Qualitative properties and standard estimates of solutions for some fourth order elliptic systems

Ruichang Pei[§,†] Jihui zhang[†] Caochuan Ma[§]

§ School of Mathematics and Statistics Tianshui Normal University, Tianshui,741001, P. R. China

† School of Mathematics and Computer Sciences, Nanjing Normal University, Nanjing, 210097, P. R. China

Abstract: In this paper, first, we make the uniform estimates for a class of fourth order elliptic system in bounded and smooth domains. Second, we study the qualitative properties of solutions with prescribed integration in R^4. Finally, we also will obtain some radially symmetric results by using moving planes methods.

Keywords: Fourth order elliptic systems; Asymptotic behavior; Uniform priori estimates; Q curvature

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1. Introduction

In this paper, we make uniform estimates to the following fourth order elliptic system:

$$\begin{cases} \Delta^2 u = Q_1(x)e^{4v}, & \text{in } \Omega \subset R^4; \\ \Delta^2 v = Q_2(x)e^{4u}, & \text{in } \Omega \subset R^4; \\ u = \triangle u = v = \triangle v = 0 & \text{on } \partial\Omega \end{cases} \tag{$*$}$$

and investigate properties of the solutions to the following fourth order elliptic system:

$$\begin{cases} \Delta^2 u = Q_1(x)e^{4v}, & x \in R^4; \\ \Delta^2 v = Q_2(x)e^{4u}, & x \in R^4. \end{cases} \tag{$**$}$$

Where Ω is a bounded smooth domain in R^4 and $\triangle^2$ is the biharmonic operator. $Q_i(x)$ $(i = 1, 2)$ are given function in $L^p(\Omega)$ for some $1 < p \leq \infty$. We assume that $u, v \in L^1(\Omega)$, $e^{4u}, e^{4v} \in L^{p'}(\Omega)$ (where p' is the conjugate exponent of p) so that $(*)$ has a meaning in the sense of distributions. A first question is whether one can conclude that all eventual solutions of $(*)$ have uniform bounds. As we will see in the Section 2 (see our main result, Theorem 2.4) the answer is positive. It is obvious that we have generalized part works in [6] when $u = v$ in $(*)$.

Recently, a series of works have been done to understand the existence and the qualitative properties of the solutions of $(**)$ when $Q_1 = Q_2$ and $u = v$. In the mean time, $(**)$ becomes

$$\Delta^2 u = Q(x)e^{4u}, \quad x \in R^4, \tag{$***$}$$

where $Q = Q_1 = Q_2$.

When $Q = 6$, Lin [1] had given a complete classification of u in terms of its growth, or of the behavior of $\triangle u$ at ∞. Xu [2] had done similar work by using moving spheres methods. Wei and Xu [4] and Martinazzi [7] also gave a complete classification of solutions for higher order conformally invariant equations compared to $(***)$. In Section 3, we consider more general functions $Q_1(x), Q_2(x)$ for problem $(**)$. This is considered as the generalization of problem $(***)$ in [1]. First, we obtain the asymptotic behavior of solutions near infinity. Consequently, we prove that all solutions satisfy an identity, which is similar to the well-known Kazdan-Warner condition (see our main result, Theorem 3.2). Finally, using the harmonic asymptotic expansion at ∞ in [1], we show that all the solutions for $(**)$ are radial symmetric provided Q_1, Q_2 is radially symmetric and non-increasing. This part can be viewed as the completion of [1].

2. Uniform estimates for problem $(*)$

Assume $\Omega \subset R^4$ is a bounded domain and let h be a solution of

$$\begin{cases} \Delta^2 h(x) = f(x), & \text{in } \Omega \subset R^4; \\ h = \triangle h = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Following the argument Brezis-Merle [8], Lin obtained the following Lemma:

Lemma 2.1. [1] *Suppose $f \in L^1(\overline{\Omega})$. For any $\delta \in (0, 32\pi^2)$, there exists a constant $C_\delta > 0$ such that the inequality,*

$$\int_\Omega exp(\frac{\delta|h|}{||f||_{L^1}})dx \le C_\delta(diam\Omega)^4,$$

where $diam\Omega$ denotes the diameter of Ω.

By using above Lemma, we obtain following consequent results:

Theorem 2.1. *Let u be a solution of (2.1) with $f \in L^1(\Omega)$. Then for every constant $k > 0$,*

$$e^{ku} \in L^1(\Omega).$$

Proof. Let $0 < \epsilon < \frac{1}{k}$, we may split f as $f = f_1 + f_2$ with $||f_1||_1 < \epsilon$ and $f_2 \in L^\infty(\Omega)$. Write u_i are the solutions of

$$\begin{cases} \Delta^2 u_i = f_i, & \text{in } \Omega; \\ u_i = \triangle u_i = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.1, we find $\int_\Omega \exp[\frac{|u_1(x)|}{||f_1||_1}] < \infty$ and thus $\int_\Omega \exp[k|u_1|] < \infty$. The conclusion follows since $|u| \le |u_1| + |u_2|$ and $u_2 \in L^\infty(\Omega)$.

Before stating our main results on a priori bounds, we state a result on the regularity of the distribution solutions of $(**)$.

