Long time behavior of solutions to a class of parabolic equations with nonstandard growth condition *

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Abstract In this paper, the author studies the long time behavior of solutions to the $p(x)$-Laplace equation $u_t = \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + f(u)$, with homogeneous Dirichlet boundary condition in a bounded domain. As for the blow-up results, it is shown, by using the energy method, that the solutions of this problem blow up in finite time for non-positive initial energy, or even for small positive initial energy. As for the extinction results, we give some sufficient conditions for the solutions to vanish in finite time. All these results generalize the ones when $p(x)$ is a constant.

Keywords $p(x)$-Laplacian; Blow-up; Extinction; Nonstandard growth condition.
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1 Introduction

In this paper, we investigate the following $p(x)$-Laplace equation with a nonlinear source

$$\begin{cases} u_t = \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + f(u), & (x,t) \in Q_T = \Omega \times (0,T], \\ u(x,t) = 0, & (x,t) \in \Gamma_T = \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with Lipschitz boundary $\partial\Omega$, f is continuous in $\mathbb{R}$. It will also be assumed throughout this paper that $p(x)$ is continuous on $\overline{\Omega}$ with the logarithmic module of continuity:

$$1 < p^- = \inf_{x\in\overline{\Omega}} p(x) \leqslant p(x) \leqslant p^+ = \sup_{x\in\overline{\Omega}} p(x) < \infty, \tag{1.2}$$

$$\forall z, \xi \in \Omega,\ |z-\xi| < 1,\ |p(z) - p(\xi)| \leqslant \omega(|z-\xi|),\ \text{where } \limsup_{\tau\to 0^+} \omega(\tau)\ln\frac{1}{\tau} = C < +\infty. \tag{1.3}$$

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When $p(x) \equiv p$, Problem (1.1) is a mathematical model describing some important features shared by many practical problems such as the flows of non-Newtonian fluids and the Smagorinsky type meteorology model. There have been many results on the existence, uniqueness, blow-up, extinction in finite time and some other properties of solutions. Interested readers may refer to, for instance, [8, 17, 21, 26] and the references therein.

When $p(x)$ is not a constant, equations in Problem (1.1) are usually referred to as equations with nonstandard growth conditions. These problems appear in the mathematical modelling of various physical phenomena such as flows of electro-rheological or thermo-rheological fluids [1, 3, 24]. They are also frequently used in the processing of digital imagines [7]. For a more detailed information on the possible applications of these models to applied sciences we refer the readers to the papers [4, 24] and the references therein. In recent years, much effort has been devoted to the study of PDEs with nonstandard growth conditions. The questions of existence, uniqueness and qualitative properties of solutions to both elliptic and parabolic equations with variable nonlinearity have been studied by many authors; see for example [2, 6].

As is known to us, extinction and blow-up are two important properties of solutions of many evolutionary equations. When $p(x) \equiv p$, Zhao [29] studied Problem (1.1) and established a global existence result for f depending on u as well as on ∇u. He also proved a blow-up result under the condition that

$$\frac{1}{p}\int_{\Omega}|\nabla u_0|^p dx - \int_{\Omega} F(u_0(x))dx \leq -\frac{4(p-1)}{pT_0(p-2)^2}\int_{\Omega} u_0^2 dx, \tag{1.4}$$

where $F(t) = \int_0^t f(s)ds$. More precisely, he showed that if there exists a $T_0 > 0$ such that (1.4) holds, then the solution to Problem (1.1) blows up in a time less than T_0. This type of results were generalized by Levine et al. [18], where the authors proved the blow-up results when the initial energy was negative, and Massaoudi [20] proved that blow-up might occur for vanishing initial energy. Recently, Liu and Wang [19] showed that solutions of Problem (1.1) could blow up in finite time even for small positive initial energy. However, there are much fewer blow-up results of parabolic equations with nonstandard growth condition. Pinasco [23] investigated the blow-up property of solutions to a semilinear parabolic equation with variable exponent source using Kaplan's eigenvalue method and Antontsev *et al.*[5] showed that the solutions to Problem (1.1) with $f(u)$ replaced by $|u|^{q-2}u$ ($q > \max\{p^+, 2\}$) may blow up in finite time when the initial energy is non-positive. For more articles concerning equations with variable exponents, see [9, 13, 15].

Extinction results of Problem (1.1) when $p(x,t) \equiv p \in (1,2)$ were also investigated in the case that $f(s) = \lambda s^q$ (see [8, 14, 28, 25, 27]). Here $q > 0$ and λ may be positive, zero or negative. When considering positive solutions, the cases $\lambda > 0$ and $\lambda < 0$ are corresponding to a function with a source and a sink, respectively. When $\lambda > 0$, some authors studied the extinction property of solutions to Problem (1.1), by using the methods of energy estimate and embedding theorem, and gave the critical exponent for extinction. They showed that when the source is weak ($q > p-1$), solutions to Problem (1.1) vanish in finite time for small initial data, while when the diffusion term dominates the sources ($q < p-1$), the solutions can not vanish for any non-negative nontrivial initial datum. As for the critical case $q = p-1$, the first eigenvalue of p-Laplace operator λ_1 plays a role. However, for variable exponent $p(x)$, things are more complicated because the first eigenvalue corresponding to the $p(x)$-Laplace operator may be 0, even if $p(x)$ is a continuous function [9]. For the extinction results of solutions to parabolic equations with variable exponents, the only reference we can find is [13], in which the authors studied an equation without sources and the boundary condition is of Neumann type. For more results on the finite time blow-up or extinction, interested reader may refer to [10, 11, 12, 16]

Motivated by the work mentioned above, we are concerned with the long time behavior of solutions to Problem (1.1) and give some sufficient conditions for the solutions to blow up or to vanish in finite time, by utilizing the energy method, integration estimates as well as embedding theorems in variable Sobolev spaces. We shall prove, for general source terms, that the solutions to Problem (1.1) blow up in finite time when the initial energy is non-positive or positive, but suitably small. As for the extinction results, we shall prove for the so-called fast diffusive equation ($1 < p^- \le p^+ < 2$) that finite time extinction may happen when the initial datum is small or when the source term is in some sense weak.

