ROOT-APPROXIMABLITY OF THE GROUP OF AUTOMORPHISMS OF THE UNIT BALL IN $\mathbb{C}^n$

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Abstract. In this paper we prove that the group of all biholomorphic maps from the open unit ball in $\mathbb{C}^n$ onto itself endowed with the compact-open topology is a root-approximable topological group.

1. Introduction

In 1906, J. Jensen and in 1915 F. Bernstein and G. Doetsch proved results that imply the continuity of a convex function under certain mild conditions. These theorems have turned out to be of most important results in the theory of convex functions with many applications in various part of mathematics. Since then many extensions of these results have been presented; see for example [4, 7, 8, 9, 11, 12].

In [1] the authors proved an analogue of the Bernstein-Doetsch theorem for mid-convex mappings defined on a root-approximable topological group. For this end, the authors introduced the notion of a root-approximable group. Let us recall the definition.

Definition 1.1. Let G be a group endowed with a topology τ. An element $x \in G$ is said to be *root-approximable* if there exists a sequence (y_n) in G such that

$$\lim_{n\to\infty} y_n = e, \quad y_n^{2^n} = x$$

for all $n \in \mathbb{N}$, where e is the identity element of G. The group G is called root-approximable group if every element of G is root-approximable.

Let A be an open subset of a group G such that $ay, ay^{-1} \in A$ for all $a \in A$ and $y \in G$, then a function $f : A \subseteq G \to \mathbb{R}$ is said to be *globally midconvex* on A if for each $a \in A$ and $y \in G$, $2f(a) \leq f(ay) + f(ay^{-1})$. The authors in [1] presented a generalization of the Bernstein-Doetsch theorem: a globally

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midconvex real-valued function in an open subset of a root-approximable topological group which is bounded from above in some neighborhood of a point is everywhere continuous. For a generalization of the Ostrowski theorem and more details, we refer the reader to [1].

So, this arises the question: which groups are root-approximable? Many classical groups, but not all, are root-approximable. For instance $\mathrm{Gl_n}(\mathbb{C})$ has this property while $\mathrm{Gl_n}(\mathbb{R})$ has not.

Our purpose in this work is to investigate this question for the group $\mathrm{Aut}(\mathbb{B})$, the group of all biholomorphic maps of the open unit ball in $\mathbb{C}^n$ into itself equipped with the compact-open topology. As one of the motivations of this work, we conclude that any globally midconvex real-valued function on $\mathrm{Aut}(\mathbb{B})$ which is bounded from above is continuous.

2. Preliminaries

Assume that the space $\mathbb{C}^n$, $n \geq 1$, is equipped with the Hermitian product

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \bar{w}_j \quad (z, w \in \mathbb{C}^n).$$

Let $\mathbb{B}$ and $\mathbb{S}$ denote the open unit ball and the unit sphere in $\mathbb{C}^n$, respectively. Also, let $\mathrm{Aut}(\mathbb{B})$ denote the group (relative to composition of maps) of *automorphisms* of $\mathbb{B}$; i.e., the group of all biholomorphic maps from $\mathbb{B}$ onto itself. It is known that $\mathrm{Aut}(\mathbb{B})$ equipped with the compact-open topology is a locally compact Hausdorff group; see Proposition 5.7 in [10]. The unitary group $\mathscr{U}$ is the group of all linear operators $U : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ preserving the Hermitian product; i.e.,

$$\langle Uz, Uw\rangle = \langle z, w\rangle \quad (z, w \in \mathbb{B}).$$

It is clear that the restriction of any $U \in \mathscr{U}$ to $\mathbb{B}$ belongs to $\mathrm{Aut}(\mathbb{B})$, so we can consider $\mathscr{U}$ as a subgroup of $\mathrm{Aut}(\mathbb{B})$.

Let us recall and fix some notations, introduced in [13], for later use. For any $a \in \mathbb{B}$, let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[a]$ generated by a, and $Q_a = I - P_a$ be the projection onto the orthogonal complement of $[a]$. To be explicit, let

$$P_0 = 0, P_a z = \frac{\langle z, a\rangle}{\langle a, a\rangle} a \quad (a \neq 0, z \in \mathbb{C}^n).$$

Also, define

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle}$$

for all $z \in \Omega = \{z : \langle z, a\rangle \neq 1\}$,where $s_a = \sqrt{1-|a|^2}$. Then $\varphi_a : \Omega \to \mathbb{C}^n$ is holomorphic and $\varphi_a^{-1} = \varphi_a$, $\varphi_a(0) = a$, $\varphi_a(a) = 0$ and $\varphi_a \in \mathrm{Aut}(\mathbb{B})$; see Theorem 2.2.2 in [13].

We need the following result whose proof can be found in [2].