Theorem 2.2. *Suppose (u, v) is a solution of equation $(**)$ with $Q_1, Q_2 \in L^p(\Omega)$ and $e^{4u}, e^{4v} \in L^{p'}(\Omega)$ for some $1 < p \le \infty$. Then $u, v \in L^\infty(\Omega)$.*

Proof. By Theorem 2.1, we know that $e^{ku} \in L^1(\Omega)$ for all k, i.e., $e^u \in L^r(\Omega)\ \forall r < \infty$. It follows that $Q_2 e^{4u} \in L^{p-\delta}\ \forall \delta > 0$ if $p < \infty$, and $Q_2 e^{4u} \in L^r(\Omega)\ \forall r < \infty$ if $p = \infty$. Standard elliptic estimates imply that $\triangle v \in L^\infty(\Omega)$. Hence, combing $v = 0$ on $\partial\Omega$, we have $v \in L^\infty(\Omega)$. Similarly, we have $u \in L^\infty(\Omega)$.

Corollary 2.1. *Suppose (u, v) is a solution of*

$$\begin{cases} \Delta^2 u = Q_1 e^{4v} + f(x), & \text{in } \Omega; \\ \Delta^2 v = Q_2 e^{4u} + g(x), & \text{in } \Omega; \\ u = g_1,\ \triangle u = g_2, & \text{on } \partial\Omega; \\ v = \psi_1,\ \triangle v = \psi_2 & \text{on } \partial\Omega \end{cases}$$

with $Q_1, Q_2 \in L^p(\Omega)$ and $e^{4u}, e^{4v} \in L^{p'}(\Omega)$ for some $1 < p \le \infty$, where $g_1,\ g_2, \psi_1, \psi_2 \in L^\infty(\partial\Omega)$ and $f, g \in L^q(\Omega)$ for some $q > 1$. Then $u, v \in L^\infty(\Omega)$.

It follows from Theorem 2.2 that, for any solution (u, v) of system $(*)$, $\int_\Omega Q_1(x) e^{4v} dx < \infty$, $\int_\Omega Q_2(x) e^{4u} dx < \infty$. Our next result states that there is a uniform bound for those integrals. For that matter, due to the fact that we are considering non-autonomous problems, we need in the theorems below geometric assumptions concerning the behavior of Q_1 and Q_2 near the boundary. So,

(H_1) There exist $r, \delta > 0$ such that $Q_1(x), Q_2(x) \in C^1(\Omega_r)$, and

$$\nabla Q_1(x) \cdot \theta \le 0 \text{ and } \nabla Q_2(x) \cdot \theta \le 0$$

for all $x \in \Omega_r (\Omega_r := \{x \in \bar{\Omega} : dist(x, \partial\bar{\Omega}) \le r\}$, and unit vectors θ such that $|\theta - \nu(\bar{x})| < \delta$, where $\bar{x}$ is the closest point to x in $\partial\Omega$ and $\nu(\bar{x})$ denotes the unit external normal to $\partial\Omega$ in the point $\bar{x}$.

With assumption (H_1) one can use the Moving Planes techniques to get bounds for the functions u and v near the boundary.

Let ϕ_1 be the eigenfunction associated to the first eigenvalue λ_1 of $(\triangle^2, H^2(\Omega)\cap H_0^1(\Omega))$.

Theorem 2.3. *Assume $Q_i(x), i=1,2$ is continuous function with $m_i \leq Q_i \leq M_i$ for some positive constants m_i and M_i and (H_1). Assume furthermore that Ω is convex. Then there exists a positive constant C, depending only on $Q_i, i=1,2$ and Ω, such that*

$$\int_\Omega Q_1(x)e^{4v}dx < C, \int_\Omega Q_2(x)e^{4u}dx < C \tag{2.2}$$

for all (u,v) solution of $(*)$.

Proof. Step 1 For each (u,v) solution of system $(*)$ we have

$$\int_\Omega Q_1(x)e^{4v}\phi_1 dx \leq C, \int_\Omega Q_2(x)e^{4u}\varphi_1 dx \leq C,$$

where the constant C depends only on Q_1, Q_2 and Ω.

From our basic assumptions for $Q_i, i=1,2$, we know that there are positive constants $a_i, i=1,2$ with $a_1 \times a_2 > \lambda_1^2$ and c such that

$$Q_1(x)e^{4t} \geq a_1 t - c \text{ and } Q_2(x)e^{4t} \geq a_2 t - c. \tag{2.3}$$

Next, multiplying the equations in $(*)$ by ϕ_1, integrating by parts and using (2.3), we obtain

$$\int_\Omega Q_1(x)e^{4v}\phi_1 dx = \lambda_1 \int_\Omega u\phi_1 dx \geq a_1 \int_\Omega v\phi_1 dx - c_1$$
$$\int_\Omega Q_2(x)e^{4u}\phi_1 dx = \lambda_1 \int_\Omega v\phi_1 dx \geq a_2 \int_\Omega u\phi_1 dx - c_1. \tag{2.4}$$

Thus

$$\lambda_1 \int_\Omega u\phi_1 dx \geq \frac{a_1 a_2}{\lambda_1} \int_\Omega u\phi_1 dx - c_1$$

which implies

$$\int_\Omega u\phi_1 dx \leq C,$$

and therefore,

$$\int_\Omega Q_1(x)e^{4v}\phi_1 dx \leq C.$$

The other inequality in (2.2) is obtained in a similar way.

Step 2 We claim that there exist $r, \delta > 0$ such that

$$\nabla u(x)\cdot\theta \leq 0 \text{ and } \nabla v(x)\cdot\theta \leq 0 \text{ for all } x \in \Omega_r,\ |\theta - \nu(x)| < \delta,$$

for each (u,v) solutions of $(*)$, where θ and ν are as in (H_1).

