LEIBNIZ ALGEBRAS ADMITTING A MULTIPLICATIVE BASIS

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ABSTRACT. In the literature, many of the descriptions of different classes of Leibniz algebras $(L, [\cdot, \cdot])$ have been made by given the multiplication table on the elements of a basis $\mathcal{B} = \{v_k\}_{k \in K}$ of L, in such a way that for any $i, j \in K$ we have that $[v_i, v_j] = \lambda_{i,j}[v_j, v_i] \in \mathbb{F}v_k$ for some $k \in K$, where $\mathbb{F}$ denotes the base field and $\lambda_{i,j} \in \mathbb{F}$. In order to give an unifying viewpoint of all these classes of algebras we introduce the more general category of Leibniz algebras admitting a multiplicative basis and study its structure. We show that if a Leibniz algebra L admits a multiplicative basis then it is the direct sum $L = \bigoplus_{\alpha} \mathcal{I}_\alpha$ with any $\mathcal{I}_\alpha$ a well described ideal of L admitting a multiplicative basis inherited from $\mathcal{B}$. Also the $\mathcal{B}$-simplicity of L is characterized in terms of the multiplicative basis.

Keywords: Leibniz algebra, infinite dimensional algebra, structure theory.
2010 MSC: 17A32, 17A60, 17B65.

1. INTRODUCTION AND PREVIOUS DEFINITIONS

The term *Leibniz algebra* was introduced in the study of a non-antisymmetric analogue of Lie algebras by Loday [27], being so the class of Leibniz algebras an extension of the one of Lie algebras. However this kind of algebras was previously studied, under the name of D-algebras, by D. Bloh [6, 7, 8]. Since the 1993 Loday's work, many researchers have been attracted by this kind of algebras being remarkable the great activity in this field developed in the last years. This activity has been mainly focussed in the frameworks of low dimensional algebras, nilpotence and related problems (see [1, 4, 5, 9, 14, 15, 16, 18, 19, 24, 25, 26, 29, 30, 31, 32]).

Definition 1.1. A *Leibniz algebra* L is a vector space over a base field $\mathbb{F}$ endowed with a bilinear product $[\cdot, \cdot]$ satisfying the *Leibniz identity*

$$[[y, z], x] = [[y, x], z] + [y, [z, x]],$$

for all $x, y, z \in L$.

In presence of anti-commutativity, Jacobi identity becomes Leibniz identity and so Lie algebras are examples of Leibniz algebras.

Let L be a Leibniz algebra, the ideal $\mathcal{S}$ generated by the squares, that is $\mathcal{S}$ is generated by the set $\{[x, x] : x \in L\}$, plays an important role in the theory since it determines the (possible) non-Lie character of L. From the Leibniz identity, this ideal satisfies

$$[L, \mathcal{S}] = 0. \tag{1}$$

In fact a Leibniz algebra L is called *simple* when $[L, L] \neq 0$ and its only ideals are $\{0\}$, $\mathcal{S}$ and L.

Supported by the PCI of the UCA 'Teoría de Lie y Teoría de Espacios de Banach', by the PAI with project numbers FQM298, FQM7156 and by the project of the Spanish Ministerio de Educación y Ciencia MTM2010-15223.

Observe that we can write

$$L = \mathcal{S} \oplus V$$

where V is a linear complement of $\mathcal{S}$ in L, (actually V is isomorphic as linear space to $L/\mathcal{S}$, the so called corresponding Lie algebra of L). Hence, by taking $\mathcal{B}_{\mathcal{S}}$ and $\mathcal{B}_V$ bases of $\mathcal{S}$ and V respectively we get

$$\mathcal{B} = \mathcal{B}_{\mathcal{S}} \dot{\cup} \mathcal{B}_V$$

a basis of L.

Definition 1.2. A basis $\mathcal{B} = \{v_k\}_{k \in K}$ of L which decomposes $\mathcal{B} = \mathcal{B}_{\mathcal{S}} \dot{\cup} \mathcal{B}_V$ as above, is said to be *multiplicative* if for any $i, j \in K$ we have that $[v_i, v_j] \in \mathbb{F} v_k$ for some $k \in K$.

Remark 1.1. Observe that if we write

$$\mathcal{B}_{\mathcal{S}} = \{e_i\}_{i \in I} \text{ and } \mathcal{B}_V = \{u_j\}_{j \in J},$$

the fact $\mathcal{S}$ is an ideal together with Equation (1) give us that the only possible non-zero products among the elements in $\mathcal{B}$ are.

(1) For any $i \in I$ and $j \in J$ we have $[e_i, u_j] \in \mathbb{F} e_k$ for some $k \in I$.
(2) For any $j, k \in J$ we have either $[u_j, u_k] \in \mathbb{F} u_l$ or $[u_j, u_k] \in \mathbb{F} e_i$ for some $l \in J$, $i \in I$.

Let us observe that if the base field $\mathbb{F}$ of a Leibniz algebra is of characteristic different to 2, then $\mathcal{S}$ is the ideal generated by the set $\{[x, y] + [y, x] : x, y \in L\}$.