Theorem 2.1. *Each element $\varphi \in \mathrm{Aut}(\mathbb{B})$ can be extended holomorphically to an open neighborhood of $\overline{\mathbb{B}}$. If $\varphi \neq \mathrm{id}_{\mathbb{B}}$, then either φ has at least one fixed point in $\mathbb{B}$ or it has no fixed points in $\mathbb{B}$ and one or two fixed points in $\mathbb{S}$. Moreover, $\mathrm{Aut}(\mathbb{B})$ acts doubly transitively on $\mathbb{S}$.*

This leads us to consider three classes of automorphisms in $\mathrm{Aut}(\mathbb{B})$.

Definition 2.2. A $\psi \in \mathrm{Aut}(\mathbb{B})$ is called

(a) *elliptic* if it has fixed points in $\mathbb{B}$.
(b) *hyperbolic* if it has no fixed points in $\mathbb{B}$ and has two distinct fixed points in $\mathbb{S}$.
(c) *parabolic* if it has no fixed points in $\mathbb{B}$ and has only one fixed point in $\mathbb{S}$.

We denote the set of elliptic, hyperbolic and parabolic automorphisms by $\mathcal{E}, \mathcal{H}$ and $\mathcal{P}$, respectively.

3. RESULTS

We begin with the following easy observation which will be needed in the sequel.

Lemma 3.1. *Let G be a group endowed with a topology. If $x \in G$ is root-approximable, then yxy^{-1} is root-approximable for all $y \in G$.*

First we deal with automorphisms in $\mathcal{E}$.

Proposition 3.2. *Every element in $\mathcal{E}$ is root-approximable.*

Proof. Let $\psi \in \mathcal{E}$ and $a \in \mathbb{B}$ be a fixed point of ψ. Then, by Cartan's theorem [13, Theorem 2.1.3], $U := \varphi_a \psi \varphi_a$ is a unitary operator, and hence $\psi = \varphi_a U \varphi_a$. By Lemma 3.1, it is sufficient to prove that every unitary operator is root-approximable.

To prove this, recall that for any $U \in \mathscr{U}$, there exist real numbers $\theta_1, \ldots, \theta_k \in [0, 2\pi)$ and orthogonal projections $P_1, \ldots, P_k$ with

$$\sum_{j=1}^{k} P_j = \mathrm{id}_{\mathbb{B}}, \quad U = \sum_{j=1}^{k} e^{i\theta_j} P_j.$$

Now define

$$U_m = \sum_{j=1}^{k} e^{i\theta_j/2^m} P_j \quad (m \geq 1),$$

and note that $\lim_{m\to\infty} U_m = \mathrm{id}_{\mathbb{B}}$ and $U_m^{2^m} = U$ for all $m \geq 1$. This completes the proof. □

Let us fix some further notations. In view of direct sum $\mathbb{C}^n = \mathbb{C} \oplus \mathbb{C}^{n-1}$, we can decompose every $z \in \mathbb{C}^n$ into (z_1, z'), where $z_1 \in \mathbb{C}$ and $z' \in \mathbb{C}^{n-1}$. We also set $e_1 = (1, 0') \in \mathbb{C}^n$, where $1 \in \mathbb{C}$ and $0' = (0, \ldots, 0) \in \mathbb{C}^{n-1}$.

For any $a \in \mathbb{B}$ set $\psi_a = -\varphi_a$.

Lemma 3.3. *Let a be a nonzero element of $\mathbb{B}$. Then $\psi_a \in \mathcal{H}$ and $\psi_{Ua} = U\psi_a U^{-1}$ for all $U \in \mathscr{U}$.*

Proof. For any nonzero element $a \in \mathbb{B}$ and $U \in \mathscr{U}$, we have

$$P_{Ua}(z) = \frac{\langle z, Ua\rangle}{\langle Ua, Ua\rangle} Ua = UP_a U^{-1}(z),$$

$$Q_{Ua} = \mathrm{id} - P_{Ua} = UU^{-1} - UP_a U^{-1} = U(\mathrm{id}_{\mathbb{B}} - P_a)U^{-1} = UQ_a U^{-1},$$

and hence

$$\begin{aligned}
\psi_{Ua}(z) &= \frac{-Ua + P_{Ua} z + s_{Ua} Q_{Ua} z}{1 - \langle z, Ua\rangle} \\
&= \frac{-Ua + UP_a U^{-1} z + s_a U Q_a U^{-1} z}{1 - \langle U^{-1} z, a\rangle} \\
&= U\psi_a U^{-1}(z)
\end{aligned}$$

for all $z \in \mathbb{B}$.