We can assume, without loss of generality, that $\Omega \subset R^4_+ := \{(x_1,x_2,x_3,x_4) \in R^4 : x_1 > 0\}$ and $(0,0,0,0) \in \partial\Omega$. Now, we consider $T_\lambda := \{(x_1,x_2,x_3,x_4) : x_1 = \lambda\}$, the cap $\Sigma_\lambda := \{(x_1,x_2,x_3,x_4) \in \Omega : x_1 < \lambda\}$ and the reflected cap $\Sigma'_\lambda := \{(2\lambda - x_1, x_2, x_3, x_4) : (x_1,x_2,x_3,x_4) \in \Sigma_\lambda\}$. It follows that there exists $\bar\lambda$ such that $\Sigma_\lambda \cup \Sigma'_\lambda \subset \Omega_r$ for each $0 < \lambda < \bar\lambda$. In fact this $\bar\lambda$ depends only on r and not on the particular point on the boundary.

For $0 < \lambda < \bar\lambda$, define in Σ_λ the auxiliary functions

$$w_\lambda(x_1,x_2,x_3,x_4) = u(2\lambda - x_1, x_2, x_3, x_4) - u(x_1,x_2,x_3,x_4),$$

$$z_\lambda(x_1,x_2,x_3,x_4) = v(2\lambda - x_1, x_2, x_3, x_4) - v(x_1,x_2,x_3,x_4).$$

Using condition (H_1) we have

$$\begin{aligned} \triangle^2 w_\lambda &= Q_1(2\lambda - x_1, x_2, x_3, x_4)e^{4v(2\lambda - x_1, x_2, x_3, x_4)} - Q_1(x_1,x_2,x_3,x_4)e^{4v(x_1,x_2,x_3,x_4)} \\ &\geq Q_1(2\lambda - x_1, x_2, x_3, x_4)e^{4v(2\lambda - x_1, x_2, x_3, x_4)} - Q_1(2\lambda - x_1, x_2, x_3, x_4)e^{4v(x_1,x_2,x_3,x_4)}. \end{aligned}$$

Now, using the mean value theorem we see that

$$\triangle^2 w_\lambda \geq c(x_1,x_2,x_3,x_4)(v(2\lambda - x_1, x_2, x_3, x_4) - v(x_1,x_2,x_3,x_4)),$$

where

$$c(x_1,x_2,x_3,x_4) = 4Q_1(2\lambda - x_1, x_2, x_3, x_4)e^{4\eta(x_1,x_2,x_3,x_4)} \geq 0$$

and $\eta(x_1,x_2,x_3,x_4)$ is real number between $v(x_1,x_2,x_3,x_4)$ and $v(2\lambda - x_1, x_2, x_3, x_4)$. Thus

$$\triangle^2 w_\lambda - c(x_1,x_2,x_3,x_4)z_\lambda \geq 0.$$

Similarly we can prove that

$$\triangle^2 z_\lambda - \bar c(x_1,x_2,x_3,x_4)w_\lambda(x_1,x_2,x_3,x_4) \geq 0,$$

where

$$\bar c(x_1,x_2,x_3,x_4) = 4Q_2(2\lambda - x_1, x_2, x_3, x_4)e^{4\xi(x_1,x_2,x_3,x_4)} \geq 0$$

and $\xi(x_1,x_2,x_3,x_4)$ is real number between $u(x_1,x_2,x_3,x_4)$ and $u(2\lambda - x_1, x_2, x_3, x_4)$.

For λ sufficiently small and positive we have that Σ_λ has small measure and so we can use the maximum principle for cooperative elliptic systems in small domains (see [9, 10]) to conclude that

$$w_\lambda \geq 0 \text{ and } z_\lambda \geq 0 \text{ in } \Sigma_\lambda.$$

Using similar arguments as in [10] we can also prove that

$$w_\lambda \geq 0 \text{ and } z_\lambda \geq 0 \text{ in } \Sigma_{\bar{\lambda}}.$$

Therefore, there exists $\epsilon > 0$ such that u and v are increasing in Ω_ϵ. Finally, the conclusion follows in a standard way as in [11].

Step 3 We claim that there exist $\epsilon > 0$ and $C > 0$ which depend only on Q_1, Q_2 and Ω such that $||u||_{L^\infty(\Omega_\epsilon)},\ ||v||_{L^\infty(\Omega_\epsilon)} \leq C$, for each (u,v) solution of $(*)$.

The conclusion follows by the same arguments as in [11], using Step 2 above.

Step 4 We claim that our theorem holds.

Let $\alpha := \inf\{\phi_1(x) : x \in \bar{\Omega} \setminus \Omega_\epsilon\}$. Using Step 3 we obtain that $Q_1(x)e^{4v}$ is bounded in Ω_ϵ. Thus

$$\begin{aligned}\int_\Omega Q_1(x)e^{4v}dx &= \int_{\Omega_\epsilon} Q_1(x)e^{4v}dx + \int_{\Omega\setminus\Omega_\epsilon} Q_1(x)e^{4v}dx \\ &\leq C + \frac{1}{\alpha}\int_{\Omega\setminus\Omega_\epsilon} Q_1(x)e^{4v}\phi_1 dx \\ &\leq C,\end{aligned}$$

where we have used Step1 to estimate the last integral. Using a similar argument we can prove the result for $Q_2(x)e^{4u}$.

Now, we presents our main result in this section.