Lemma 1.1. *Let* $(L, [\cdot, \cdot])$ *be a Leibniz algebra over a base field* $\mathbb{F}$ *of characteristic different to* 2*. If* $\mathcal{B} = \{v_k\}_{k \in K}$ *is a basis of* L *satisfying that for any* $i, j \in K$ *we have* $[v_i, v_j] = \lambda_{i,j} [v_j, v_i] \in \mathbb{F} v_k$ *for some* $k \in K$ *and some* $\lambda_{i,j} \in \mathbb{F}$ *then* L *admits* $\mathcal{B}$ *as multiplicative basis.*

Proof. By the above observation, we can assert that $\mathcal{S}$ is generated as linear space by $\{v_j : j \in J \subset K\}$. From here, we can find a basis $\mathcal{B}_{\mathcal{S}}$ of $\mathcal{S}$ formed by elements of $\mathcal{B}$ and a basis $\mathcal{B}_V := \mathcal{B} \setminus \mathcal{B}_{\mathcal{S}}$ of V which make of $\mathcal{B}$ a multiplicative basis. □

Lemma 1.1 gives us the way of checking easily that the expositions of many of the classes of Leibniz algebras which have been described in the literature, have been made by presenting a multiplication table of the Leibniz algebra in terms of a multiplicative basis, becoming so examples of Leibniz algebras admitting a multiplicative basis. This is the case for instance of the two and three dimensional nilpotent Leibniz algebras (see [27, 2]), of the non-Lie Leibniz algebras L with $L/\mathcal{S}$ abelian described in [2], of the classes of (complex) finite-dimensional naturally graded filiform Leibniz algebras and n-dimensional filiform graded filiform Leibniz algebras of length $n-1$ (see [3]), of the categories of finite-dimensional 0-filiform Leibniz algebras, of finite-dimensional nonsplit graded filiform Leibniz algebras, and of different types of finite-dimensional 2-filiform nonsplit Leibniz algebras (see [17]), of the class of four-dimensional solvable Leibniz algebras with three-dimensional rigid nilradical (see [20]), of the families of four-dimensional solvable Leibniz algebras with two-dimensional nilradical and of certain types, respect to its radical, of four-dimensional solvable Leibniz algebras (see [23]), of several types of solvable Leibniz algebras with naturally graded filiform nilradical considered in [22], of the solvable Leibniz algebras whose nilradical is NF_n (see [21]), of the family of (complex) finite-dimensional Leibniz algebras with Lie quotient sl_2 (see [28]), and so on. From here, the class of Leibniz algebras admitting a multiplicative basis becomes a wide class of Leibniz algebras. Let us concrete one example of the above ones from [28].

Example 1.1. Let $L = \mathcal{S} \oplus V$ be the n-dimensional complex Leibniz algebra where

$$\mathcal{B}_{\mathcal{S}} = \{e_1^1, ..., e_{t_1}^1, e_1^2, ..., e_{t_2}^2, ..., e_1^p, ..., e_{t_p}^p\}$$

is a basis of $\mathcal{S}$, the set

$$\mathcal{B}_V = \{v_1, v_2, v_3\}$$

is a basis of V and the non-zero products respect to the elements in the basis

$$\mathcal{B} = \mathcal{B}_{\mathcal{S}} \dot{\cup} \mathcal{B}_V$$

of L are:

$$\begin{gathered}
[v_1, v_3] = 2v_1, \quad [v_2, v_3] = -2v_2, \quad [v_1, v_2] = v_3, \\
[v_3, v_1] = -2v_1, \quad [v_3, v_2] = 2v_2, \quad [v_2, v_1] = -v_3, \\
[e_k^j, v_3] = (t_j - 2k)e_k^j, \quad k = 0, ..., t_j, \\
[e_k^j, v_2] = e_{k+1}^j, \quad k = 0, ..., t_j - 1, \\
[e_k^j, v_1] = -k(t_j + 1 - k)e_{k-1}^j, \quad k = 0, ..., t_j,
\end{gathered}$$

$1 \leq j \leq p$.

Then L becomes a Leibniz algebra admitting $\mathcal{B}$ as a multiplicative basis.

The present paper is devoted to the study of Leibniz algebras L of arbitrary dimension and over an arbitrary base field $\mathbb{F}$ admitting a multiplicative basis, by focussing on its structure.

The paper is organized as follows. In §2 and by inspiring in the connections of root techniques developed for split Leibniz algebras and superalgebras in [12, 13] we introduce connections techniques on the set of indexes of the multiplicative basis $\mathcal{B}$ so as to obtain a powerful tool for the study of this class of algebras. By making use of these techniques we show that any Leibniz algebra L admitting a multiplicative basis is of the form $L = \bigoplus\limits_{\alpha} \mathcal{I}_\alpha$ with any $\mathcal{I}_\alpha$ a well described ideal of L admitting a multiplicative basis inherited from $\mathcal{B}$. In §3 the $\mathcal{B}$-simplicity of these ideals is characterized in terms of the multiplicative basis.

Finally, we would like to note that the techniques we develop in the preset paper are far away from the ones introduced in the study of the previously mentioned classes of Leibniz algebras having a multiplicative basis. The above references concerning these classes of Leibniz algebras are mainly centered in the finite dimensional setup an so linear algebra tools are fundamental in their arguments, but many times these argument do not hold in the infinite-dimensional case or when the base field is not algebraically close. Our techniques also hold in the infinite-dimensional case and over arbitrary base fields, being adequate enough to provide us of a second Wedderburn-type theorem in this general framework (Theorems 2.1 and 3.1). Indeed, although we make use of the ideal $\mathcal{S}$ which is inherent to Leibniz theory, we hope these techniques can be useful in the study of the structure of other wider categories of algebras.

Let us introduce the following infinite-dimensional Leibniz algebras which will be considered later. We will denote by $\mathbb{N}$ the set of non-negative integers.

