Reflexive–EP elements in rings

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Abstract

We define and characterize reflexive–EP elements in rings, that is elements which commute with their image-kernel (p,q)-inverse.

Key words and phrases: EP elements, outer inverse, group inverse, ring.

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1 Introduction

Let $\mathcal{R}$ be a ring with the unit 1. We use $\mathcal{R}^\bullet$ to denote the set of all idempotents of $\mathcal{R}$. Let $a \in \mathcal{R}$. We say that $b \in \mathcal{R}$ is an outer inverse of a provided that $bab = b$ holds. An element $b \in \mathcal{R}$ is an inner inverse of a, if $aba = a$ holds. In this case a is inner regular (or relatively regular). The set of all inner regular elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^-$. If b is both inner and outer generalized inverse of a, then b is a reflexive generalized inverse of a.

The outer inverse is not unique in general, but it is unique if we fix the corresponding idempotents ([9]): let $a \in \mathcal{R}$, and let $p, q \in \mathcal{R}^\bullet$. An element $b \in \mathcal{R}$ satisfying

$$bab = b, \qquad ba = p, \qquad 1 - ab = q,$$

will be called (p,q)–outer generalized inverse of a, written $b = a^{(2)}_{p,q}$. If $a^{(2)}_{p,q}$ exists, it is unique.

Instead of prescribing the idempotents ab and ba, we may prescribe certain kernel and image ideals related to these idempotents([12]): let $p, q \in \mathcal{R}^\bullet$, an element $b \in \mathcal{R}$ is the image-kernel (p,q)-inverse of a if

$$bab = b, \quad ba\mathcal{R} = p\mathcal{R} \quad \text{and} \quad (1-ab)\mathcal{R} = q\mathcal{R}.$$

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The image-kernel (p, q)-inverse b is unique if it exists, and it will be denoted by $a^{\times}$.

Observe that the image-kernel (p, q)-inverse of Kantún-Montiel ([12]) coincides with the (p, q, l)-outer generalized inverse of Cao and Xue ([6]), but his approach is different.

If the image-kernel (p, q)-inverse b of a satisfies the equations $a = aba$, then b is a reflexive image-kernel (p, q)-inverse of a and it is denote by $a^{(1,\times)}$. It follows that $a^{(1,\times)}$ is also unique in the case when it exists.

An element $a \in \mathcal{R}$ is group invertible if there is $a^{\#} \in \mathcal{R}$ such that

$$aa^{\#}a = a, \quad a^{\#}aa^{\#} = a^{\#}, \quad aa^{\#} = a^{\#}a.$$

Recall that $a^{\#}$ is uniquely determined by previous equations and it is called the group inverse of a. We use $\mathcal{R}^{\#}$ to denote the set of all group invertible elements of $\mathcal{R}$. The group inverse $a^{\#}$ double commutes with a, that is, $ax = xa$ implies $a^{\#}x = xa^{\#}$ [2]. Recall that the group inverse is a particular case of the generalized Drazin inverse (for more details see [14, 15, 16]).

For $u, v \in \mathcal{R}^{\bullet}$, notice that $u^{\circ} = (1-u)\mathcal{R}$ and ${}^{\circ}u = \mathcal{R}(1-u)$. Also, we have

$$u\mathcal{R} = v\mathcal{R} \;\Leftrightarrow\; {}^{\circ}u =^{\circ} v$$

and

$$\mathcal{R}u = \mathcal{R}v \;\Leftrightarrow\; u^{\circ} = v^{\circ}.$$

Complex matrices and Hilbert spaces operators with closed ranges A with the property that the ranges of A and A^* coincides, are known as EP or range-Hermitian (EP for equal projections onto $R(A)$ and $R(A^*)$). EP matrices, EP linear operators on Banach or Hilbert spaces and EP elements of C^*–algebras or Banach algebras have been investigated by many authors (see [1, 3, 7, 8, 10]). In rings with involution EP elements are those elements for which the group and the Moore–Penrose inverse exist and coincide [17, 20]. The EP elements are important since they are characterized by commutativity with their Moore–Penrose inverse.

Tian and Wang [21] defined weighted–EP matrices as matrices that commute with their weighted Moore-Penrose inverse. Similar objects in the contexts of C^*-algebra elements, Banach space operators and Banach algebra elements are investigated in [5, 18].