Theorem 2.4. *Assume $Q_i(x), i = 1,2$ is continuous function with $m_i \leq Q_i \leq M_i$ for some positive constants m_i and M_i and (H_1). Assume furthermore that Ω is convex. Then there exists a constant $C > 0$ such that*

$$||u||_{L^\infty} \text{ and } ||v||_{L^\infty} \leq C,$$

for all eventual solutions (u,v) of system $(*)$.

Proof. In view of $\int_\Omega Q_1(x)e^{4v}dx < C$ and $\int_\Omega Q_2(x)e^{4u}dx < C$, we may assume that there exist two nonnegative bounded measures μ and ν such that

$$Q_1(x)e^{4v_n} \to \mu \text{ and } Q_2(x)e^{4u_n} \to \nu. \tag{2.5}$$

We also observe that, as a consequence of Theorem 2.3, the solutions $((u_n, v_n))$ of $(*)$ are bounded in $L^1(\Omega)$:

$$||u_n||_{L^1},\ ||v_n||_{L^1} \leq C,\ \forall n. \tag{2.6}$$

A point $x \in \Omega$ is called a $8\pi^2$ regular point with respect to μ if there is a function $\psi \in C_c(\Omega),\ 0 \leq \psi \leq 1$, with $\psi = 1$ in a neighborhood of x such that

$$\int_\Omega \psi d\mu < 8\pi^2.$$

We define

$$\Omega_\mu(8\pi^2) = \{x \in \Omega : x \text{ is not a } 8\pi^2 \text{ regular point with respect to} \mu\}$$

Since $\int d\mu < c$, it follows that $\Omega_\mu(8\pi^2)$ is a finite set. Let S_u be the blow-up set for the sequence (u_n) , that is

$$S_u := \{x \in \Omega : \exists (x_n) \subset \Omega \text{ such that } x_n \to x \text{ and } u_n(x_n) \to +\infty\}.$$

In fact, our theorem will be proved if we can show that $S_u = S_v = \varnothing$.

Next, we prove our above conclusion by four steps.

Step 1 We claim that for x_0 is a regular point for the measure μ (or for the measure ν), then there exist constants $\rho > 0$ and C, independent of n, such that

$$||u_n||_{L^\infty(B_\rho(x_0))} \leq C, \ ||v_n||_{L^\infty(B_\rho(x_0))} \leq C.$$

Using the fact that x_0 is a regular point of the measure μ we have a function $\psi \in C_c(\Omega), \ \ 0 \leq \psi \leq 1$, with $\psi \equiv 1$ in some neighborhood V_{x_0} of x_0, such that $\int \psi d\mu < 8\pi^2$. Thus, $\int_{V_{x_0}} d\mu < 8\pi^2$, which implies that there exist $R > 0, \ \delta > 0$ and n_0 such that for all $n \geq n_0$

$$\int_{B_R(x_0)} Q_2(x) e^{4u_n} \leq 8\pi^2 - \delta. \tag{2.7}$$

Using this estimates, we first work with the second equation in $(*)$. Let us write $v_n := v_{1,n} + v_{2,n}$, where

$$\triangle^2 v_{1,n} = Q_2(x) e^{4u_n}, \text{ in } B_R(x_0) \text{ and } v_{1,n} = \triangle v_{1,n} = 0 \text{ on } \partial B_R(x_0).$$

Notice that $\triangle^2 v_{2,n} = 0$ in $B_R(x_0)$.

Using Lemma 2.1 and (2.7), we obtain

$$C \geq \int_{B_R} e^{(32\pi^2 - \frac{\delta}{2}) \frac{v_{1,n}}{\int Q_2(x) e^{4u_n}}} \geq \int_{B_R} e^{p4v_{1,n}}, \tag{2.8}$$

where $p > 1$ is a constant depending only on δ. It follows from $t < e^t$ that

$$||v_{1,n}||_{L^p(B_R(x_0))} \leq C. \tag{2.9}$$

Since the function $\triangle v_{2,n}$ is harmonic , we have

$$||v_{2,n}||_{L^1(B_R)} \leq ||v_n||_{L^1(B_R)} + ||v_{1,n}||_{L^1(B_R)} \leq C,$$

and so

$$||v_{2,n}||_{L^\infty(B_{\frac{R}{4}})} \leq C. \tag{2.10}$$

On the other hand, from $Q_1(x)e^{4v_n} \leq ce^{4v_{1,n}}e^{4v_{2,n}}$, we have

$$||Q_1(x)e^{4v_n}||_{L^p(B_{\frac{R}{4}})} \leq c||e^{4v_n}||_{L^p(B_{\frac{R}{4}})} \leq C, \quad \text{for some } p > 1. \tag{2.11}$$

In order to prove that $||v_n||_{L^\infty(B_\rho)} \leq C$, for some $\rho < \frac{R}{4}$, it is enough to prove a similar bounds as (2.10) for $v_{1,n}$, namely

$$||v_{1,n}||_{L^\infty(B_\rho)} \leq C. \tag{2.12}$$

For that matter, we use the first equation in $(*)$. Let us write $u_n = u_{1,n} + u_{2,n}$, where

$$\triangle^2 u_{1,n} = Q_1(x)e^{4v_n}, \text{ in } B_{\frac{R}{4}} \text{ and } u_{1,n} = \triangle u_{1,n} = 0 \text{ on } \partial B_{\frac{R}{4}}.$$