Example 1.2. Let $L = \mathcal{S} \oplus V$ be the Leibniz algebra, over a base field with characteristic different to 2, where

$$\mathcal{B}_{\mathcal{S}} = \{e_n : n \in \mathbb{N}\}$$

is a basis of $\mathcal{S}$, the set

$$\mathcal{B}_V = \{v_a, v_b, v_c, v_d\}$$

is a basis of V and the non-zero products respect to the elements in the basis

$$\mathcal{B} = \mathcal{B}_{\mathcal{S}} \dot{\cup} \mathcal{B}_V$$

of L are:

$$[v_b, v_c] = v_a, \quad [v_c, v_b] = -v_a, \quad [v_d, v_d] = e_0,$$
$$[e_0, v_d] = e_1, \quad [e_n, v_a] = e_n \text{ for } n \geq 2, \quad [e_n, v_b] = e_{n+1} \text{ for } n \geq 2, \quad \text{and}$$
$$[e_n, v_c] = (n-2)e_{n-1} \text{ for } n \geq 3.$$

Then L becomes a Leibniz algebra admitting $\mathcal{B}$ as a multiplicative basis.

Example 1.3. Let $L = \mathcal{S} \oplus V$ be the Leibniz algebra, over a base field with characteristic different to 2, where

$$\mathcal{B}_{\mathcal{S}} = \{e_{n,m} : (n,m) \in \mathbb{N} \times \mathbb{N}\}$$

is a basis of $\mathcal{S}$, the set

$$\mathcal{B}_V = \{v_n : n \in \mathbb{N}\}$$

is a basis of V and the non-zero products respect to the elements in the basis

$$\mathcal{B} = \mathcal{B}_{\mathcal{S}} \dot{\cup} \mathcal{B}_V$$

of L are:

$$[v_1, v_2] = v_0, \quad [v_2, v_1] = -v_0,$$
$$[v_3, v_4] = v_0, \quad [v_4, v_3] = -v_0,$$
$$[e_{n,m}, v_0] = e_{n,m} \text{ for } n, m \geq 1, \quad [e_{n,m}, v_1] = e_{n+1,m} \text{ for } n, m \geq 1,$$
$$[e_{n,m}, v_2] = (n-1)e_{n-1,m} \text{ for } n \geq 2 \text{ and } m \geq 1,$$
$$[e_{n,m}, v_3] = e_{n,m+1}, \text{ for } n, m \geq 1,$$
$$[e_{n,m}, v_4] = (m-1)e_{n,m-1} \text{ for } n \geq 1 \text{ and } m \geq 2,$$
$$[e_{0,0}, v_5] = e_{0,0}, \quad [e_{n,0}, v_{4+2n}] = e_{n,0} \text{ for } n \geq 1 \text{ and}$$
$$[e_{0,m}, v_{5+2m}] = e_{0,m} \text{ for } m \geq 1$$

Then L becomes a Leibniz algebra admitting $\mathcal{B}$ as a multiplicative basis.

2. Decomposition as direct sum of ideals

In what follows $L = \mathcal{S} \oplus V$ denotes a Leibniz algebra over a base field $\mathbb{F}$ admitting a multiplicative basis $\mathcal{B} = \mathcal{B}_{\mathcal{S}} \dot{\cup} \mathcal{B}_V$, with bases $\mathcal{B}_{\mathcal{S}} = \{e_i\}_{i \in I}$ and $\mathcal{B}_V = \{u_j\}_{j \in J}$ of $\mathcal{S}$ and V respectively.

We begin this section by developing connection techniques among the elements in the sets of indexes I and J as the main tool in our study.

By renaming if necessary we can suppose $I \cap J = \emptyset$. Now, for each $k \in I \dot{\cup} J$, a new assistant variable $\overline{k} \notin I \dot{\cup} J$ is introduced and we denote by

$$\overline{I} := \{\overline{i} : i \in I\} \text{ and } \overline{J} := \{\overline{j} : j \in J\}$$

the sets consisting of all these new symbols. Also, given any $\overline{k} \in \overline{I} \dot{\cup} \overline{J}$ we will denote

$$\overline{(\overline{k})} := k.$$

Finally, we will write by $\mathcal{P}(A)$ the power set of a given set A.

Next, we consider the following operation which recover, in a sense, certain multiplicative relations among the elements of the basis $\mathcal{B}$:

$$\star : (I\dot{\cup}J) \times (I\dot{\cup}J\dot{\cup}\overline{I}\dot{\cup}\overline{J}) \to \mathcal{P}(I\dot{\cup}J),$$

given by

- For $i, k \in I$,
$$i \star k = \emptyset$$
- For $i \in I$ and $j \in J$,
$$i \star j = j \star i = \begin{cases} \emptyset, & \text{if } [e_i, u_j] = 0 \\ \{k\}, & \text{if } 0 \neq [e_i, u_j] \in \mathbb{F}e_k \text{ with } k \in I \end{cases}$$
- For $i \in I$ and $\overline{k} \in \overline{I}$,
$$i \star \overline{k} = \{j \in J : 0 \neq [e_k, u_j] \in \mathbb{F}e_i\}$$
- For $i \in I$ and $\overline{j} \in \overline{J}$,
$$i \star \overline{j} =$$
$$\{k \in J : 0 \neq [u_j, u_k] \in \mathbb{F}e_i\} \cup \{l \in J : 0 \neq [u_l, u_j] \in \mathbb{F}e_i\} \cup \{m \in I : 0 \neq [e_m, u_j] \in \mathbb{F}e_i\}$$
- For $j, k \in J$,
$$j \star k = k \star j = \alpha \cup \beta$$
where
$$\alpha = \begin{cases} \emptyset, & \text{if } [u_j, u_k] = 0 \\ \{l\}, & \text{if } 0 \neq [u_j, u_k] \in \mathbb{F}u_l \\ \{i\}, & \text{if } 0 \neq [u_j, u_k] \in \mathbb{F}e_i \end{cases} \text{ and } \beta = \begin{cases} \emptyset, & \text{if } [u_k, u_j] = 0 \\ \{m\}, & \text{if } 0 \neq [u_k, u_j] \in \mathbb{F}u_m \\ \{r\}, & \text{if } 0 \neq [u_k, u_j] \in \mathbb{F}e_r \end{cases}$$
- For $j \in J$ and $\overline{i} \in \overline{I}$,
$$j \star \overline{i} = \emptyset$$
- For $j \in J$ and $\overline{k} \in \overline{J}$,
$$j \star \overline{k} = \{l \in J : 0 \neq [u_l, u_k] \in \mathbb{F}u_j\} \cup \{m \in J : 0 \neq [u_k, u_m] \in \mathbb{F}u_j\}.$$