The outline of this paper is organized as follows. In Section 2, we shall introduce the function spaces of Orlicz-Sobolev type, give the definition of weak solutions to Problem (1.1) as well as the existence result, for completeness. In Section 3, the blow-up results will be proved and in Section 4 some sufficient conditions for the solutions to vanish in finite time will be given.

2 Preliminaries

In this section, we introduce some definitions and notations. It is well known that the equation in Problem (1.1) may be degenerate or singular at the points where $|\nabla u| = 0$, and hence there is no classical solution in general. To state the definition of the weak solutions, we first introduce some function spaces and their properties which will be used throughout this paper.

2.1 The function spaces.

Let $p(x)$ be a measurable bounded function defined on $\overline{\Omega}$. We introduce the set of functions

$$L^{p(\cdot)}(Q_T) = \Big\{ u(x,t) : \ u \text{ is measurable in } Q_T, A_{p(.)}(u) \equiv \int_{Q_T} |u(x,t)|^{p(x)} dxdt < \infty \Big\}.$$

The set $L^{p(\cdot)}(Q_T)$ equipped with the norm (Luxemburg's norm)

$$\|u\|_{p(.),Q_T} = \inf \Big\{ \lambda > 0 : \ \int_{Q_T} |\frac{u(x,t)}{\lambda}|^{p(x)} dxdt < 1 \Big\}$$

becomes a Banach space. The set $C^\infty(Q_T)$ is dense in $L^{p(\cdot)}(Q_T)$, provided that the exponent $p(x) \in C^0(\overline{\Omega})$. For the sake of simplicity, we state some results about the properties of the Luxemburg's norm.

Lemma 2.1. *[22] For any* $u \in L^{p(\cdot)}(Q_T)$,

$$(1)\ \|u\|_{p(\cdot)} < 1\ (= 1; > 1) \Leftrightarrow A_{p(\cdot)}(u) < 1\ (= 1; > 1);$$

$$(2)\ \|u\|_{p(\cdot)} < 1\ \Rightarrow \|u\|_{p(\cdot)}^{p^+} \leqslant A_{p(\cdot)}(u) \leqslant \|u\|_{p(\cdot)}^{p^-};$$

$$\|u\|_{p(\cdot)} \geqslant 1\ \Rightarrow \|u\|_{p(\cdot)}^{p^-} \leqslant A_{p(\cdot)}(u) \leqslant \|u\|_{p(\cdot)}^{p^+};$$

$$(3)\ \|u\|_{p(\cdot)} \to 0 \Leftrightarrow A_{p(\cdot)}(u) \to 0;\ \|u\|_{p(\cdot)} \to \infty \Leftrightarrow A_{p(\cdot)}(u) \to \infty.$$

Lemma 2.2. *(Hölder's inequality) [22] For any* $u \in L^{p(\cdot)}(Q_T)$ *and* $v \in L^{q(\cdot)}(Q_T)$,

$$\Big|\int_{Q_T} uv dx d\tau\Big| \leqslant (\frac{1}{p^-} + \frac{1}{q^-})\|u\|_{p(\cdot)}\|v\|_{q(\cdot)} \leqslant 2\|u\|_{p(\cdot)}\|v\|_{q(\cdot)},$$

where $q(x)$ *satisfies for a.e.* $x \in \Omega$

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1.$$

Lemma 2.3. *(Poincare's inequality) [22] Assume that* $p(x)$ *satisfies the log-continuity condition* (1.3)*. Then there exists a constant* $C(p^{\pm}, |\Omega|) > 0$ *such that for any* $u \in W_0^{1,p(x)}(\Omega)$,

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)}.$$

From Lemma 2.3 we know that $\|u\|_{p(x)} + \|\nabla u\|_{p(x)}$ and $\|\nabla u\|_{p(x)}$ are two equivalent norms of u in $W_0^{1,p(x)}(\Omega)$. We will use the latter throughout this paper.

We denote by $W(Q_T)$ the Banach space

$$W(Q_T) = \{u(x,t) :\ u \in L^2(Q_T),\ |\nabla u| \in L^{p(x,t)}(Q_T),\ u = 0\ \ on\ \Gamma_T\},$$

equipped with the norm

$$\|u\|_{W(Q_T)} = \|u\|_{2,Q_T} + \|\nabla u\|_{p(\cdot),Q_T},$$

and by $W'(Q_T)$ the dual space of $W(Q_T)$ with respect to the inner product in $L^2(Q_T)$. It is known that $C_0^\infty(Q_T)$ is dense in $W(Q_T)$ when $p(x,t)$ fulfills the log-continuity condition in $\overline{Q}_T$ (1.3).

2.2 Existence of weak solutions.

Definition 2.1. *A function* $u(x,t) \in W(Q_T) \cap L^\infty(0,T; L^2(\Omega))$ *is called a weak solution of Problem* (1.1) *if for every test function* $\zeta \in \{\eta(x,t) :\ \eta \in W(Q_T) \cap L^\infty(0,T; L^2(\Omega)),\ \eta_t \in W'(Q_T)\}$*, and every* $t_1, t_2 \in [0,T]$*, the following identity holds:*

$$\int_{t_1}^{t_2}\int_\Omega (u\zeta_t - |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\zeta + f(u)\zeta) dx dt = \int_\Omega u\zeta dx\ |_{t_1}^{t_2}. \tag{2.1}$$

Problem (1.1) admits at least one weak solution under some additional conditions. For completeness, we state the following local existence result for Problem (1.1) with a more general nonlinearity $f(x,t,u)$.