Observe that in view of (2.11), by standard elliptic regularity we have

$$||u_{1,n}||_{L^\infty(B_{\frac{R}{8}})} \leq C. \tag{2.13}$$

Notice that $\triangle^2 u_{2,n} = 0$ in $B_{\frac{R}{4}}$. Thus $\triangle u_{2,n}$ is harmonic in $B_{\frac{R}{4}}$, and it follows that

$$||u_{2,n}||_{L^\infty(B_{\frac{R}{16}})} \leq C. \tag{2.14}$$

From (2.13) and (2.14) we have

$$||u_n||_{L^\infty(B_{\frac{R}{16}})} \leq C. \tag{2.15}$$

Now we go back to the second equation in $(*)$. Using (2.15) and elliptic regularity we have

$$||v_{2,n}||_{L^\infty(B_{\frac{R}{16}})} \leq C. \tag{2.16}$$

From (2.12) and (2.16) we have

$$||v_n||_{L^\infty(B_{\frac{R}{16}})} \leq C,$$

which together with (2.15) proves our Step 1, taking $\rho = \frac{R}{16}$.

Step 2 We claim that $S_\mu \subset \Omega_\mu$ and $S_\nu \subset \Omega_\nu$.

In fact, This follows directly from Step 1 and the definition of the sets $\Omega_\mu,\ S_\mu,\ S_\nu$ and Ω_ν.

Step 3 We claim that $\Omega_\mu \subset S_\nu$ and $\Omega_\nu \subset S_\mu$.

Let $x_0 \in \Omega_\mu$. We claim that for each $R > 0$ we have

$$\lim_{n \to +\infty} ||u_n||_{L^\infty(B_R(x_0))} = +\infty. \tag{2.17}$$

Suppose by contradiction that there exists $R_0 > 0$ and a subsequence, which we denote also by (u_n), such that

$$||u_n||_{L^\infty(B_{R_0}(x_0))} \leq C.$$

So,

$$||Q_2(x)e^{4u_n}||_{L^\infty(B_{R_0}(x_0))} \leq C,$$

which implies that for $R < R_0$ we have

$$\int_{B_R(x_0)} Q_2(x)e^{4u_n} \leq CR^4.$$

Thus, there exists $R_1 > 0$, such that

$$\int_{B_{R_1}(x_0)} Q_2(x)e^{4u_n} \leq 8\pi^2.$$

This implies that x_0 is a regular point of μ, which is a contradiction.

Now we observe that there exists $R > 0$ such that x_0 is the only non-regular point in $B_R(x_0)$.

Next, we use (2.17) to prove that $x_0 \in S_\nu$. Indeed, from (2.17) there exists $(x_n) \subset B_R(x_0)$ such that $x_n \to \tilde{x}$ and $v(x_n) \to +\infty$. So, one needs to prove $\tilde{x} = x_0$. Indeed if this were not the case, then $\tilde{x}$ would be a regular point, which is not possible, since u_n is bounded in a neighborhood of a regular point.

With similar arguments as in the proof we just completed, we can prove that $\Omega_\nu \subset S_\mu$.

As a consequence of Step 2 and Step 3 we conclude that those four sets coincide:

$$S_\mu = \Omega_\mu = S_\nu = \Omega_\nu.$$

Step 4 We claim that $S_\mu = \varnothing$.

We prove this claim by contradiction. Suppose that there exist $x_0 \in S_\mu$. Since x_0 is isolated, we can take $R > 0$ such that $\overline{B_R(x_0)} \cap (S_\mu \setminus \{x_0\}) = \varnothing$.

Next, we consider the Navier boundary value problems in $B_R(x_0)$,

$$\triangle^2 z_n = Q_2(x)e^{4u_n}, \text{ in } B_R(x_0) \text{ and } z_n = \triangle z_n = 0 \text{ on } \partial B_R(x_0).$$

We know that the function u_n satisfies

$$\triangle^2 v_n = Q_2(x)e^{4u_n}, \text{ in } B_R(x_0) \text{ and } v_n \geq 0, \triangle v_n \leq 0 \text{ on } \partial B_R(x_0).$$

Thus, by the maximum principle we have

$$0 \leq z_n \leq v_n \text{ in } \overline{B_R(x_0)}.$$

Taking the limit we have that $z_n \to z$, where z is a solution of the problem

$$\triangle^2 z = \mu, \text{ in } B_R(x_0) \text{ and } z = \triangle z = 0 \text{ on } \partial B_R(x_0).$$

On the other hand the problem

$$\triangle^2 w = 8\pi^2 \delta_0, \text{ in } B_R(x_0) \text{ and } w = \triangle w = 0 \text{ on } \partial B_R(x_0)$$

has the solution

$$w(x) = ln \frac{R}{|x - x_0|}.$$

Since x_0 is not a regular point it follows that $\mu > 8\pi^2 \delta_0$. So

$$z(x) \geq ln|x - x_0|^{-1} + \circ(1), \; x \to x_0.$$

Now with the hypothesis $Q_1(x)e^{4t} \geq Ce^{4t}$, we have

$$\lim_{n \to +\infty} \int_{B_R(x_0)} Q_1(x) e^{4v_n} \geq C \int_{B_R(x_0)} e^{4w} = \infty,$$

which is impossible.

3. Qualitative properties of solutions of problem $(**)$

In this section, we study the qualitative properties of solutions of problem $(**)$.