The mapping $\star$ is not still good enough for our purposes and so we need to introduce the following one:
$$\phi : \mathcal{P}(I\dot{\cup}J) \times (I\dot{\cup}J\dot{\cup}\overline{I}\dot{\cup}\overline{J}) \to \mathcal{P}(I\dot{\cup}J),$$
as

- $\phi(\emptyset, I\dot{\cup}J\dot{\cup}\overline{I}\dot{\cup}\overline{J}) = \emptyset$,
- For any $\emptyset \neq K \in \mathcal{P}(I\dot{\cup}J)$ and $a \in I\dot{\cup}J\dot{\cup}\overline{I}\dot{\cup}\overline{J}$,
$$\phi(K, a) = \bigcup_{k \in K} (k \star a).$$

It is straightforward to verify that for any $i, j \in I\dot{\cup}J$ and $a \in I\dot{\cup}J\dot{\cup}\overline{I}\dot{\cup}\overline{J}$ we have that $j \in i \star a$ if and only if $i \in j \star \overline{a}$. This fact implies that for any $K \in \mathcal{P}(I\dot{\cup}J)$ and $a \in I\dot{\cup}J\dot{\cup}\overline{I}\dot{\cup}\overline{J}$ we have

$$i \in \phi(K, a) \text{ if and only if } \phi(\{i\}, \overline{a}) \cap K \neq \emptyset. \tag{2}$$

Definition 2.1. Let i and j be distinct elements in the set of indexes $I\dot{\cup}J$. We say that i is *connected* to j if there exists a subset

$$\{i_1, i_2, ..., i_{n-1}, i_n\} \subset I\dot{\cup}J\dot{\cup}\overline{I}\dot{\cup}\overline{J}$$

with $n \geq 2$ such that the following conditions hold:

1. $i_1 = i$.

2. $\phi(\{i_1\}, i_2) \neq \emptyset$,
   $\phi(\phi(\{i_1\}, i_2), i_3) \neq \emptyset$,
   $\phi(\phi(\phi(\{i_1\}, i_2), i_3), i_4) \neq \emptyset$,
   $\cdots\cdots\cdots$
   $\phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-2}), i_{n-1}) \neq \emptyset$.

3. $j \in \phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-1}), i_n)$.

The subset $\{i_1, i_2, ..., i_{n-1}, i_n\}$ is called a *connection* from i to j and we accept i to be connected to itself.

Proposition 2.1. *The relation $\sim$ in $I\dot{\cup}J$, defined by $i \sim j$ if and only if i is connected to j, is an equivalence relation.*

Proof. By definition $i \sim i$, that is, the relation $\sim$ is reflexive.

Let us see the symmetric character of $\sim$: If $i \sim j$ with $i \neq j$ then there exists a connection

$$\{i_1, i_2, ..., i_{n-1}, i_n\}$$

from i to j satisfying Definition 2.1. Let us show that the set

$$\{j, \bar{i}_n, \bar{i}_{n-1}, ..., \bar{i}_3, \bar{i}_2\}$$

gives rise to a connection from j to i.

Indeed, by taking $K := \phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-1})$ we can apply the relation given by (2) to the expression

$$j \in \phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-1}), i_n)$$

to get

$$\phi(\{j\}, \overline{i_n}) \cap \phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-2}), i_{n-1}) \neq \emptyset$$

and so

$$\phi(\{j\}, \bar{i}_n) \neq \emptyset.$$

By taking

$$k \in \phi(\{j\}, \bar{i}_n) \cap \phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-2}), i_{n-1}),$$

the relation given by (2) and the fact $k \in \phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-2}), i_{n-1})$ allow us to assert

$$\phi(\phi(\{j\}, \bar{i}_n), \overline{i_{n-1}}) \cap \phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-3}), i_{n-2}) \neq \emptyset$$

and consequently

$$\phi(\phi(\{j\}, \bar{i}_n), \bar{i}_{n-1}) \neq \emptyset.$$

By iterating this process we get

$$\phi(\phi(\cdots(\phi(\{j\}, \bar{i}_n), \cdots), \bar{i}_{n-r+1}), \bar{i}_{n-r}) \cap$$
$$\phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-r-2}), i_{n-r-1}) \neq \emptyset$$

for $0 \leq r \leq n-3$. Observe that this relation in the case $r = n-3$ reads as

$$\phi(\phi(\cdots(\phi(\{j\}, \bar{i}_n), \cdots), \bar{i}_4), \bar{i}_3) \cap \phi(\{i_1\}, i_2) \neq \emptyset.$$

Since $i_1 = i$, if we write $K := \phi(\phi(\cdots(\phi(\{\bar{j}\}, \bar{i}_n), \cdots), \bar{i}_4), \bar{i}_3)$, the previous observation allows us to assert that $\phi(\{i\}, i_2) \cap K \neq \emptyset$. Hence the relation (2) applies to get

$$i \in \phi(\phi(\cdots(\phi(\{j\}, \bar{i}_n), \cdots), \bar{i}_3), \bar{i}_2)$$

and concludes $\sim$ is symmetric.