Theorem 2.1. *[5] Assume that* (1.2),(1.3) *are fulfilled and suppose that the nonlinearity* $f(x,t,u)$ *satisfies the growth condition*

$$|f(x,t,s)| \leq c_0|s|^\delta + h(x,t) \text{ with some constants } c_0 > 0 \text{ and } \delta > 1, \tag{2.2}$$

where $h \in L^1(0,T;L^\infty(\Omega))$. *Then for every* $u_0 \in L^\infty(\Omega)$ *there exists* $\theta \in (0,T]$, *depending on* $\delta, c_0, \|u_0\|_{L^\infty}(\Omega), \|h\|_{L^1(0,\theta;L^\infty(\Omega))}$, *such that Problem* (1.1) *has at least one weak solution* $u \in W(Q_\theta)$ *with* $u_t \in W'(Q_\theta)$ *and* $\|u\|_{\infty,Q_\theta} < \infty$. *The solution can be continued to the interval* $[0,T^*)$ *where*

$$T^* = \sup\{\theta > 0 : \ \|u\|_{\infty,Q_\theta} < \infty\}.$$

Remark 2.1. *[5] Suppose that* $p(x)$ *satisfies the log-continuity condition* (1.3). *Then for every* $u,v \in W(Q_T)$ *with* $u_t, v_t \in W'(Q_T)$ *the formula of integration by parts holds*

$$\int_{Q_T} uv_t dxdt + \int_{Q_T} u_t v dxdt = \int_\Omega uvdx \,|_{t_1}^{t_2} .$$

In this case, the identity (2.1) *can be rewritten in the form*

$$\int_{t_1}^{t_2}\int_\Omega (u_t\zeta + |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\zeta - f(x,t,u)\zeta)dxdt = 0. \tag{2.3}$$

We assume throughout this paper that the data of Problem (1.1) satisfy the conditions of Theorem 2.1, so that Problem (1.1) admits at least one weak local solution in the sense of Definition 2.1. Moreover, under these conditions, we can apply the formula of integration by parts to derive some a priori estimates which are needed in the forthcoming proof.

3 Blow-up results.

In this section, we shall investigate the blow-up properties of solutions to Problem (1.1), using energy methods. It will be shown that the solutions to Problem (1.1) may blow up in finite time for non-positive initial energy, or even for positive initial energy, under some additional conditions imposed on the initial data.

Throughout this section, we define for $t \geq 0$

$$E(t) = \int_\Omega \frac{1}{p(x)}|\nabla u(x,t)|^{p(x)}dx - \int_\Omega F(u(x,t))dx, \tag{3.1}$$

where $F(t) = \int_0^t f(s)ds$. The first result in this direction is the following

Theorem 3.1. *Assume that* $p(x)$ *satisfies conditions* (1.2) *and* (1.3), *that* f *is continuous on* $\mathbb{R}$ *fulfilling* (2.2) *and*

$$|s|^r \leq rF(s) \leq sf(s), \quad r > \max\{p^+, 2\}. \tag{3.2}$$

Then for any nonzero initial datum $u_0 \in L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega)$ *such that* $E(0) \leq 0$, *the solutions of Problem* (1.1) *blow up in finite time, namely, there exists a* $T^* < \infty$ *such that* $\|u(\cdot,t)\|_{\infty,\Omega} \to \infty$ *as* $t \to T^*$.

Remark 3.1. *The most typical example of f satisfying (3.2) is $f(s) = |s|^{r-2}s$, for $r > \max\{p^+, 2\}$. This shows, in a sense, that the source has to dominate the diffusion term.*

Proof. Multiplying the equation (1.1) by u_t, integrating by parts and using the fact that

$$\partial_t\Big(\frac{1}{p(x)}|\nabla u(x,t)|^{p(x)}\Big) = |\nabla u|^{p(x)-2}\nabla u \nabla u_t, \quad \partial_t(F(u)) = f(u)u_t,$$

we have

$$E'(t) = \frac{d}{dt}(\int_\Omega \frac{1}{p(x)}|\nabla u(x,t)|^{p(x)}dx - \int_\Omega F(u(x,t))dx) = -\int_\Omega u_t^2 dx \leq 0, \tag{3.3}$$

which implies that $E(t) \leq E(0)$.

Next define $g(t) = (\int_\Omega u^2 dx)/2$. By choosing u as a test function in the definition of weak solutions, integrating over the cylinder $\Omega \times (\tau, \tau + h)$ with some $h > 0$ sufficiently small and then dividing the resulting equality by h, we arrive at

$$\frac{1}{h}\int_\tau^{\tau+h}\int_\Omega uu_t dxdt = -\frac{1}{h}\int_\tau^{\tau+h}\int_\Omega |\nabla u|^{p(x)}dxdt + \frac{1}{h}\int_\tau^{\tau+h}\int_\Omega uf(u)dxdt. \tag{3.4}$$

Since

$$\int_\Omega |\nabla u|^{p(x)}dx, \ \int_\Omega uf(u)dx \in L^1(0,T),$$

we know by the Lebesgue differentiation theorem that each of the two terms on the right-hand side has a limit as $h \to 0$ for a.e. $\tau > 0$. So does the term on the left-hand side. Thus by taking limit on both sides of equation (3.4) and noticing (3.1) and (3.2) we see that

$$\begin{aligned} g'(t) &= \int_\Omega uu_t dx = -\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega uf(u)dx \\ &= -\int_\Omega p(x)\frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega uf(u)dx \\ &\geq -p^+\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega uf(u)dx \\ &= -p^+\Big(E(t) + \int_\Omega F(u)dx\Big) + \int_\Omega uf(u)dx \\ &\geq \int_\Omega uf(u)dx - p^+\int_\Omega F(u)dx \\ &\geq (\frac{r-p^+}{r})\int_\Omega rF(u)dx \\ &\geq (\frac{r-p^+}{r})\int_\Omega |u|^r dx. \end{aligned} \tag{3.5}$$