From [8], Brezis-Merle implies that u is bounded from above when u satisfies $-\triangle u = V(x)e^u$ and other conditions. This result is used to study the qualitative properties and classification of solutions for some second order elliptic equation (See [12, 13]). Now, one naturally ask: is any solution (u, v) to system $(**)$ with $\int_{R^4} Q_1(x)e^{4v} < +\infty$ and $\int_{R^4} Q_2(x)e^{4u} < +\infty$ bounded from above? We will partially answer this problem and obtain the following result:

Theorem 3.1. *Assume $Q_i(x), i = 1, 2$ is a positive bounded away from 0 and bounded from above function and (u, v) is a C^2 solution of $(**)$ with $\int_{R^4} e^{4u} < +\infty$, $u(x) = \circ(|x|^2)$ and $\int_{R^4} e^{4v} < +\infty$, $v(x) = \circ(|x|^2)$ Then $u^+ \in L^\infty(R^4)$ and $v^+ \in L^\infty(R^4)$.*

Before we begin our proof, we need following lemmas:

Lemma 3.1. [3, 5] *Suppose (u, v) is a C^2 function on R^4 such that*

(a) $Q_1 e^{4v}$ and $Q_2 e^{4u}$ are in $L^1(R^4)$ with $0 < m_i \leq Q_i \leq M_i, i = 1, 2$ for some constants m_i, M_i;

(*b*) *in the sense of weak derivative, u, v respectively satisfies the following equations:*

$$\triangle u + \frac{2}{\beta_0}\int_{R^4} \frac{Q_1(y)e^{4v(y)}}{|x-y|^2}dy = 0$$

and

$$\triangle v + \frac{2}{\beta_0}\int_{R^4} \frac{Q_2(y)e^{4u(y)}}{|x-y|^2}dy = 0.$$

Then there are two constants $c_1, c_2 > 0$, respectively depending on v, u, such that $|\triangle u|(x) \le c_1$ on R^4 and $|\triangle v|(x) \le c_2$ on R^4. Where β_0 being given by $(-\triangle_x)^2(ln\frac{1}{|x-y|})$ $= \beta_0\delta_y(x)$. In fact, $\beta_0 = 8\pi^2$.

Lemma 3.2. [5] *Suppose S is C^2 function on R^4 such that $0 \le (-\triangle)S(x) \le A$ on R^4 for some constant A and $\int_{R^4} Q(y)e^{4S(y)}dy = \alpha < \infty$ with $0 < m \le Q \le M$. Then there exists a constant B, depending only on A, m, M and α such that $S(x) \le B$ on R^4.*

Lemma 3.3. *Suppose (u, v) is a solution of $(**)$. Let*

$$w_1(x) = \frac{1}{8\pi^2}\int_{R^4} ln\frac{|x-y|}{|y|+1}Q_1(y)e^{4v(y)}dy$$

and

$$w_2(x) = \frac{1}{8\pi^2}\int_{R^4} ln\frac{|x-y|}{|y|+1}Q_2(y)e^{4u(y)}dy.$$

Then there exist two constants c_1, c_2 such that

$$w_1(x) \le \beta_1 ln(|x|+1) + c_1$$

and

$$w_2(x) \le \beta_2 ln(|x|+1) + c_2,$$

where $\beta_1 = \frac{1}{8\pi^2}(\int_{R^4} Q_1(y)e^{4v(y)}dy)$ and $\beta_2 = \frac{1}{8\pi^2}(\int_{R^4} Q_2(y)e^{4u(y)}dy)$.

Proof. For $|x| \ge 4$, we decompose $R^4 = A_1 \cup A_2$, where $A_1 = \{y||y-x| \le \frac{|x|}{2}\}$ and $A_2 = \{y||y-x| \ge \frac{|x|}{2}\}$. For $y \in A_1$, we have $|y| \ge |x| - |x-y| \ge \frac{|x|}{2} \ge |x-y|$, which implies

$$ln\frac{|x-y|}{|y|+1} \le 0.$$

Since $|x-y| \le |x| + |y| \le |x|(|y|+1)$ for $|x|, |y| \ge 2$ and $ln|x-y| \le ln|x| + c$ for

$|x| \geq 4$ and $|y| \leq 2$, we have

$$\begin{aligned} w_1(x) &\leq \frac{1}{8\pi^2}\int_{A_2} ln\frac{|x-y|}{|y|+1}Q_1(y)e^{4v(y)}dy \\ &\leq \frac{1}{8\pi^2}(\int_{R^4} Q_1(y)e^{4v(y)}dy)ln|x| + c_1 \\ &= \beta_1 ln(|x|+1) + c_1. \end{aligned}$$

Similarly, we have

$$w_2(x) \leq \beta_2 ln(|x|+1) + c_2.$$

Lemma 3.4. *Suppose (u, v) is a solution of $(**)$ with $u(x) = \circ(|x|^2)$ and $v(x) = \circ(|x|^2)$. Then $\triangle u(x)$ and $\triangle v(x)$ can be represented by*

$$\triangle u(x) = -\frac{1}{4\pi^2}\int_{R^4}\frac{Q_1(y)e^{4v(y)}}{|x-y|^2}dy \tag{3.1}$$

and

$$\triangle v(x) = -\frac{1}{4\pi^2}\int_{R^4}\frac{Q_2(y)e^{4u(y)}}{|x-y|^2}dy. \tag{3.2}$$

Proof. Let $k = u + w_1$. It is obvious that $\triangle^2 k \equiv 0$ in R^4. Similar proof of Lin [1], we have for any $x_0 \in R^4$ and $r > 0$

$$2\pi^2 r^3 \exp(\frac{r^2}{2}\triangle k(x_0)) \leq e^{-4k(x_0)}\int_{|x-x_0|=r} e^{4k}d\sigma.$$