Finally, let us verify the transitive character of $\sim$. Suppose $i \sim j$ and $j \sim k$. If $i = j$ or $j = k$ it is trivial, so suppose $i \neq j$ and $j \neq k$ and write $\{i_1, ..., i_n\}$ for a connection from i to j and $\{j_1, ..., j_m\}$ for a connection from j to k. Then we clearly have that $\{i_1, ..., i_n, j_2, ..., j_m\}$ is a connection from j to k. We have shown the connection relation is an equivalence relation. $\square$

By the above Proposition we can consider the next quotient set on the set of indexes $I\dot{\cup}J$,

$$I\dot{\cup}J/\sim = \{[i] : i \in I\dot{\cup}J\},$$

becoming $[i]$ the set of elements in $I\dot{\cup}J$ which are connected to i.

Our next goal in this section is to associate an ideal $\mathcal{I}_{[i]}$ of L to any $[i]$. Fix $i \in I\dot{\cup}J$, we start by defining the linear subspaces $\mathcal{S}_{[i]} \subset \mathcal{S}$ and $V_{[i]} \subset V$ as follows

$$\mathcal{S}_{[i]} := \bigoplus_{j \in [i] \cap I} \mathbb{F}e_j \subset \mathcal{S},$$

$$V_{[i]} := \bigoplus_{k \in [i] \cap J} \mathbb{F}u_k \subset V$$

Finally, we denote by $\mathcal{I}_{[i]}$ the direct sum of the two subspaces above, that is,

$$\mathcal{I}_{[i]} := \mathcal{S}_{[i]} \oplus V_{[i]}.$$

Definition 2.2. Let L be a Leibniz algebra with a multiplicative basis $\mathcal{B}$. It is said that a subalgebra A of L admits a multiplicative basis $\mathcal{B}_A$ *inherited* from $\mathcal{B}$ if $\mathcal{B}_A$ is a multiplicative basis of A satisfying $\mathcal{B}_A \subset \mathcal{B}$.

Proposition 2.2. *For any $i \in I\dot{\cup}J$, the linear subspace $\mathcal{I}_{[i]}$ is an ideal of L admitting a multiplicative basis inherited from the one of L.*

Proof. We can write

$$[\mathcal{I}_{[i]}, L] = [\mathcal{S}_{[i]} \oplus V_{[i]}, (\bigoplus_{l \in I} \mathbb{F}e_l) \oplus (\bigoplus_{j \in J} \mathbb{F}u_j)].$$

In case $[e_k, u_j] \neq 0$ for some $k \in [i] \cap I$ and $j \in J$, we have that $0 \neq [e_k, u_j] \in \mathbb{F}e_m$ with $m \in I$ and so the connection $\{k, j\}$ gives us $k \sim m$, so $m \in [i]$ and then $0 \neq [e_k, u_j] \in \mathcal{S}_{[i]}$. Hence we get

$$[\mathcal{S}_{[i]}, \bigoplus_{j \in J} \mathbb{F}u_j] \subset \mathcal{S}_{[i]}.$$

In a similar way we have $[V_{[i]}, \bigoplus_{j \in J} \mathbb{F}u_j] \subset \mathcal{I}_{[i]}$ and so

$$[\mathcal{I}_{[i]}, L] \subset \mathcal{I}_{[i]}.$$

On the other hand,

$$[L, \mathcal{I}_{[i]}] = [(\bigoplus_{l \in I} \mathbb{F}e_l) \oplus (\bigoplus_{j \in J} \mathbb{F}u_j), \mathcal{S}_{[i]} \oplus V_{[i]}]$$

and in case $0 \neq [e_l, u_k] \in \mathbb{F}e_m$ for some $l \in I$ and $k \in [i] \cap J$ we have that the connection $\{k, l\}$ gives us $k \sim m$ and so $[\bigoplus_{l \in I} \mathbb{F}e_l, V_{[i]}] \subset \mathcal{S}_{[i]}$. In a similar way $[\bigoplus_{j \in J} \mathbb{F}u_j, V_{[i]}] \subset \mathcal{I}_{[i]}$ and then

$$[L, \mathcal{I}_{[i]}] \subset \mathcal{I}_{[i]}.$$

Hence $\mathcal{I}_{[i]}$ is an ideal of L.

Finally, observe that the set

$$\mathcal{B}_{\mathcal{I}_{[i]}} := \{e_j : j \in [i] \cap I\} \dot\cup \{u_k : k \in [i] \cap J\}$$

is a multiplicative basis of $\mathcal{I}_{[i]}$ satisfying $\mathcal{B}_{\mathcal{I}_{[i]}} \subset \mathcal{B}$. From here we have that $\mathcal{I}_{[i]}$ admits a multiplicative basis inherited from the one of L. □

Corollary 2.1. *If L is simple, then there exists a connection between any couple of elements in the set of indexes $I \dot\cup J$.*

Proof. The simplicity of L implies $[L, L] \neq 0$ and so $J \neq \emptyset$. From here, Proposition 2.2 gives us $\mathcal{I}_{[j_0]} = L$ for some $j_0 \in J$ being then $[j_0] = I \dot\cup J$. That is, any couple of elements in $I \dot\cup J$ are connected. □

Theorem 2.1. *A Leibniz algebra L with a multiplicative basis decomposes as the direct sum*

$$L = \bigoplus_{[i] \in (I \dot\cup J)/\sim} \mathcal{I}_{[i]},$$

where any $\mathcal{I}_{[i]}$ is one of the ideals, admitting a multiplicative basis inherited from the one of L, given in Proposition 2.2.