By using Hölder's inequality we have

$$g^{r/2}(t) \leq (\frac{1}{2})^{r/2}|\Omega|^{(r-2)/2}\int_\Omega |u|^r dx. \tag{3.6}$$

Thus it is deduced immediately by combining (3.5) and (3.6) that

$$g'(t) \geq C_0 g^{r/2}(t),$$

where $C_0 = 2^{r/2}(1-p^+/r)|\Omega|^{(2-r)/2} > 0$. A direct integration of the above inequality over $(0,t)$ then yields

$$g^{r/2-1}(t) \geq \frac{1}{g^{1-r/2}(0) - C_0(r/2-1)t},$$

which implies that $g(t)$ blows up at a finite time $T^* \leq g^{1-r/2}(0)/[C_0(r/2-1)]$, and so does u. The proof is complete. □

The validity of Theorem 3.1 is based on the fact that $r > \max\{p^+, 2\}$. In the case $r = p^+ > 2$, we can still prove that all non-stationary solutions blow up in finite time when the initial energy is non-positive.

Theorem 3.2. *Assume that all the conditions in Theorem 3.1 hold with the exception that* (3.2) *is replaced by*

$$|s|^r \leq rF(s) \leq sf(s), \quad r = p^+ > 2. \tag{3.7}$$

Then every non-stationary weak solution of Problem (1.1) *blows up in finite time.*

Proof. Define $E(t)$ and $g(t)$ as in Theorem 3.1 and set $G(t) = \int_0^t g(s)ds$. It is easy to see from (3.3) that

$$E(t) + \int_0^t \int_\Omega u_\tau^2 dx d\tau = E(0). \tag{3.8}$$

By using the arguments similar to that in Theorem 3.1 we obtain

$$\frac{1}{2}\int_\Omega u^2 dx + \int_0^t \int_\Omega \Big[|\nabla u|^{p(x)} - uf(u)\Big] dx d\tau = \frac{1}{2}\int_\Omega u_0^2 dx, \tag{3.9}$$

and

$$G''(t) = g'(t) = \int_\Omega u u_t dx = \int_\Omega \Big[-|\nabla u|^{p(x)} + uf(u)\Big] dx. \tag{3.10}$$

Multiplying (3.10) by $1/r$ and adding the result to (3.8) we get

$$\frac{1}{r}\int_\Omega \Big[-|\nabla u|^{p(x)} + uf(u)\Big] dx + \int_\Omega \Big[\frac{1}{p(x)}|\nabla u|^{p(x)} dx - F(u)\Big] dx + \int_0^t \int_\Omega u_\tau^2 dx d\tau = E(0) + \frac{1}{r}G''(t).$$

By applying $p(x) \leq p^+$, $E(0) \leq 0$ and (3.7) to the above equality we arrive at

$$(\frac{1}{p^+} - \frac{1}{r})\int_\Omega |\nabla u|^{p(x)} dx + \int_0^t \int_\Omega u_\tau^2 dx d\tau \leq \frac{1}{r}G''(t),$$

which implies

$$\int_0^t \int_\Omega u_\tau^2 dx d\tau \leq \frac{1}{r}G''(t), \tag{3.11}$$

since $r = p^+$.

Suppose on the contrary that u is a global solution of (1.1). Since u is non-stationary, there exists an $\varepsilon_0 > 0$ and a moment $t_0 > 0$ such that $G''(t) \geq \varepsilon_0$ and $G(t) \geq \varepsilon_0$ for all $t \geq t_0$. Using Hölder's inequality and the definition of $G(t)$, we obtain the following relations

$$(G'(t) - G'(t_0))^2 \quad = \quad \Big(\int_{t_0}^t \int_\Omega u u_t dx d\tau\Big)^2 \leq \|u_t\|^2_{2,\Omega\times(t_0,t)} \|u\|^2_{2,\Omega\times(t_0,t)}$$

$$\leq \quad \frac{2}{r}G''(t)G(t) \quad \text{for all } t > t_0. \tag{3.12}$$

Since $G(t), G''(t) \geq \varepsilon_0$, $G'(t) > 0$ for all $t > t_0$, it is necessary that $G'(t) \nearrow \infty$ as $t \to \infty$, which implies, for any fixed $1 < \nu < r/2$, that

$$1 - \left(\frac{2\nu}{r}\right)^{\frac{1}{2}} \geq \frac{G'(t_0)}{G'(t)} \quad \text{as } t \to \infty.$$

It follows that for every fixed $\nu \in (1, r/2)$ there exists a moment $t^* > t_0$ such that, for all $t \geq t^*$

$$(G'(t) - G'(t_0))^2 \geq \frac{2\nu}{r}(G'(t))^2, \quad G(t^*) > 0. \tag{3.13}$$

By combining (3.12) and (3.13), we see, for all $t > t^*$, that

$$\nu(G'(t))^2 \leq \frac{r}{2}(G'(t) - G'(t_0))^2 \leq G''(t)G(t).$$

A straightforward integration of the above inequality over (t^*, t) leads to the inequality

$$G^{\nu-1}(t) \geq \frac{G^{\nu-1}(t^*)}{1 - (t - t^*)(\nu - 1)\frac{G'(t^*)}{G(t^*)}} \to \infty \quad \text{as } t \to t^* + \frac{G(t^*)}{(\nu - 1)G'(t^*)},$$

which contradicts the assumption that u is a global solution. This completes the proof of this theorem. $\square$

When the $L^{p(x)}$-norm of ∇u_0 has a positive lower bound, blow-up may also occur when the initial energy is positive. To prove this result, we assume that

$$\inf\left\{\int_\Omega F(u)dx : \ |u| = 1\right\} > 0,$$

and let B be the optimal constant of the embedding inequality

$$\left(\int_\Omega rF(u)dx\right)^{1/r} \leq B\|\nabla u\|_{p(.),\Omega}, \quad u \in W_0^{1,p(x)}(\Omega), \tag{3.14}$$

that is

$$B^{-1} = \inf_{0 \neq u \in W_0^{1,p(x)}(\Omega)} \frac{\|\nabla u\|_{p(.)}}{(\int_\Omega rF(u)dx)^{1/r}},$$

where $r \in (p^+, Np^-/(N - p^-)]$ is a fixed positive constant.