The factorization of EP objects is a central topic of this area. In fact, factorizations of EP elements were considered for matrices, Hilbert and Banach space operators and Banach and C^*- algebra elements [4, 8, 19].

The main objective of this article is to introduce and study the reflexive–EP elements, that is elements which commute with their image-kernel (p, q)-inverse. In this way we extend EP objects from C^*-algebras or rings with involution to rings, a context where no involution is available, using an idea that is similar to the one that led to introduce EP Banach space operators and EP Banach algebra elements. Also, we characterize reflexive–EP elements in rings using one kind of factorization.

2 Reflexive–EP elements

In the beginning of this section, we state the definition of reflexive–EP elements in rings.

Definition 2.1. Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^\bullet$ such that $a^{(1,\times)}$ exists. The element a is reflexive–EP if $aa^{(1,\times)} = a^{(1,\times)}a$.

In a ring with involution $\mathcal{R}$, let $a \in \mathcal{R}$ be a Moore-Penrose invertible element and $b = a^\dagger$, the Moore-Penrose inverse of a. Note that b is the reflexive image-kernel (p, q)-inverse of a, where $p = a^\dagger a$ and $q = 1 - aa^\dagger$, so that this notion consists in an extension of the Moore-Penrose inverse to rings. The extension of the EP objects from C^*-algebras or rings with involution to rings, is then clear.

Now, we present a lot of equivalent conditions for an element of a ring to be reflexive–EP.

Theorem 2.1. *Let $p, q \in \mathcal{R}^\bullet$ and let $a \in \mathcal{R}$ such that $a^{(1,\times)}$ exists. Then the following statements are equivalent:*

(i) $aa^{(1,\times)} = a^{(1,\times)}a$,

(ii) $a \in \mathcal{R}^\#$ *and* $a^{(1,\times)} = a^\#$,

(iii) $a\mathcal{R} = p\mathcal{R}$ *and* $\mathcal{R}a = \mathcal{R}(1 - q)$,

(iv) $a\mathcal{R} \subset p\mathcal{R}$ *and* $\mathcal{R}a \subset \mathcal{R}(1 - q)$,

(v) $a\mathcal{R} \supset p\mathcal{R}$ *and* $\mathcal{R}a \supset \mathcal{R}(1 - q)$,

(vi) ${}^\circ a = {}^\circ p$ *and* $a^\circ = (1 - q)^\circ$,

(vii) $a \in \mathcal{R}^\#$ *and* $aa^\# a^{(1,\times)} = a^{(1,\times)}a^\# a$,

(viii) $a \in \mathcal{R}^\#$ *and* $a^k = a^{(1,\times)}aa^k = a^k aa^{(1,\times)}$, *for any/some integer* $k \geq 1$,

(ix) $a \in \mathcal{R}^{\#}$ *and* $a^{\#}a^{(1,\times)} = a^{(1,\times)}a^{\#}$,

(x) $a \in \mathcal{R}^{\#}$ *and* $aa^{(1,\times)}a^{(1,\times)}a = a^{(1,\times)}aaa^{(1,\times)}$,

(xi) $a \in \mathcal{R}^{\#}$ *and* $(a^{(1,\times)})^2a^{\#} = a^{(1,\times)}a^{\#}a^{(1,\times)} = a^{\#}(a^{(1,\times)})^2$,

(xii) $a \in a^{(1,\times)}\mathcal{R}^{-1} \cap \mathcal{R}^{-1}a^{(1,\times)}$,

(xiii) *there exist* x, $y \in R$ *such that* $a = a^{(1,\times)}x = ya^{(1,\times)}$ *and* $x\mathcal{R} = \mathcal{R}$ *and* $y^{\circ} = \{0\}$,

(xiv) $a \in a^{(1,\times)}\mathcal{R} \cap \mathcal{R}a^{(1,\times)}$,

(xv) $a \in \mathcal{R}^{\#}$ *and* $a^ka^{(1,\times)} = a^{(1,\times)}a^k$, *for any/some integer* $k \geq 1$,

(xvi) $a \in \mathcal{R}^{\#}$ *and* $(a^{(1,\times)})^k = (a^{\#})^k$, *for any/some integer* $k \geq 1$,

(xvii) $(a^{(1,\times)})^{(1,\times)} = a$,

(xviii) $(a + \lambda a^{(1,\times)})\mathcal{R} = (\lambda a + a^3)\mathcal{R}$ *and* $\mathcal{R}(a + \lambda a^{(1,\times)}) = \mathcal{R}(\lambda a + a^3)$, *for any/some complex number* $\lambda \neq 0$.