Since $k = u + w_1 \leq u(x) + \beta ln|x| + c$ follows from Lemma 3.3, we have

$$r^{3-4\beta}\exp(\frac{\triangle k(x_0)}{2}r^2) \in L^1[1, +\infty].$$

Thus $\triangle k(x_0) \leq 0$ for all $x_0 \in R^4$. By Liouville's theorem, $\triangle k(x) \equiv -c_1$ in R^4 for some constant $c_1 \geq 0$. Hence, we have

$$\triangle u(x) = -\frac{1}{4\pi^2}\int_{R^4}\frac{Q_1(y)e^{4v(y)}}{|x-y|^2}dy - c_1. \tag{3.3}$$

Now, we claim that $c_1 = 0$. Otherwise, we have $\triangle u(x) \leq -c_1 < 0$ for $|x| \geq R_0$ where R_0 is sufficiently large. Let

$$h(y) = u(y) + \epsilon|y|^2 + A(|y|^{-2} - R_0^{-2}), \tag{3.4}$$

where ϵ is small such that

$$\triangle h(y) = \triangle u + 8\epsilon < -\frac{c_1}{2} < 0 \tag{3.5}$$

for $|y| > R_0$, and A is sufficiently large so that $\inf_{|y|\geq R_0} h(y)$ is achieved by some $y_0 \in R^4$ with $|y_0| > R_0$. Applying the maximum principle to (3.5) at y_0, we have a contradiction. Hence, our claim is proved.

Similarly, we can prove that (3.2) holds.

Proof of Theorem 3.1. By Lemma 3.2 and Lemma 3.4, our conclusion holds.

Now, we study the qualitative properties of solutions of equation $(**)$. Following our Theorem 3.1 and Chen [13], we obtain the following results:

Theorem 3.2. *Assume that* $Q_i(x), i = 1, 2$ *is a positive* C^1 *function bounded away from* 0 *and from above and* (u, v) *is a* C^2 *solution of equation* $(**)$ *with* $\int_{R^4} e^{4u}dx < \infty$ *and* $\int_{R^4} e^{4v}dx < \infty$, $u(x) = \circ(|x|^2)$ *and* $v(x) = \circ(|x|^2)$. *Then*

$$-\beta_1 ln(|x| + 1) - c \leq u(x) \leq -\beta_1 ln(|x| + 1) + c \tag{3.6}$$

with $\beta_1 > 1$ *and*

$$-\beta_2 ln(|x| + 1) - c \leq v(x) \leq -\beta_2 ln(|x| + 1) + c \tag{3.7}$$

with $\beta_2 > 1$.

Furthermore, we have the following identity

$$\int_{R^4} [(x, \nabla Q_1)e^{4v} + (x, \nabla Q_2)e^{4u}]dx = 32\pi^2[\beta_1\beta_2 - (\beta_1 + \beta_2)]. \tag{3.8}$$

Theorem 3.3. *Suppose* (u, v) *satisfies the assumptions of Theorem 3.2 and* $Q_i, i = 1, 2$ *is radially symmetric and monotone decreasing, then* u *and* v *are radially symmetric and monotone decreasing.*

Lemma 3.5. *Assume* (u, v) *satisfies the assumptions of Theorem 3.2, then*

$$\frac{w_i(x)}{ln|x|} \to \beta_i, \text{ uniformly as } |x| \to \infty.$$

Proof. Here we prove $w_1(x) \to \beta_1 ln|x|$ as $|x| \to \infty$. We need only to verify that

$$I = \int_{R^4} \frac{ln|x-y| - ln(|y|+1) - ln|x|}{ln|x|} Q_1(y)e^{4v(y)}dy \to 0 \text{ as } |x| \to \infty.$$

Write $I = I_1 + I_2 + I_3$, are the integrals on the regions $D_1 = \{y : |x - y| \leq 1\}$, $D_2 = \{y : |x - y| > 1 \text{ and } |y| \leq k\}$ and $D_3 = \{y : |x - y| > 1 \text{ and } |y| > k\}$

respectively. We may assume that $|x| \geq 3$.

(a) To estimate I_1, we simply notice that

$$|I_1| \leq C \int_{|x-y|\leq 1} Q_1(y)e^{4v(y)}dy - \frac{1}{ln|x|}\int_{|x-y|\leq 1} ln|x-y|Q_1(y)e^{4v(y)}dy.$$

Then by the boundedness of Qe^{4v} (See Theorem 3.1) and $\int_{R^4} Q_1(y)e^{4v(y)}dy$, we see that $I_1 \to 0$ as $|x| \to \infty$.

(b) For each fixed k, in region D_2, we have, as $|x| \to \infty$,

$$\frac{ln|x-y| - ln(|y|+1) - ln|x|}{ln|x|} \to 0.$$

Hence $I_2 \to 0$.

(c) To see $I_3 \to 0$, we use the fact that for $|x-y| > 1$

$$|\frac{ln|x-y| - ln(|y|+1) - ln|x|}{ln|x|}| \leq c.$$

Then let $k \to \infty$.

Similarly, we have $w_2(x) \to \beta_2 ln|x|$ as $|x| \to \infty$.

Lemma 3.6. *Assume (u, v) satisfies the assumptions of Theorem 3.2, then*

$$u(x) = \frac{1}{8\pi^2}\int_{R^4} ln\frac{|y|+1}{|x-y|}Q_1(y)e^{4v(y)}dy + c_0$$

and

$$v(x) = \frac{1}{8\pi^2}\int_{R^4} ln\frac{|y|+1}{|x-y|}Q_2(y)e^{4u(y)}dy + \tilde{c_0}$$

where c_0 and $\tilde{c_0}$ are two constants.