Set

$$E_1 = (\frac{1}{p^-} - \frac{1}{r})B^r\alpha_1^{\frac{r}{p^-}} + (\frac{1}{p^+} - \frac{1}{r})B^r\alpha_1^{\frac{r}{p^+}} > 0, \tag{3.15}$$

where α_1 satisfies

$$\frac{1}{p^+}\alpha_1 - B^r(\frac{1}{p^-}\alpha_1^{\frac{r}{p^-}} + \frac{1}{p^+}\alpha_1^{\frac{r}{p^+}}) = 0. \tag{3.16}$$

We have the following blow-up result.

Theorem 3.3. *Assume that the exponent* $p(x)$ *satisfies* (1.2),(1.3) *and* (3.2) *holds. Suppose also that the following conditions hold:*

(H1) For E_1 *defined in* (3.15),

$$E(0) < E_1, \quad \int_\Omega |\nabla u_0|^{p(x)} dx > \alpha_1;$$

(H2) $\max\{1, 2N/(N+2)\} < p^- < N, \quad \max\{2, p^+ + \sqrt{p^+(p^+ - p^-)}\} < r < Np^-/(N-p^-)$. *Then the solutions to Problem* (1.1) *blow up in finite time.*

We begin the proof of this theorem with two lemmas. The first lemma gives a lower bound estimate of the $L^{p(x)}$-norm of $\nabla u(x,t)$.

Lemma 3.1. *Suppose that* u *is a solution of Problem* (1.1). *Assume also that (H1) holds. Then there exists a positive constant* $\alpha_2 > \alpha_1$*, such that*

$$\int_\Omega |\nabla u|^{p(x)} dx \geq \alpha_2, \quad t \geq 0, \tag{3.17}$$

and

$$\int_\Omega rF(u)dx \geq B^r(\alpha_2^{\frac{r}{p^-}} + \alpha_2^{\frac{r}{p^+}}), \quad t \geq 0. \tag{3.18}$$

Proof. From (3.1) and (3.14) and Lemma 2.1 we see that

$$\begin{aligned} E(t) &\geqslant \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \frac{1}{r}\int_\Omega rF(u)dx \\ &\geqslant \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \frac{B^r}{r}\|\nabla u\|^r_{p(.)} \\ &\geqslant \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \frac{B^r}{r}\max\{(\int_\Omega |\nabla u|^{p(x)}dx)^{\frac{1}{p^-}}, (\int_\Omega |\nabla u|^{p(x)}dx)^{\frac{1}{p^+}}\}^r \\ &\geqslant \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \frac{B^r}{r}((\int_\Omega |\nabla u|^{p(x)}dx)^{\frac{r}{p^-}} + (\int_\Omega |\nabla u|^{p(x)}dx)^{\frac{r}{p^+}}) \\ &\triangleq \frac{1}{p^+}\alpha - \frac{B^r}{r}(\alpha^{\frac{r}{p^-}} + \alpha^{\frac{r}{p^+}}) \triangleq l(\alpha), \end{aligned} \tag{3.19}$$

where $\alpha = \alpha(t) = \int_\Omega |\nabla u|^{p(x)}dx$. It is easily verified that l is increasing for $\alpha \in (0, \alpha_1)$, decreasing for $\alpha > \alpha_1$; $l(\alpha) \to -\infty$ as $\alpha \to +\infty$ and $l(\alpha_1) = E_1$, where α_1 is given in (3.16). Since $E(0) < E_1$, there exists an $\alpha_2 > \alpha_1$ such that $l(\alpha_2) = E(0)$.

Set $\alpha_0 = \int_\Omega |\nabla u_0|^{p(x)}dx$. We see from (3.19) that $l(\alpha_0) \leq E(0) = l(\alpha_2)$, which implies $\alpha_0 \geq \alpha_2$ by (H1). To establish (3.17), we suppose by contradiction that $\int_\Omega |\nabla u(x,t_0)|^{p(x)}dx < \alpha_2$ for some $t_0 > 0$. By the continuity of $\int_\Omega |\nabla u(x,t)|^{p(x)}dx$ we can choose t_0 such that $\int_\Omega |\nabla u(x,t_0)|^{p(x)}dx > \alpha_1$. It follows from (3.19) that

$$E(t_0) \geq l(\alpha(t_0)) > l(\alpha_2) = E(0),$$

which is a contradiction since $E(t) \leq E(0)$ for all $t \geq 0$. Hence (3.17) is established.