Proof. (i) $\Leftrightarrow$ (ii): Since the group inverse is unique, this equivalence holds.

(i) $\Rightarrow$ (iii): Observe that $a\mathcal{R} = aa^{(1,\times)}\mathcal{R} = a^{(1,\times)}a\mathcal{R} = p\mathcal{R}$ and $\mathcal{R}a = \mathcal{R}a^{(1,\times)}a = \mathcal{R}aa^{(1,\times)} = \mathcal{R}(1-q)$.

(iii) $\Rightarrow$ (iv): It is obvious.

(iv) $\Rightarrow$ (i): The hypothesis $a\mathcal{R} \subset p\mathcal{R}$ can be write as $aa^{(1,\times)}\mathcal{R} \subset a^{(1,\times)}a\mathcal{R}$ which implies $aa^{(1,\times)} = a^{(1,\times)}ax$ for some $x \in \mathcal{R}$. Hence, $aa^{(1,\times)} = a^{(1,\times)}aa^{(1,\times)}ax = a^{(1,\times)}aaa^{(1,\times)}$. In the similar way, $\mathcal{R}a \subset \mathcal{R}(1-q)$ gives $a^{(1,\times)}a = a^{(1,\times)}aaa^{(1,\times)}$. Therefore, $a^{(1,\times)}a = aa^{(1,\times)}$.

(iii) $\Rightarrow$ (v) $\Rightarrow$ (i): Analogy as (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i), we can prove these implications.

(iii) $\Leftrightarrow$ (vi): For idempotents $aa^{(1,\times)}$, $a^{(1,\times)}a$, p, $1-q$, we have

$$a\mathcal{R} = p\mathcal{R} \;\Leftrightarrow\; aa^{(1,\times)}\mathcal{R} = p\mathcal{R} \;\Leftrightarrow\; {}^{\circ}(aa^{(1,\times)}) = {}^{\circ}p \;\Leftrightarrow\; {}^{\circ}a = {}^{\circ}p$$

and

$$\begin{aligned}\mathcal{R}a = \mathcal{R}(1-q) \;&\Leftrightarrow\; \mathcal{R}a^{(1,\times)}a = \mathcal{R}(1-q) \;\Leftrightarrow\; (a^{(1,\times)}a)^{\circ} = (1-q)^{\circ}\\ &\Leftrightarrow\; a^{\circ} = (1-q)^{\circ}.\end{aligned}$$

(ii) $\Rightarrow$ (vii) $\wedge$ (viii): This is trivial.

(vii) $\Rightarrow$ (i): The equality $aa^{\#}a^{(1,\times)} = a^{(1,\times)}a^{\#}a$ gives

$$aa^{(1,\times)} = aaa^{\#}a^{(1,\times)} = aa^{(1,\times)}a^{\#}a = aa^{(1,\times)}aa^{\#} = aa^{\#}$$

and

$$a^{(1,\times)}a = a^{(1,\times)}a^{\#}aa = aa^{\#}a^{(1,\times)}a = a^{\#}a.$$

So, $aa^{(1,\times)} = a^{\#}a = a^{(1,\times)}a$.

(viii) $\Rightarrow$ (i): If $a \in \mathcal{A}^{\#}$ and $a^k = a^{(1,\times)}aa^k = a^k aa^{(1,\times)}$, for any/some integer $k \geq 1$, then

$$aa^{\#} = a^k(a^{\#})^k = a^{(1,\times)}aa^k(a^{\#})^k = a^{(1,\times)}aaa^{\#} = a^{(1,\times)}a$$

and

$$a^{\#}a = (a^{\#})^k a^k = (a^{\#})^k a^k aa^{(1,\times)} = a^{\#}aaa^{(1,\times)} = aa^{(1,\times)}.$$

So, $a^{(1,\times)}a = aa^{\#} = aa^{(1,\times)}$.

(i) $\Rightarrow$ (ix): Since the group inverse $a^{\#}$ double commutes with a, we conclude that this implication is true.

(ix) $\Rightarrow$ (i): Consequently, by $(a^{\#})^{\#} = a$ and double commutativity.

(i) $\Rightarrow$ (x): Obviously.