Proof. Since we can write

$$L = \mathcal{S} \oplus V$$

and

$$\mathcal{S} = \bigoplus_{[i] \in (I \dot\cup J)/\sim} \mathcal{S}_{[i]}, \quad V = \bigoplus_{[i] \in (I \dot\cup J)/\sim} V_{[i]}$$

we clearly have

$$L = \bigoplus_{[i] \in (I \dot\cup J)/\sim} \mathcal{I}_{[i]}.$$

□

Example 2.1. Consider the Leibniz algebra $L = \mathcal{S} \oplus V$ in Example 1.2. We have

$$I = \mathbb{N} \text{ and } J = \{a, b, c, d\}.$$

From the multiplication table of L it is not difficult to write the operation $\star$ in detail. For instance, we have that given $n \in \mathbb{N}$ and $j \in J$,

$$n \star j = j \star n = \begin{cases} \{1\}, & \text{if } n = 0 \text{ and } j = d \\ \emptyset, & \text{if } n = 0 \text{ and } j \in \{a, b, c\} \\ \emptyset, & \text{if } n = 1 \text{ and any } j \in J \\ \{n\}, & \text{if } n \geq 2 \text{ and } j = a \\ \{n+1\}, & \text{if } n \geq 2 \text{ and } j = b \\ \{n-1\}, & \text{if } n \geq 3 \text{ and } j = c \\ \emptyset, & \text{if } n = 2 \text{ and } j = c \end{cases}$$

From here, we can also obtain an explicit expression of the mapping

$$\phi : \mathcal{P}(\mathbb{N} \cup J) \times (\mathbb{N} \dot\cup J \dot\cup \overline{\mathbb{N}} \dot\cup \overline{J}) \to \mathcal{P}(\mathbb{N} \cup J).$$

Observe that the connection $\{0, d\}$ gives us $0 \sim 1$, the connection $\{0, \overline{d}\}$ gives us $0 \sim d$, the connection $\{b, c\}$ gives us $b \sim a$, the connection $\{b, c, \overline{b}\}$ gives us $b \sim c$, the connection $\{b, c, n\}$ gives us $b \sim n$ for $n \geq 2$ and that $0 \nsim b$. Hence, $(\mathbb{N} \cup J)/\sim = \{[0], [2]\}$ where $[0] = \{0, 1, d\}$ and $[2] = \{n \in \mathbb{N} : n \geq 2\} \cup \{a, b, c\}$ and so Theorem 2.1 allows us to assert that

$$L = L_1 \oplus L_2,$$

where any L_i is an ideal of L and where L_1 has as (multiplicative) basis $\{e_0, e_1, v_d\}$ and L_2 has as (multiplicative) basis $\{e_i : i \geq 2\} \cup \{v_a, v_b, v_c\}$.

Example 2.2. Consider the Leibniz algebra $L = \mathcal{S} \oplus V$ in Example 1.3. We have

$$I = \mathbb{N} \times \mathbb{N} \text{ and } J = \mathbb{N}.$$

From the multiplication table of L it is routine to describe $\star$ in detail. For instance, we get that for any $(n, m) \in \mathbb{N} \times \mathbb{N}$ and $p \in \mathbb{N}$,

$$(n,m) \star p = p \star (n,m) = \begin{cases} \{(0,0)\}, & \text{if } n = 0,\, m = 0 \text{ and } p = 5 \\ \emptyset, & \text{if } n = 0,\, m = 0 \text{ and } p \neq 5 \\ \{(n,0)\}, & \text{if } n \geq 1,\, m = 0 \text{ and } p = 4 + 2n \\ \emptyset, & \text{if } n \geq 1,\, m = 0 \text{ and } p \neq 4 + 2n \\ \{(0,m)\}, & \text{if } n = 0,\, m \geq 1 \text{ and } p = 5 + 2m \\ \emptyset, & \text{if } n = 0,\, m \geq 1 \text{ and } p \neq 5 + 2m \\ \{(n,m)\}, & \text{if } n \geq 1,\, m \geq 1 \text{ and } p = 0 \\ \{(n+1,m)\}, & \text{if } n \geq 1,\, m \geq 1 \text{ and } p = 1 \\ \{(n-1,m)\}, & \text{if } n \geq 2,\, m \geq 1 \text{ and } p = 2 \\ \emptyset, & \text{if } n = 1,\, m \geq 1 \text{ and } p = 2 \\ \{(n,m+1)\}, & \text{if } n \geq 1,\, m \geq 1 \text{ and } p = 3 \\ \{(n,m-1)\}, & \text{if } n \geq 1,\, m \geq 2 \text{ and } p = 4 \\ \emptyset, & \text{if } n \geq 1,\, m = 1 \text{ and } p = 4 \end{cases}$$

As in Example 2.1 it is now straightforward to compute the equivalence classes in

$$((\mathbb{N} \times \mathbb{N}) \cup \mathbb{N})/\sim .$$

We obtain

$$((\mathbb{N} \times \mathbb{N}) \cup \mathbb{N})/\sim = [(1,1)] \cup [(0,0)] \cup \{[(n,0)] : n \geq 1\} \cup \{[(0,m)] : m \geq 1\},$$

being $[(1,1)] = \{(n,m) : n \geq 1, m \geq 1\} \cup \{0,1,2,3,4\}$, $[(0,0)] = \{(0,0),5\}$, $[(n,0)] = \{(n,0), 4+2n\}$ for any $n \geq 1$ and $[(0,m)] = \{(0,m), 5+2m\}$ for any $m \geq 1$.

From here, Theorem 2.1 allows us to assert that L decomposes as the direct sum of the ideals

$$L = L_{(1,1)} \oplus L_{(0,0)} \oplus (\bigoplus_{n \geq 1} L_{(n,0)}) \oplus (\bigoplus_{m \geq 1}) L_{(0,m)}$$

where $L_{(1,1)}$ admits as (multiplicative) basis $\{v_0, v_1, v_2, v_3\} \cup \{e_{(n,m)} : n \geq 1, m \geq 1\}$, the ideal $L_{(0,0)}$ has as (multiplicative) basis $\{v_5, e_{(0,0)}\}$, any $L_{(n,0)}$, $n \geq 1$, admits as (multiplicative) basis $\{v_{4+2n}, e_{(n,0)}\}$ and any $L_{(0,m)}$, $m \geq 1$, has as (multiplicative) basis $\{v_{5+2m}, e_{(0,m)}\}$.

