1 of P such that $P = NP_1$. By the hypotheses, P_1 is a CAP*-subgroup of G. Hence P_1 covers or avoids K/N. Since K/N is a non-abelain simple group, we have P_1 avoids K/N. Therefore, $P_1 \cap K = P_1 \cap N$. Noticing $P \leq K$, we have $P_1 = P_1 \cap N$. So $P = NP_1 = N$, a contradiction. Hence $O_p(G) = 1$.

Step 7. The final contradiction.

By Lemma 2.6, we have N is a non-abelain simple group. Now we can pick a maximal subgroup P_1 of P such that $N_p \leq P_1$ by Step 3. Since P_1 is a CAP*-subgroup of G and $N/1$ is a non-Frattini G-chief factor, P_1 covers or avoids $N/1$. If P_1 avoids $N/1$, then $N_p = P_1 \cap N = P_1 \cap 1 = 1$, contrary to Step 3. If P_1 covers $N/1$, then $N \leq P_1$ and N is solvable, a contradiction.

This completes the proof of this theorem. □

Corollary 3.4 *([10, Theorem 4.2]) Let p be a prime dividing the order of a group G and P a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if every maximal subgroup of P is a CAP*-subgroup of G.*

Proof. By Theorem 3.5 we know that G is p-supersolvable . Since the p-length of p-supersolvable groups is at most 1, we have $PO_{p'}(G)$ is normal in G. Set $\overline{G} = G/O_{p'}(G)$. Then $\overline{G} = N_{\overline{G}}(\overline{P}) = N_G(P)O_{p'}(G)/O_{p'}(G)$ is p-nilpotent by hypotheses. Hence G is p-nilpotent, as desired. □

Corollary 3.5 *([10, Theorem 4.1]) Let p be a prime dividing the order of G and P a Sylow p-subgroup of G. Then the following statements are equivalent:*

(1) G is p-supersolvable.

(2) P and every maximal subgroup of P is a CAP-subgroup of G.*

Proof. We only need to consider (2) ⇒ (1).

By Theorem 3.5 we know that P is of order p or G is p-supersolvable from the hypothesis that every maximal subgroup of P is a CAP*-subgroup of G. Now suppose that P is of order p. By [1, Lemma 3.1], we know that $O^{p'}(G/O_{p'}(G))$ is a simple group. Denote $O^{p'}(G/O_{p'}(G)) = K/O_{p'}(G)$, then $P \leq K$ and $K/O_{p'}(G)$ is a non-Frattini G-chief factor. Since P is a CAP*-subgroup of G by hypotheses, we have P covers $K/O_{p'}(G)$. This implies that $K = PO_{p'}(G)$. Hence G is p-supersolvable, as desired. □

It is easy to see that [10, Lemma 3.1], i.e., Lemma 2.7, is a corollary of Theorem 3.3, we do not repeat this here.

References

[1] A. Ballester-Bolinches, L. M. Ezquerro and A. N. Skiba, Local embeddings of some families of subgroups of finite group, *Acta Math. Sinica, Eng. Ser.*, **25**(2009), 869-882.

[2] A. Ballester-Bolinches, L. M. Ezquerro and A. N. Skiba, On second maximal subgroups of Sylow subgroups of finite groups, *J. Pure Appl. Algebra*, **215**(2011), 705-714.

[3] J. H. Conway, R. T. Curtis, S.P. Norton, R. A. Parker and R.A.Wilson, *ATLAS of Finite Groups*, Clarendon Press, Oxford, 1985.

[4] K. Doerk and T. Hawkes, *Finite Soluble Groups*, Berlin-New York: Walter de Gruyter, 1992.

[5] L. M. Ezquerro, A contribution to the theory of finite supersolvable groups, *Rend. Sen. Mat. Univ. Padova*, **89** (1993), 161-170.

[6] W. Gaschütz, Praefrattini gruppen, *Arch. Math.*, **13**(1962), 418-426.

[7] J. D. Gillam, Cover-avoid subgroup in finite solvable groups, *J. Algebra*, **29**(1974), 324-329.

[8] B. Huppert, *Endliche Gruppen I*, Berlin-Heidelberg-New York: Springer-Verlag, 1967.

[9] B. Huppert and N. Blackburn, *Finite Groups III*, Berlin-New York: Springer-Verlag, 1982.

[10] S. Li J. Liu, X. Guo, A generalization of cover-avoiding properties in finite groups, *Comm. Algebra*, **39**(2011), 1455-1464.

[11] J. Liu, X. Guo and S. Li, The influence of CAP*-subgroup on the solvablitiy of finite groups, *Bull. Malay. Math. Sci. Soc.*, 35 (2012), 227-237

[12] Y. Li and G. Wang, A note on the CAP-maximal subgroups of finite group, *Alg., Groups Geom.*, **23(4)**(2006), 285-290.

[13] Y. Li, Y. Wang and H. Wei, The influence of π-quasinormality of some subgroups of a finite group, *Arch Math.*, **81**(2003), 245-252.

[14] J. A. Petrillo, The cover-avoidance property in finite groups, PhD thesis, State University of New York at Binghamton, 2003.

[15] L. Shemetkov and A. Skiba, On the $\chi\phi$-hypercenter of finite groups, *J. Algebra*, **322**(2009), 2106-2117.

[16] M. J. Tomkinson, Cover-avoidance properties in finite solvable groups, *Canad. Math. Bull.*, **19**(2)(1976), 213-216.

[17] M. Weinstein(editor), *Between nilpotent and solvable*, Polygonal Publishing House, Passaic, 1982.