(x) $\Rightarrow$ (viii): Let $a \in \mathcal{R}^{\#}$ and $aa^{(1,\times)}a^{(1,\times)}a = a^{(1,\times)}aaa^{(1,\times)}$. Then

$$\begin{aligned} a^{(1,\times)}aaa^{(1,\times)} &= aa^{(1,\times)}a^{(1,\times)}a = aa^{\#}(aa^{(1,\times)}a^{(1,\times)}a)a^{\#}a \\ &= a^{\#}aa^{(1,\times)}aaa^{(1,\times)}aa^{\#} = a^{\#}a. \end{aligned} \tag{1}$$

Multiplying the equality (1) by a^k, for any/some integer $k \geq 1$, from the left side, we get $a^k aa^{(1,\times)} = a^k$. Also, multiplying the equality (1) by a^k, for any/some integer $k \geq 1$, from the right side, we obtain $a^{(1,\times)}aa^k = a^k$. Hence, (viii) holds.

(ix) $\Rightarrow$ (xi): Trivial.

(xi) $\Rightarrow$ (vii): If $a \in \mathcal{R}^{\#}$ and $(a^{(1,\times)})^2a^{\#} = a^{(1,\times)}a^{\#}a^{(1,\times)} = a^{\#}(a^{(1,\times)})^2$, we get

$$\begin{aligned} (a^{(1,\times)})^2a &= (a^{(1,\times)})^2a^{\#}a^2 = a^{(1,\times)}a^{\#}a^{(1,\times)}a^2 \\ &= a^{(1,\times)}(a^{\#})^2aa^{(1,\times)}a^2 = a^{(1,\times)}a^{\#}a \end{aligned}$$

and

$$a(a^{(1,\times)})^2 = a^2a^{\#}(a^{(1,\times)})^2 = a^2a^{(1,\times)}a^{\#}a^{(1,\times)} = aa^{\#}a^{(1,\times)}.$$

Since the group inverse $(a^{\#})^{\#} = a$ double commutes with $a^{\#}$, $(a^{(1,\times)})^2a^{\#} = a^{\#}(a^{(1,\times)})^2$ implies $(a^{(1,\times)})^2a = a(a^{(1,\times)})^2$ which gives $a^{(1,\times)}a^{\#}a = aa^{\#}a^{(1,\times)}$.

(i) $\Rightarrow$ (xii): From the hypothesis $aa^{(1,\times)} = a^{(1,\times)}a$, we can obtain $a = (a^2 + 1 - a^{(1,\times)}a)a^{(1,\times)}$ and

$$(a^2 + 1 - a^{(1,\times)}a)^{-1} = (a^{(1,\times)})^2 + 1 - a^{(1,\times)}a.$$

So, $a \in \mathcal{R}^{-1}a^{(1,\times)}$. In the same way, we have $a = a^{(1,\times)}(a^2+1-aa^{(1,\times)})$ and $(a^2+1-aa^{(1,\times)})^{-1} = (a^{(1,\times)})^2+1-aa^{(1,\times)}$. Therefore, $a \in a^{(1,\times)}\mathcal{R}^{-1}$ and the statement (xii) is satisfied.

(xii) $\Rightarrow$ (xiii) $\Rightarrow$ (xiv): Obviously.

(xiv) $\Rightarrow$ (iv): This implication follows by $a^{(1,\times)}\mathcal{R} = a^{(1,\times)}a\mathcal{R} = p\mathcal{R}$ and $\mathcal{R}a^{(1,\times)} = \mathcal{R}aa^{(1,\times)} = \mathcal{R}(1-q)$.

(i) $\Rightarrow$ (xv): It is obvious.

(xv) $\Rightarrow$ (iv): Assume that $a \in \mathcal{R}^{\#}$ and $a^k a^{(1,\times)} = a^{(1,\times)}a^k$, for any/some integer $k \geq 1$. Since $a = a^k(a^{\#})^{k-1} = a^k a^{(1,\times)}a(a^{\#})^{k-1}$, we have

$$a\mathcal{R} \subset a^k a^{(1,\times)}\mathcal{R} = a^{(1,\times)}a^k\mathcal{R} \subset a^{(1,\times)}a\mathcal{R} = p\mathcal{R}.$$

From $a = (a^{\#})^{k-1}aa^{(1,\times)}a^k = (a^{\#})^{k-1}aa^k a^{(1,\times)}$, we deduce that $\mathcal{R}a \subset \mathcal{R}aa^{(1,\times)} = \mathcal{R}(1-q)$. Hence, (iv) holds.