Proof. By the Lemma 3.4, we have $\triangle(u + w_1) = 0$ in R^4. By Theorem 3.1, we have $u^+ \in L^\infty$. So, combing lemma 3.3, we have $u + w_1 \leq cln|x| + c$, since $u + w_1$ is harmonic function, by the gradient estimates of harmonic functions, we have $u(x) + w_1(x) \equiv c_0$. Similarly, we have $v(x) + w_2(x) \equiv \tilde{c_0}$.

Lemma 3.7. *Suppose (u, v) satisfies the assumptions of Theorem 3.2, then $u_1(x) \geq -\beta_1 ln(|x|+1) - c_1$ with $\beta_1 > 1$ and $u_2(x) \geq -\beta_2 ln(|x|+1) - c_2$ with $\beta_2 > 1$.*

Proof. By Lemma 3.3 and Lemma 3.6, we have

$$u(x) > -\beta_1 ln(|x|+1) - c_1$$

and

$$v(x) > -\beta_2 ln(|x|+1) - c_2.$$

From above inequality, $\int_{R^4} e^{4v}dx < +\infty$ and $\int_{R^4} e^{4u}dx < +\infty$, we have $\beta_1 > 1, \beta_2 > 1$.

Lemma 3.8. *Suppose (u, v) satisfies the assumptions of Theorem 3.2, then $u(x) \le -\beta_1 ln(|x|+1) + c_1$ and $v(x) \le -\beta_2 ln(|x|+1) + c_2$.*

Proof. In fact, for $|x-y| \ge 1$, we have

$$|x| \le |x-y|(|y|+1).$$

Then

$$ln|x| - 2ln(|y|+1) \le ln|x-y| - ln(|y|+1).$$

Consequently,

$$\begin{aligned}
w_1(x) &\ge \frac{1}{8\pi^2}\int_{|x-y|\ge 1}(ln|x| - 2ln(|y|+1))Q_1(y)e^{4v(y)}dy \\
&+ \frac{1}{8\pi^2}\int_{|x-y|\le 1}(ln|x-y| - ln(|y|+1))Q_1(y)e^{4v(y)}dy \\
&\ge \beta_1 ln|x| - \frac{ln|x|}{8\pi^2}\int_{|x-y|\le 1} Q_1(y)e^{4v(y)}dy \\
&+ \frac{1}{8\pi^2}\int_{|x-y|\le 1} ln|x-y|Q_1(y)e^{4v(y)}dy \\
&- \frac{1}{8\pi^2}\int_{R^4} ln(|y|+1)Q_1(y)e^{4v(y)}dy \\
&= \beta_1 ln|x| + I_1 + I_2 + I_3.
\end{aligned}$$

Taking into account of the fact (see Lemma 3.5) that

$$\frac{u(x)}{ln|x|} \to -\beta_1 \text{ and } \beta_1 > 1$$

and by the boundedness of $Q_1(x)$, we have

$$I_1, I_2 \to 0 \text{ as } |x| \to \infty$$

and I_3 is finite. Therefore

$$w_1(x) \ge \beta_1 ln(|x|+1) - c_1.$$

By lemma 3.6, we have

$$u(x) \leq -\beta_1 ln(|x|+1) + c_1.$$

Similarly, we have

$$v(x) \leq -\beta_2 ln(|x|+1) + c_2.$$

Proof of Theorem 3.2. By Lemma 3.7 and Lemma 3.8, then (3.6) and (3.7) hold. By Lin's Lemma 2.6 and lemma 2.7 in [1], we can similarly infer that (3.8) hold.

Proof of Theorem 3.3. By Theorem 3.2, we have $u(x) \to -\beta_1 ln|x|$ as $|x| \to \infty$, where $\beta_1 > 1$. Let $\tilde{v}(x) = -\triangle u(x)$. By revised Lin's Lemma 2.8 [1], $\tilde{v}(x)$ has a harmonic asymptotic expansion at ∞ :

$$\begin{cases} \tilde{v}(x) & = \frac{1}{|x|^2}(2\beta_1 + \sum_{j=1}^{4} \frac{a_j}{|x|^2}) + \bigcirc(\frac{1}{|x|^4}), \\ \tilde{v}_{x_i} & = -\frac{4\beta_1 x_i}{|x|^4} + \bigcirc(\frac{1}{|x|^4}), \\ \tilde{v}_{x_i x_j} & = \bigcirc(\frac{1}{|x|^4}). \end{cases} \tag{3.9}$$

Where a_j $(j = 1$ to $4)$ are constants. Let $\tilde{u}(x) = -\triangle v(x)$. Similarly, we have

$$\begin{cases} \tilde{u}(x) & = \frac{1}{|x|^2}(2\beta_2 + \sum_{j=1}^{4} \frac{b_j}{|x|^2}) + \bigcirc(\frac{1}{|x|^4}), \\ \tilde{u}_{x_i} & = -\frac{4\beta_2 x_i}{|x|^4} + \bigcirc(\frac{1}{|x|^4}), \\ \tilde{u}_{x_i x_j} & = \bigcirc(\frac{1}{|x|^4}). \end{cases} \tag{3.10}$$

Where b_j $(j = 1$ to $4)$ are constants.

Remained proof essentially equals to Lin's proof. We omit it here.

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