To prove (3.18), we exploit (3.1), (3.3) and (3.19) that

$$\int_\Omega F(u)dx \quad \geq \quad \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - E(t) \geq \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - E(0)$$

$$\geq \quad \frac{1}{p^+}\alpha_2 - l(\alpha_2) = \frac{B^r}{r}(\alpha_2^{\frac{r}{p^-}} + \alpha_2^{\frac{r}{p^+}}).$$

Therefore (3.18) is concluded and the proof is complete. □

Next define $H(t) = E_1 - E(t)$, then we have the following

Lemma 3.2. *For all $t \geq 0$,*

$$0 < H(0) \leq H(t) \leq \int_\Omega F(u)dx. \tag{3.20}$$

Proof. We see from (3.3) that $H'(t) \geq 0$, which implies that

$$H(t) \geq H(0) = E_1 - E(0) > 0, \quad t \geq 0. \tag{3.21}$$

On the other hand, from (3.1) we can obtain

$$H(t) = E_1 - \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega F(u)dx.$$

Combining (3.15),(3.16) and (3.17) we get

$$E_1 - \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \leq E_1 - \frac{\alpha_1}{p^+} \leq 0.$$

Hence $H(t) \leq \int_\Omega F(u)dx$. The proof of this lemma is complete. □

Based on the above two lemmas, we are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. Define $g(t) = (\int_\Omega u^2 dx)/2$. As in the proof of Theorem 3.1 we see that

$$\begin{aligned} g'(t) &= \int_\Omega uu_t dx = -\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega uf(u)dx \\ &= -\int_\Omega p(x)\frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega uf(u)dx \\ &\geq -p^+\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega uf(u)dx \\ &= -p^+(E(t) + \int_\Omega F(u)dx) + \int_\Omega uf(u)dx \\ &= \int_\Omega uf(u)dx - p^+\int_\Omega F(u)dx - p^+(E_1 - H(t)) \quad \text{(by (3.2) and (3.20))} \\ &\geq (\frac{r-p^+}{r})\int_\Omega rF(u)dx - p^+E_1. \end{aligned} \tag{3.22}$$

On the other hand, by (3.18) we obtain

$$\begin{aligned} p^+E_1 &= \frac{p^+E_1}{B^r(\alpha_2^{\frac{r}{p^-}} + \alpha_2^{\frac{r}{p^+}})}\Big(B^r(\alpha_2^{\frac{r}{p^-}} + \alpha_2^{\frac{r}{p^+}})\Big) \\ &\leq \frac{p^+E_1}{B^r(\alpha_2^{\frac{r}{p^-}} + \alpha_2^{\frac{r}{p^+}})}\int_\Omega rF(u)dx. \end{aligned} \tag{3.23}$$

Combining (3.2), (3.22) and (3.23) and utilizing Hölder's inequality, we have

$$g'(t) \geq C^* g^{\frac{r}{2}}(t), \tag{3.24}$$

where

$$C^* = \frac{r-p^+}{r}\Big[1 - \frac{\alpha_1^{\frac{r}{p^-}} + \alpha_1^{\frac{r}{p^+}}}{\alpha_2^{\frac{r}{p^-}} + \alpha_2^{\frac{r}{p^+}}}\Big] 2^{\frac{r}{2}} |\Omega|^{\frac{2-r}{2}} + \frac{p^- - p^+}{p^-}\Big[\frac{\alpha_1^{\frac{r}{p^-}}}{\alpha_2^{\frac{r}{p^-}} + \alpha_2^{\frac{r}{p^+}}}\Big] 2^{\frac{r}{2}} |\Omega|^{\frac{2-r}{2}}$$

is known to be positive, by (3.15), (3.16) and (H2). Integrating (3.24) over $(0, t)$ again yields

$$g(t) \geqslant \Big[g^{1-\frac{r}{2}}(0) - (\frac{r}{2} - 1)C^* t\Big]^{\frac{2}{2-r}},$$

which implies that $G(t)$, and hence u, blows up at a time $T^* \leq g^{1-r/2}(0)/[C^*(r/2-1)]$. The proof of this theorem is complete. □

4 Extinction results.

When $p(x) \equiv p \in (1, 2)$, the equation in Problem (1.1) is usually called a fast diffusion equation. It has been shown that the weak solution to Problem (1.1) vanishes in finite time when $p \in (1, 2)$ and $f(s) = as^q (a, q > 0)$ for $q > p - 1$. In this section, for $f(s) \equiv as^q (a, q > 0)$, we shall prove the extinction results and give some sufficient conditions for the solutions to Problem (1.1) to vanish in finite time, by using integral estimates and the embedding theorems in variable Sobolev spaces. In addition, we assume that the initial datum u_0 is non-negative. By using similar method to that of [15] we know that the solution to Problem (1.1) $u(x, t)$ in this case is also non-negative. The first result in this direction is the following

Theorem 4.1. *Assume that* $0 \leq u_0(x) \in L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega)$ *and* $2N/(N+2) \leq p^- \leq p^+ < 2$.
(i) If $q = p^+ - 1$, *then every solution to Problem* (1.1) *vanishes in finite time provided that the coefficient* a *or* $|\Omega|$ *is suitably small.*
(ii) If $p^+ - 1 < q < 1$, *then every solution to Problem* (1.1) *vanishes in finite time provided that* u_0 *or* a *or* $|\Omega|$ *is suitably small.*
(iii) If $q \geq 1$, *then every bounded solution to Problem* (1.1) *vanishes in finite time provided that* u_0 *or* a *or* $|\Omega|$ *is suitably small.*

Proof. We first prove the case (ii). Choosing u to be the test function in (2.3), integrating over the cylinder $\Omega \times (t, t+h)$ with $h > 0$ and dividing the resulting equality by h we obtain

$$\frac{1}{h}\int_t^{t+h}\int_\Omega u u_t dx d\tau = -\frac{1}{h}\int_t^{t+h}\int_\Omega |\nabla u|^{p(x)} dx d\tau + \frac{a}{h}\int_t^{t+h}\int_\Omega u^{q+1} dx d\tau. \tag{4.1}$$

By letting $h \to 0^+$ on both sides of the above equality and using Lebesgue differentiation theorem we have, for a.e. $t \in (0, T)$,

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega u^2 dx\Big) + \int_\Omega |\nabla u|^{p(x)} dx = a\int_\Omega u^{q+1} dx. \tag{4.2}$$

Applying Young's inequality on the right hand side of (4.2) and noticing $q < 1$, we get