(ii) $\Rightarrow$ (xvi): We can easy verify.

(xvi) $\Rightarrow$ (iv): If $a \in \mathcal{R}^{\#}$ and $(a^{(1,\times)})^k = (a^{\#})^k$, for any/some integer $k \geq 1$, then

$$a\mathcal{R} = (a^{\#})^k a^{k+1}\mathcal{R} = (a^{(1,\times)})^k a^{k+1}\mathcal{R} \subset a^{(1,\times)}\mathcal{R} = p\mathcal{R}$$

and

$$\mathcal{R}a \subset \mathcal{R}a^{k+1}(a^{\#})^k \subset \mathcal{R}(a^{(1,\times)})^k \subset \mathcal{R}a^{(1,\times)} = \mathcal{R}(1-q).$$

(xvii) $\Rightarrow$ (iii): From $(a^{(1,\times)})^{(1,\times)} = a$, we have $a\mathcal{R} = aa^{(1,\times)}\mathcal{R} = p\mathcal{R}$ and $\mathcal{R}a = \mathcal{R}a^{(1,\times)}a = \mathcal{R}(1-q)$.

(iii) $\Rightarrow$ (xvii): The condition (iii) gives $aa^{(1,\times)}\mathcal{R} = a\mathcal{R} = p\mathcal{R}$ and $\mathcal{R}a^{(1,\times)}a = \mathcal{R}a = \mathcal{R}(1-q)$. So, $(a^{(1,\times)})^{(1,\times)} = a$.

(xviii) $\Rightarrow$ (v): Since $(a+\lambda a^{(1,\times)})\mathcal{R} = (\lambda a + a^3)\mathcal{R}$, $\lambda \neq 0$, there exists $x \in \mathcal{R}$ such that $a + \lambda a^{(1,\times)} = (\lambda a + a^3)x$. Now, we have

$$\begin{aligned} a + \lambda aa^{(1,\times)}a^{(1,\times)} &= aa^{(1,\times)}(a+\lambda a^{(1,\times)}) = aa^{(1,\times)}(\lambda a + a^3)x \\ &= (\lambda a + a^3)x = a + \lambda a^{(1,\times)}, \end{aligned}$$

which gives $a^{(1,\times)} = aa^{(1,\times)}a^{(1,\times)}$. Therefore,

$$p\mathcal{R} = a^{(1,\times)}\mathcal{R} = aa^{(1,\times)}a^{(1,\times)}\mathcal{R} \subset a\mathcal{R}.$$

Analogously, by $\mathcal{R}(a+\lambda a^{(1,\times)}) = \mathcal{R}(\lambda a + a^3)$, we obtain $a^{(1,\times)} = a^{(1,\times)}a^{(1,\times)}a$ which implies $\mathcal{R}(1-q) = \mathcal{R}a^{(1,\times)} \subset \mathcal{R}a$.

(ii) $\Rightarrow$ (xviii): Applying the hypothesis $a^{(1,\times)} = a^{\#}$, from

$$a + \lambda a^{\#} = (a^3 + \lambda a)(a^{\#})^2 = (a^{\#})^2(a^3 + \lambda a)$$

and

$$a^3 + \lambda a = (a + \lambda a^{\#})a^2 = a^2(a + \lambda a^{\#}),$$

we deduce that condition (xviii) holds.

□

We can check the following result.

Corollary 2.1. *Let $p, q \in \mathcal{R}^\bullet$ and let $a \in \mathcal{R}$ such that $a^{(1,\times)}$ exists. Then the following statements are equivalent:*

(i) $aa^{(1,\times)} = a^{(1,\times)}a$,

(ii) $a^{(1,\times)} = a(a^{(1,\times)})^2 = (a^{(1,\times)})^2 a$;

(iii) $a \in \mathcal{R}^{\#}$ *and* $a^{\#} = a(a^{(1,\times)})^2 = (a^{(1,\times)})^2 a$,

(iv) *if* $b \in \mathcal{R}$ *is such that* $ab = ba$, *then* $a^{(1,\times)}b = ba^{(1,\times)}$,