3. THE $\mathcal{B}$-SIMPLE COMPONENTS

In this section our target is to characterize the minimality of the ideals which give rise to the decomposition of L in Theorem 2.1, in terms of connectivity properties in the set of indexes $I \dot{\cup} J$. Since a Leibniz algebra L is called *simple* when its only ideals are $\{0\}$, $\mathcal{S}$ and L (see [2]), we introduce the next concept in a natural way.

Definition 3.1. *A Leibniz algebra L admitting a multiplicative basis $\mathcal{B}$ is called $\mathcal{B}$-simple if its only ideals admitting a multiplicative basis inherited from $\mathcal{B}$ are $\{0\}$, $\mathcal{S}$ and L.*

Observe that we can find in a Leibniz algebra admitting a multiplicative basis $\mathcal{B}$ ideals which do not admit a multiplicative basis inherited from $\mathcal{B}$. Indeed, consider the Leibniz algebra L in Example 1.2. The linear subspace with basis $\{v_a, v_a+v_b+v_c\} \cup \mathcal{B}_{\mathcal{S}}$ is actually an ideal of L which does not admit any multiplicative basis inherited from $\mathcal{B}$. However, it is clear that any simple Leibniz algebra admitting a multiplicative basis $\mathcal{B}$ is $\mathcal{B}$-simple.

As in the previous section, $L = \mathcal{S} \oplus V$ will denote a Leibniz algebra over an arbitrary base field $\mathbb{F}$ and of arbitrary dimension, admitting a multiplicative basis $\mathcal{B} = \mathcal{B}_{\mathcal{S}} \dot{\cup} \mathcal{B}_V$ with $\mathcal{B}_{\mathcal{S}} = \{e_i\}_{i \in I}$ and $\mathcal{B}_V = \{u_j\}_{j \in J}$ bases of $\mathcal{S}$ and V respectively.

We will have the opportunity of restricting the connectivity relation to the set I and to the set J by just allowing that the connections are formed by elements in $J \dot{\cup} \overline{J}$. Then we will say that two indexes $k, l \in K$, where either $K = I$ or $K = J$, are J-connected.

Definition 3.2. Let k and l be two distinct elements in K with either $K = I$ or $K = J$. We say that k is *J-connected* to l, denoted by $k \sim_J l$, if there exists a connection $\{i_1, i_2, ..., i_n\}$ from k to l such that

$$i_2, ..., i_n \in J \dot{\cup} \overline{J}.$$

We will also say that the set $\{i_1, i_2, ..., i_n\}$ is a J-connection from k to l and accept k to be J-connected to itself.

We observe that it is straightforward to verify that the arguments in Proposition 2.1 allow us to assert that the relation $\sim_J$ is an equivalence relation in I and in J.

Let us introduce the notion of $\star$-multiplicativity in the framework of Leibniz algebras with multiplicative bases, in a similar way to the ones of closed-multiplicativity for graded Lie algebras, graded Lie superalgebras, split Leibniz algebras and split Leibniz superalgebras (see [10, 11, 12, 13] for these notions and examples). From now on, for any $\overline{j} \in \overline{J}$ we will denote $u_{\overline{j}} = 0$.

Definition 3.3. We say that a Leibniz algebra $L = \mathcal{S} \oplus V$ admits a *$\star$-multiplicative basis* $\mathcal{B}$ if it is multiplicative and the conditions below hold.

(1) Given $j \in J$ and $k \in I \dot{\cup} J$ such that $k \in j \star a$ for some $a \in J \dot{\cup} \overline{J}$ then $v_k \in \mathbb{F}[u_j, u_a + u_{\overline{a}}]$, where $v_k \in \{e_k, u_k\}$ depending on $k \in I$ or $k \in J$.

(2) Given $i, j \in I$ such that $j \in i \star a$ for some $a \in J\dot{\cup}\overline{J}$ then $e_j \in \mathbb{F}[e_i, u_a + u_{\overline{a}}]$.

Proposition 3.1. *Suppose L admits a $\star$-multiplicative basis $\mathcal{B}$ and I and J have respectively all of their elements J-connected, then any nonzero ideal $\mathcal{I}$ of L with a multiplicative basis inherited from $\mathcal{B}$ such that $\mathcal{I} \subsetneqq \mathcal{S}$ satisfies $\mathcal{I} = L$.*

Proof. Since $\mathcal{I} \subsetneqq \mathcal{S}$ we can take some $j_0 \in J$ such that

$$0 \neq u_{j_0} \in \mathcal{I}. \tag{3}$$

We know that J has all of their elements J-connected. From here, given any $k \in J$, we can consider a J-connection

$$\{j_0, j_2, ..., j_n\} \subset J\dot{\cup}\overline{J} \tag{4}$$

from j_0 to k.

We know

$$\phi(\{j_0\}, j_2) \neq \emptyset$$

and so for any $k_1 \in \phi(\{j_0\}, j_2)$ we have $k_1 \in j_0 \star j_2$. Taking now into account Equation (3) and the $\star$-multiplicativity of $\mathcal{B}$ we get

$$0 \neq u_{k_1} \in \mathbb{F}[u_{j_0}, u_{l_2}] \subset \mathcal{I}$$

if $k_1 \in J$ or

$$0 \neq e_{k_1} \in \mathbb{F}[u_{j_0}, u_{l_2}] \subset \mathcal{I}$$

if $k_1 \in I$, for $l_2 = \{j_2, \overline{j}_2\} \cap J$.