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega u^2 dx\Big) + \int_\Omega |\nabla u|^{p(x)} dx \le \frac{a(q+1)}{2}\int_\Omega u^2 dx + \frac{a(1-q)}{2}|\Omega|, \tag{4.3}$$

where $|\Omega|$ denotes the N-d Lebesgue measure of Ω. Dropping the second term of the left hand side and applying Gronwall's inequality one obtains

$$\int_\Omega u^2 dx \le C(\|u_0\|_2, |\Omega|, a, q, T). \tag{4.4}$$

Integrating (4.3) over $(t_1, t_2) \subset (0, T)$ and applying (4.4) we have

$$\int_{t_1}^{t_2}\int_\Omega |\nabla u|^{p(x)} dx \le C(\|u_0\|_2, |\Omega|, a, q, T). \tag{4.5}$$

Choose u_t as the test function in (2.3) and integrating over $\Omega \times (0, t)$, one arrives at

$$\int_0^t\int_\Omega u_t^2 dx d\tau = -\int_0^t\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla u_t dx d\tau + a\int_0^t\int_\Omega u^q u_t dx d\tau, \tag{4.6}$$

which implies that

$$\int_0^t\int_\Omega u_t^2 dx d\tau + \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx = \int_\Omega \frac{1}{p(x)}|\nabla u_0|^{p(x)} dx + \frac{a}{q+1}\Big(\int_\Omega u^{q+1} dx - \int_\Omega u_0^{q+1} dx\Big).$$

By dropping the first term of the left hand side and the non-positive part of the second term of the right hand side, and applying Hölder's inequality to the positive part of the second term we finally get

$$\frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)} dx \le \frac{1}{p^-}\int_\Omega |\nabla u_0|^{p(x)} dx + \frac{1+q}{2}|\Omega|^{\frac{1-q}{2}}\Big(\int_\Omega u^2 dx\Big)^{\frac{1+q}{2}}. \tag{4.7}$$

Combining (4.7) and (4.4) we see that

$$\int_\Omega |\nabla u|^{p(x)} dx \le C(\|u_0\|_2, \|u_0\|_{p(x)}, p^\pm, |\Omega|, a, q, T). \tag{4.8}$$

By the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,p^-}(\Omega) \hookrightarrow L^2(\Omega)$ and Lemma 2.1 we obtain the following relation

$$\begin{aligned}\int_\Omega |\nabla u|^{p(x)} dx &\ge \min\{\|\nabla u\|_{p(x)}^{p^-}, \|\nabla u\|_{p(x)}^{p^+}\}\\ &= \min\{\|\nabla u\|_{p(x)}^{p^- - p^+}, 1\}\|\nabla u\|_{p(x)}^{p^+}\\ &\ge C(u_0, a, T, q, p^\pm, |\Omega|)\|\nabla u\|_{p(x)}^{p^+}\\ &\ge C(u_0, a, T, q, p^\pm, |\Omega|)\|\nabla u\|_{p^-}^{p^+}\\ &\ge C(u_0, a, T, q, p^\pm, |\Omega|)\|u\|_2^{p^+}.\end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.2) and applying Hölder's inequality to the right hand side to yield

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega u^2 dx\Big) + C(\int_\Omega u^2 dx)^{\frac{p^+}{2}} \le a|\Omega|^{\frac{1-q}{2}}(\int_\Omega u^2 dx)^{\frac{1+q}{2}}. \tag{4.10}$$

Set $J_r(t) = \int_\Omega u^r(x,t)dx$. Now for $r = 2$ we see from (4.10) that

$$\frac{d}{dt}J_2(t) + CJ_2^{p^+/2}(t) \le 2a|\Omega|^{(1-q)/2}J_2^{(1+q)/2}(t),$$

which is equivalent to

$$\frac{d}{dt}J_2(t) + \Big(C - 2a|\Omega|^{(1-q)/2}J_2^{(1+q-p^+)/2}(t)\Big)J_2^{p^+/2}(t) \le 0. \tag{4.11}$$

Although the constant C in (4.11) depends on a, $|\Omega|$ and u_0, it does not tend to 0 when any one of the above three parameters tends to 0. Thus, from (4.11) we know that if a or $|\Omega|$ or u_0 is so small that $C_1 = C - 2a|\Omega|^{(1-q)/2}J_2^{(1+q-p^+)/2}(0) > 0$, then by integrating (4.11) over $(0,t)$ we obtain

$$J_2(t) \le \Big(J_2^{\frac{2-p^+}{2}}(0) - \frac{2-p^+}{2}C_1 t\Big)^{\frac{2}{2-p^+}},$$

which enures the finite time extinction of $J_2(t)$. This is the conclusion of (ii).

When $q = p^+ - 1$, we see from (4.11) that C_1 is also positive if a or $|\Omega|$ is suitably small, which implies the finite time extinction of $J_2(t)$ as in case (ii). The conclusion of case (i) holds.

When $q \ge 1$ and u is bounded, we deduce from (4.2) that

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega u^2dx\Big) + \int_\Omega |\nabla u|^{p(x)}dx \le aM^{q-1}\int_\Omega u^2dx,$$

where $M > 0$ is a constant such that $\|u\|_\infty \le M$. The rest of the arguments are similar to that of case (ii) and so the details are omitted. This is case (iii) and the whole proof of this theorem is complete. □

Theorem 4.2. *Assume that* $1 < p^- < 2N/(N+2)$, $1 < p^+ < Np^-/(N-p^-)$ *and* $0 \le u_0 \in L^\infty(\Omega) \cap W_0^{1,p(x)}(\Omega)$. *Denote* $\beta = (2-p^-)(N-p^-)/(p^-)^2 > 1$.
(i) If $q > p^+ - 1 + \beta(p^+ - p^-)$, *then every bounded solution of Problem* (1.1) *vanishes in finite time provided that* a *or* $|\Omega|$ *or* u_0 *is suitably small.*
(ii) If $q = p^+ - 1 + \beta(p^+ - p^-)$, *then every bounded solution of Problem* (1.1) *vanishes in finite time provided that* a *or* $|\Omega|$ *is suitably small.*