(v) $a \in \mathcal{R}^{\#}$ *and* $a^{(1,\times)}a^{\#}a + aa^{\#}a^{(1,\times)} = 2a^{(1,\times)}$,

(vi) $a \in \mathcal{R}^{\#}$ *and* $a^k aa^{(1,\times)} + a^{(1,\times)}aa^k = 2a^k$, *for any/some integer* $k \geq 1$,

(vii) $a \in \mathcal{R}^{\#}$ *and* $a^{2k-1} = a^{(1,\times)}a^{2k+1}a^{(1,\times)}$, *for any/some integer* $k \geq 1$,

(viii) $a \in \mathcal{R}^{\#}$ *and* $(a^{\#})^k a^{(1,\times)} = a^{(1,\times)}(a^{\#})^k$, *for any/some integer* $k \geq 1$,

(ix) $aa^{(1,\times)}(a + \lambda a^{(1,\times)}) = (a + \lambda a^{(1,\times)})aa^{(1,\times)}$ *and* $a^{(1,\times)}a(a + \lambda a^{(1,\times)}) = (a + \lambda a^{(1,\times)})a^{(1,\times)}a$, *for any/some complex number* $\lambda \neq 0$,

(x) $a \in \mathcal{A}^{\#}$ *and* $(a^{\#})^k a^{(1,\times)}a = (a^{(1,\times)})^k$, *for any/some integer* $k \geq 1$,

(xi) $a \in \mathcal{A}^{\#}$ *and* $(a^{\#})^{k+l-1} = (a^{(1,\times)})^l(a^{\#})^{k-1} = (a^{\#})^{k-1}(a^{(1,\times)})^l$, *for any/some integers* $k, l \geq 1$.

(xii) $a \in \mathcal{A}^{\#}$ *and* $a(a^{(1,\times)})^{k+1} = (a^{\#})^k = (a^{(1,\times)})^{k+1}a$, *for any/some integer* $k \geq 1$,

(xiii) $a \in \mathcal{A}^{\#}$ *and* $(a^{(1,\times)})^{k+1} = (a^{\#})^k a^{(1,\times)} = a^{(1,\times)}(a^{\#})^k$, *for any/some integer* $k \geq 1$.

Let $\mathcal{A}$ be a complex unital Banach algebra. One important characterization of reflexive–EP elements in a Banach algebra will be given in the next result.

Theorem 2.2. *Let $a \in \mathcal{A}^-$ and $p, q \in \mathcal{A}^\bullet$ such that $a^{(1,2)}_{p,q}$ exists. Then, the following statements are equivalent:*

(i) $aa^{(1,\times)} = a^{(1,\times)}a$,

(ii) *there exists some holomorphic function* $f\colon U \to \mathbb{C}$*, where* U *is an open neighbourhood of* $\sigma(a)$*, such that* $a^{(1,\times)} = f(a)$.

Proof. (i) $\Rightarrow$ (ii): Since $a^{(1,\times)} = a^{\#}$, by [13, Theorem 4.4], $a^{\#} = f(a)$, where f is holomorphic in a neighbourhood of $\sigma(a)$, and $f(\lambda) = 0$ in a neighbourhood of 0, $f(\lambda) = \lambda^{-1}$ in a neighbourhood of $\sigma(a)\backslash\{0\}$. Consequently, the statements (ii) holds.

(ii) $\Rightarrow$ (i): If $a^{(1,\times)} = f(a)$, for some function f holomorphic in a neighbourhood of $\sigma(a)$, by a property of the holomorphic calculus, $a^{(1,\times)}$ commutes with a. $\square$

3 Factorization $a = uv$

Let $p, q \in \mathcal{R}^{\bullet}$. Consider a factorization of $a \in \mathcal{R}$ of the form

$$a = uv, \quad u'u = 1 = vv', \quad \mathcal{R}u' = \mathcal{R}(1-q), \quad v'\mathcal{R} = p\mathcal{R}. \tag{2}$$

There exist $u', v' \in \mathcal{R}$ such that $u'u = 1 = vv'$ if and only if $\mathcal{R}u = \mathcal{R} = v\mathcal{R}$. Therefore, $u' = u_{1,q}^{(1,\times)}$ and $v' = v_{p,1}^{(1,\times)}$, where $u_{1,q}^{(1,\times)}$ is the image-kernel $(1,q)$-inverse of u and $v_{p,1}^{(1,\times)}$ is the image-kernel $(p,1)$-inverse of v.