Since $k \in J$, necessarily $\phi(\{j_0\}, j_2) \cap J \neq \emptyset$ and we have

$$0 \neq \bigoplus_{j \in \phi(\{j_0\}, j_2) \cap J} \mathbb{F}u_j \subset \mathcal{I}. \tag{5}$$

Since

$$\phi(\phi(\{j_0\}, j_2), j_3) \neq \emptyset.$$

we can argue as above, taking into account Equation (5), to get

$$0 \neq \bigoplus_{j \in \phi(\phi(\{j_0\}, j_2), j_3) \cap J} \mathbb{F}u_j \subset \mathcal{I}.$$

By reiterating this process with the J-connection (4) we obtain

$$0 \neq \bigoplus_{j \in \phi(\phi(\cdots(\phi(j_0, j_2), \cdots), j_{n-1}), j_n) \cap J} \mathbb{F}u_j \subset \mathcal{I}.$$

Since $k \in \phi(\phi(\cdots(\phi(j_0, j_2), \cdots), j_{n-1}), j_n) \cap J$ we conclude $u_k \in \mathcal{I}$ and so

$$V = \bigoplus_{j \in J} \mathbb{F}u_j \subset \mathcal{I}. \tag{6}$$

Taking now into account that $\mathcal{S} \subset [\mathcal{S}, V] + [V, V]$, Equation (6) allows us to assert

$$\mathcal{S} \subset \mathcal{I}. \tag{7}$$

Finally, since $L = \mathcal{S} \oplus V$, Equations (6) and (7) give us $\mathcal{I} = L$. $\square$

Proposition 3.2. *Suppose L admits a $\star$-multiplicative basis $\mathcal{B}$ and I has all of its elements J-connected, then any nonzero ideal $\mathcal{I}$ of L with a multiplicative basis inherited from $\mathcal{B}$ such that $\mathcal{I} \subset \mathcal{S}$ satisfies $\mathcal{I} = \mathcal{S}$.*

Proof. Taking into account $\mathcal{I} \subset \mathcal{S}$ we can fix some $i_0 \in I$ satisfying

$$0 \neq e_{i_0} \in \mathcal{S}.$$

Since I has of of its elements J-connected, we can argue from i_0 with the $\star$-multiplicativity of $\mathcal{B}$ as it is done in Proposition 3.1 from j_0, to get $\mathcal{S} \subset \mathcal{I}$ and then $\mathcal{I} = \mathcal{S}$. □

Theorem 3.1. *Suppose L admits a $\star$-multiplicative basis $\mathcal{B}$. Then L is $\mathcal{B}$-simple if and only if I and J have respectively all of their elements J-connected.*

Proof. Suppose L is $\mathcal{B}$-simple. If $\mathcal{S} \neq \{0\}$ and we take $i \in I$, we have that the linear space $\bigoplus\limits_{k\in I:k\sim_J i} \mathbb{F}e_k$ is an ideal of L with a multiplicative basis inherited from $\mathcal{B}$. Indeed, we have

$$[L, \bigoplus_{k\in I:k\sim_J i} \mathbb{F}e_k] + [\bigoplus_{k\in I:k\sim_J i} \mathbb{F}e_k, \mathcal{S}] \subset [L, \mathcal{S}] = 0$$

and

$$[\bigoplus_{k\in I:k\sim_J i} \mathbb{F}e_k, u_j] \subset \bigoplus_{k\in I:k\sim_J i} \mathbb{F}e_k$$

for any $j \in J$ because given any $h \in \bigoplus\limits_{k\in I:k\sim_J i} \mathbb{F}e_k$ such that $0 \neq [e_k, u_j] = e_l$ we have $e_l \in e_k \star u_j$ and so $\{k, j\}$ is a J-connection from k to l. By the symmetry and transitivity of $\sim_J$ in I we get $e_l \in \bigoplus\limits_{k\in I:k\sim_J i} \mathbb{F}e_k$. Hence $[e_k, u_l] \subset \bigoplus\limits_{k\in I:k\sim_J i} \mathbb{F}e_k$ as desired. We conclude $\bigoplus\limits_{k\in I:k\sim_J i} \mathbb{F}e_k$ is an ideal of L with a multiplicative basis inherited from $\mathcal{B}$ and so, by $\mathcal{B}$-simplicity, necessarily $\bigoplus\limits_{k\in I:k\sim_J i} \mathbb{F}e_k = \mathcal{S}$. Consequently any couple of indexes in I are J-connected.

Consider now any $j \in J$ and the linear subspace

$$\mathcal{S} \oplus (\bigoplus_{l\in J:l\sim_J j} \mathbb{F}u_l).$$

A similar argument to the above one gives us that this linear subspace is actually an ideal of L which admits a multiplicative basis inherited from $\mathcal{B}$. From here, $\mathcal{S} \oplus (\bigoplus\limits_{l\in J:l\sim_J j} \mathbb{F}u_l) = L$ which implies in particular $\bigoplus\limits_{l\in J:l\sim_J j} \mathbb{F}u_l = \bigoplus\limits_{m\in J} \mathbb{F}u_m$ and so we get that any couple of indexes in J are also J-connected.

Conversely, consider $\mathcal{I}$ a nonzero ideal of L admitting a multiplicative basis inherited by the one of L. We have two possibilities for $\mathcal{I}$, either $\mathcal{I} \not\subseteq \mathcal{S}$ or $\mathcal{I} \subset \mathcal{S}$. In the first one, Proposition 3.1 gives us $\mathcal{I} = L$, while in the second one Proposition 3.2 shows $\mathcal{I} = \mathcal{S}$. We have proved L is $\mathcal{B}$-simple. □

Acknowledgment. The author would like to thank the referees for their exhaustive reviews of the paper as well as for many suggestions which have helped to improve the work.

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