Proof. We first prove case (i) when $q < 1$. By choosing the test function to be u^{r-1} ($r = N(2-p^-)/p^- > 2$) and applying the arguments similar to the proof of Theorem 4.1, we arrive at

$$\frac{1}{r}\frac{d}{dt}\int_\Omega u^r(x,t)dx + (r-1)\int_\Omega u^{r-2}|\nabla u|^{p(x)}dx = a\int_\Omega u^{q+r-1}(x,t)dx. \tag{4.12}$$

Noticing that u is bounded, we see that there exists an $M > 0$ such that $\|u\|_\infty \le M$, which implies that

$$(r-1)\int_\Omega u^{r-2}|\nabla u|^{p(x)}dx \ge C(r,N,p^\pm,M)\int_\Omega |\nabla u^\beta|^{p(x)}dx. \tag{4.13}$$

By Lemma 2.1 and the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,p^-}(\Omega) \hookrightarrow L^{Np^-/(N-p^-)}(\Omega)$ one gets

$$\begin{aligned}\int_\Omega |\nabla u^\beta|^{p(x)}dx &\geq \min\{\|\nabla u^\beta\|_{p(x)}^{p^-}, \|\nabla u^\beta\|_{p(x)}^{p^+}\} \\ &= \min\{\|\nabla u^\beta\|_{p(x)}^{p^- - p^+}, 1\}\|\nabla u^\beta\|_{p(x)}^{p^+} \\ &\geq C(\|u_0\|_\infty, a, M, q, p^\pm, |\Omega|)\|\nabla u^\beta\|_{p(x)}^{p^+} \\ &\geq C(\|u_0\|_\infty, a, M, q, p^\pm, |\Omega|)\|\nabla u^\beta\|_{p^-}^{p^+} \\ &\geq C(\|u_0\|_\infty, a, M, q, p^\pm, |\Omega|)\|u^\beta\|_{Np^-/(N-p^-)}^{p^+}.\end{aligned} \tag{4.14}$$

Substituting (4.13) and (4.14) into (4.12) and applying Hölder's inequality (since $q < 1$) to the right hand side of (4.12) we have

$$\frac{1}{r}\frac{d}{dt}\int_\Omega u^r(x,t)dx + C\Big(\int_\Omega u^r(x,t)dx\Big)^{\frac{Np^-p^+}{N-p^-}} \leq a|\Omega|^{\frac{1-q}{r}}\Big(\int_\Omega u^r dx\Big)^{\frac{q+r-1}{r}},$$

which is equivalent to

$$\frac{1}{r}\frac{d}{dt}\int_\Omega u^r(x,t)dx + \Big[C - a|\Omega|^{\frac{1-q}{r}}(\int_\Omega u^r dx)^{\frac{q+r-1}{r} - \frac{Np^-p^+}{N-p^-}}\Big]\Big(\int_\Omega u^r dx\Big)^{\frac{Np^-p^+}{N-p^-}} \leq 0. \tag{4.15}$$

Noticing that $J_r(t) = \int_\Omega u^r(x,t)dx$, we have

$$\frac{d}{dt}J_r(t) + r\Big[C - a|\Omega|^{\frac{1-q}{r}}J_r^{\frac{q+r-1}{r} - \frac{Np^-p^+}{N-p^-}}(t)\Big]J_r^{\frac{Np^-p^+}{N-p^-}}(t) \leq 0. \tag{4.16}$$

Moreover, it is not difficult to verify that $(Np^-p^+)/(N-p^-) < 1$ and $(q+r-1)/r - Np^-p^+/(N-p^-) > 0$ by the definition of β and the conditions imposed on $p^\pm, q$ and N. Thus, if a or $|\Omega|$ or u_0 is suitably small that $C_2 \triangleq C - a|\Omega|^{(1-q)/r}J_r^{(q+r-1)/r-(Np^-p^+)/(N-p^-)}(0) > 0$, then we know that $J_r(t)$ vanishes in finite time, by applying the treatments similar to that used in the proof of Theorem 4.1.

When $q \geq 1$, (4.12) can be rewritten as

$$\frac{1}{r}\frac{d}{dt}\int_\Omega u^r(x,t)dx + (r-1)\int_\Omega u^{r-2}|\nabla u|^{p(x)}dx \leq aM^{q-1}\int_\Omega u^r(x,t)dx,$$

to which the above arguments can be applied and the extinction result holds.

Next we prove case (ii). When $q = p^+ - 1 + \beta(p^+ - p^-)$, the constant C_2 in (4.16) equals to $C - a|\Omega|^{(1-q)/r}$, which is positive for small a or $|\Omega|$. The conclusion of case (ii) can be proved by applying the arguments in case (i). The proof is complete. □

Remark 4.1. *Although the above two theorems are concerned with the case $N \geq 2$, extinction can also happen for the case $N = 1$. Indeed, since $p^- > 1$, the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,p^-}(\Omega) \hookrightarrow L^2(\Omega)$, and hence (4.9) still holds. Thus, all the three cases in Theorem 4.1 can be applied to the case $N = 1$, and the details are omitted.*

Remark 4.2. *It can be seen from Theorems 4.1,4.2 that when $p^+ = p^-$, namely, $p(x) \equiv p$, the range of q for the solution to vanish in finite time is $q \geq p-1$, which is compatible with the* *known results (see [25, 27]). So our results are partial generalizations of theirs. However, for the moment, we are not able to show whether they are the critical extinction exponents since we can not use the first eigenvector of the $p(x)$-Laplace operator as a sub-solution to show that the solutions do not vanish in finite time when q is suitably small. We shall explore some other methods to deal with this problem in our future work.*

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