Theorem 3.1. *Let* $p, q \in \mathcal{R}^{\bullet}$ *and let* $a \in \mathcal{R}$ *have a factorization (2). Then* $a^{(1,\times)}$ *exists and* $a^{(1,\times)} = v_{p,1}^{(1,\times)}u_{1,q}^{(1,\times)}$.

Proof. Denote by $a' = v_{p,1}^{(1,\times)}u_{1,q}^{(1,\times)}$. Observe that

$$a'aa' = v_{p,1}^{(1,\times)}u_{1,q}^{(1,\times)}uvv_{p,1}^{(1,\times)}u_{1,q}^{(1,\times)} = v_{p,1}^{(1,\times)} \cdot 1 \cdot u_{1,q}^{(1,\times)} = a'$$

and

$$aa'a = uvv_{p,1}^{(1,\times)}u_{1,q}^{(1,\times)}uv = u \cdot 1 \cdot v = a.$$

Also, $a'a\mathcal{R} = v_{p,1}^{(1,\times)}v\mathcal{R} = p\mathcal{R}$ and $\mathcal{R}aa' = \mathcal{R}uu_{1,q}^{(1,\times)} = \mathcal{R}(1-q)$. Thus, $a' = a^{(1,\times)}$. $\square$

Lemma 3.1. *Let* $p, q \in \mathcal{R}^{\bullet}$ *and let* $a \in \mathcal{R}$ *have a factorization (2). Then*

(i) $v_{p,1}^{(1,\times)}\mathcal{R} = v_{p,1}^{(1,\times)}v\mathcal{R} = p\mathcal{R}$,

(ii) ${}^{\circ}v_{p,1}^{(1,\times)} = {}^{\circ}v_{p,1}^{(1,\times)}v = {}^{\circ}p$,

(iii) $\mathcal{R}u_{1,q}^{(1,\times)} = \mathcal{R}uu_{1,q}^{(1,\times)} = \mathcal{R}(1-q)$,

(iv) $(u_{1,q}^{(1,\times)})^\circ = (uu_{1,q}^{(1,\times)})^\circ = (1-q)^\circ$.

We study characterizations of elements which satisfy $aa^{(1,\times)} = a^{(1,\times)}a$ in rings using factorization of the form $a = uv$.

Theorem 3.2. *Let $p, q \in \mathcal{R}^\bullet$ and let $a \in \mathcal{R}$ have a factorization (2). Then the following statements are equivalent:*

(i) $aa^{(1,\times)} = a^{(1,\times)}a$,

(ii) $uu_{1,q}^{(1,\times)} = v_{p,1}^{(1,\times)}v$,

(iii) $\mathcal{R}v = \mathcal{R}(1-q)$ *and* $u\mathcal{R} = p\mathcal{R}$,

(iv) $v^\circ = (1-q)^\circ$ *and* ${}^\circ u = {}^\circ p$,

(v) $\exists\ c \in \mathcal{R}^{-1} : v = cu_{1,q}^{(1,\times)}$ *and* $u = v_{p,1}^{(1,\times)}c$,

(vi) $\mathcal{R}^{-1}v = \mathcal{R}^{-1}u_{1,q}^{(1,\times)}$ *and* $u\mathcal{R}^{-1} = v_{p,1}^{(1,\times)}\mathcal{R}^{-1}$,

(vii) $a^{(1,\times)} \in u\mathcal{R} \cap \mathcal{R}v$.

Proof. We can verify that the statements (i), (ii), (iii) and (iv) are equivalent, by the proof of Theorem 3.1.

(ii) $\Rightarrow$ (v): Let $c = vu$ and $d = u_{1,q}^{(1,\times)}v_{p,1}^{(1,\times)}$. Then, from

$$v = vv_{p,1}^{(1,\times)}v = vuu_{1,q}^{(1,\times)} = cu_{1,q}^{(1,\times)}$$

and

$$u = uu_{1,q}^{(1,\times)}u = v_{p,1}^{(1,\times)}vu = v_{p,1}^{(1,\times)}c,$$

we deduce that

$$cd = cu_{1,q}^{(1,\times)}v_{p,1}^{(1,\times)} = vv_{p,1}^{(1,\times)} = 1 = u_{1,q}^{(1,\times)}u = u_{1,q}^{(1,\times)}v_{p,1}^{(1,\times)}c = dc.$$

Hence, $c \in \mathcal{R}^{-1}$ and (iv) is satisfies.

Obviously, the following implications hold: (v) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (vii).

(vii) $\Rightarrow$ (i): By (vii), we get $a^{(1,\times)} = a(a^{(1,\times)})^2 = (a^{(1,\times)})^2a$ which gives (i). $\square$

In the next result which can be proved easy, we give new equivalent condition for an element a of a ring to be reflexive–EP.

Corollary 3.1. *Let $p, q \in \mathcal{R}^\bullet$ and let $a \in \mathcal{R}$ have a factorization (2). Then the following statements are equivalent:*

(i) $aa^{(1,\times)} = a^{(1,\times)}a$,

(ii) $\exists\, c \in \mathcal{R} : c^\circ = \{0\} = {}^\circ c, v = cu_{1,q}^{(1,\times)}$ *and* $u = v_{p,1}^{(1,\times)}c$,

(iii) $\exists\, c \in \mathcal{R} : c\mathcal{R} = \mathcal{R} = \mathcal{R}c, v = cu_{1,q}^{(1,\times)}$ *and* $u = v_{p,1}^{(1,\times)}c$,

(iv) $\exists\, t \in \mathcal{R}^{-1} : u_{1,q}^{(1,\times)} = tv$ *and* $v_{p,1}^{(1,\times)} = ut$,

(v) $\exists\, t \in \mathcal{R} : t^\circ = \{0\} = {}^\circ t, u_{1,q}^{(1,\times)} = tv$ *and* $v_{p,1}^{(1,\times)} = ut$,

(vi) $\exists\, t \in \mathcal{R} : t\mathcal{R} = \mathcal{R} = \mathcal{R}t, u_{1,q}^{(1,\times)} = tv$ *and* $v_{p,1}^{(1,\times)} = ut$,

(vii) $\exists\, c, c_1, t, t_1 \in \mathcal{R} : v = cu_{1,q}^{(1,\times)}, u = v_{p,1}^{(1,\times)}c_1, u_{1,q}^{(1,\times)} = tv$ *and* $v_{p,1}^{(1,\times)} = ut_1$,

(viii) $u = v_{p,1}^{(1,\times)}vu$, $v = vuu_{1,q}^{(1,\times)}$, $u_{1,q}^{(1,\times)} = u_{1,q}^{(1,\times)}v_{p,1}^{(1,\times)}v$ *and* $v_{p,1}^{(1,\times)} = uu_{1,q}^{(1,\times)}v_{p,1}^{(1,\times)}$.

Let $\mathcal{R}$ be a ring with involution and let $a \in \mathcal{R}$ be a Moore-Penrose invertible element. Then $b = a^\dagger$ is the reflexive image-kernel (p, q)-inverse of a, where $p = a^\dagger a$ and $q = 1 - aa^\dagger$. If, for mentioned p and q, the element a has a factorization (2), we get $p = a^\dagger a = v'u'uv = v'v$ and $q = 1 - aa^\dagger = 1 - uu'$ implying $u' = u^\dagger$ and $v' = v^\dagger$. So, observe that the result of this section recovers some results related to the corresponding factorization presented in [4].

4 Final remarks

According to [11, Thorem 6], the condition of being inner regular is equivalent to the one of being Moore-Penrose invertible, for C^*-algebra elements. In rings with involution the inner regularity is not enough to ensure the existence of a Moore–Penrose inverse. EP elements present a particular class of Moore-Penrose invertible elements which have been intensively studied in different contexts such as matrices, Hilbert space bounded and linear maps, C^*-algebras, rings with involution.

Boasso in [3] made further inroads into the theory when he gave a definition of EP elements of a Banach algebra in the absence of involution. His definition relies on the characterization of Hermitian elements using the topology of the underlying algebra due to Palmer and Vidav. However, there are no obvious candidates for Hermitian elements in a ring (or algebra) without involution, and so this avenue does not seem to be accessible from the purely algebraic point of view.

In this article, we introduce and study the reflexive–EP elements as objects similar to EP elements in the contexts of ring elements. Notice that the reflexive–EP elements are characterized using the concept of group inverse, since the reflexive–EP ring elements consists in a particular class of group invertible elements. The aim of this paper is to observe that a lot of characterizations which are true for EP elements in C^*-algebra are also true for the reflexive–EP elements in ring.

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