For the other claim, first note that the fixed points of ψ_{re_1}, for $0 < r < 1$, are $\pm e_1$. This is because that if $z = (z_1, z') \in \mathbb{C} \oplus \mathbb{C}^{n-1}$ is a fixed point of ψ_{re_1}, then

$$\psi_{re_1}(z) = \left(\frac{z_1 - r}{1 - rz_1}, \frac{\sqrt{1 - r^2}}{1 - rz_1} z' \right) = (z_1, z'),$$

which implies that $z = \pm e_1$. Now, since $\mathscr{U}$ acts transitively on $\mathbb{S}$, so there exists a unitary operator U with $a = Ure_1$, where $r = |a|$. It follows that $\pm Ue_1$ are two fixed points of ψ_a, and whence $\psi_a \in \mathcal{H}$. □

Lemma 3.4. *Suppose $\psi \in \mathrm{Aut}(\mathbb{B})$ has exactly two fixed points $\pm\zeta$ in $\mathbb{S}$. Then there exist an element $a \in \mathbb{B}$ and an operator $U \in \mathscr{U}$ such that $\psi = U\psi_a = \psi_a U$ and $Ua = a$.*

Proof. First consider the special case $\zeta = e_1$. It is known that there is a unique operator $U \in \mathscr{U}$ such that $\psi = U\psi_a$, where $\psi(a) = 0$: see Theorem 2.2.5 in [13]. Then

$$U\Big(\frac{-a + P_a e_1 + s_a Q_a e_1}{1- < e_1, a >}\Big) = e_1, \quad U\Big(\frac{-a - P_a e_1 - s_a Q_a e_1}{1+ < e_1, a >}\Big) = -e_1.$$

By a simple computation, we can show that $Ua = \overline{a}_1 e_1$, where $a = (a_1, a') \in \mathbb{C} \oplus \mathbb{C}^{n-1}$. But U is a unitary operator, so $|a| = |Ua| = |a_1|$. Consequently, $a = (a_1, 0)$ and

$$Ue_1 = \frac{1 - \overline{a}_1}{1 - a_1} e_1 = \frac{1 + \overline{a}_1}{1 + a_1} e_1.$$

Hence a_1 is real, and therefore $a = a_1 e_1, Ua = a$. By Lemma 3.3, we get that

$$U\psi_a = U\psi_a U^{-1} U = \psi_{Ua} U = \psi_a U.$$

Now for the general case, note that there is a $V \in \mathscr{U}$ such that $Ve_1 = \zeta$ and $V^{-1}\psi V(\pm e_1) = \pm e_1$. Hence for $\kappa = V^{-1}\psi V$, the above discussion shows that there exist a unique $U' \in \mathscr{U}$ and $a \in \mathbb{B}$ such that $\kappa = U'\psi_a = \psi_a U'$ and $U'a = a$. A little manipulation shows that $\psi = U\psi_b = \psi_b U$ and $Ub = b$, where $b = Va$ and $U = VU'V^{-1}$. □

In the next result we consider an upper-half space in $\mathbb{C}^n$; namely, we set

$$\Omega = \{w = (w_1, w') \in \mathbb{C}^n : \text{Im } w_1 > |w_2|^2 + \cdots + |w_n|^2 \text{ where } w' = (w_2, ..., w_n)\}.$$

Recall that the *Cayley transform* ϕ defined as

$$\phi(z) = i\frac{e_1 + z}{1 - z_1}$$

for all $z \in \mathbb{C}^n$ with $z_1 \neq 1$. Then the map ϕ is a biholomorphic map from $\mathbb{B}$ onto Ω. If $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of Ω in $\mathbb{C}^n$, then $w \in \partial\Omega$ if and only if

$$\text{Im } w_1 = |w'|^2.$$

Let $\overline{\Omega} \cup \{\infty\}$ be the one-point compactification of $\overline{\Omega}$, and let $\phi(e_1) = \infty$. Therefore ϕ is a homeomorphism from $\overline{\mathbb{B}}$ onto $\overline{\Omega} \cup \{\infty\}$, and ϕ induces an isomorphism between $\text{Aut}(\mathbb{B})$ and $\text{Aut}(\Omega)$. For more details, see Section 2.3 in [13].

Lemma 3.5. *Let $H = \{h_a : a \in \partial\Omega\}$, where $h_a : \Omega \to \Omega$ defined by*

$$h_a(w) = (w_1 + a_1 + 2i\langle w', a' \rangle, w' + a').$$

Then H is a root-approximable subgroup of $\mathrm{Aut}(\Omega)$ *and each element of* $H \setminus \{\mathrm{id}_\Omega\}$ *has no fixed points in* $\overline{\Omega}$*. Furthermore, for every* $\zeta \in \mathbb{S}$*, there is an* $a \in \partial\Omega$ *such that* $\phi^{-1} h_a \phi(\zeta) = -e_1$*.*

Proof. Recall from Part 2.3.3 in [13] that H is a subgroup of $\mathrm{Aut}(\Omega)$ that consists of the translations of Ω and each element of $H \setminus \{\mathrm{id}_\Omega\}$ has no fixed points in $\overline{\Omega}$. For any $a \in \partial\Omega$, define

$$a^{(m)} = \left(\frac{\mathrm{Re}\ a_1}{2^m} + i\frac{\mathrm{Im}\ a_1}{2^{2m}}, \frac{a'}{2^m} \right) \quad (m = 0, 1, 2, \ldots).$$

Then $a^{(m)} \in \partial\Omega$ and $a^{(m)} \longrightarrow 0$ as $m \longrightarrow \infty$. By a simple computation, we find that $(h_{a^{(m)}})$ converges to the identity map id_Ω, and

$$\left(h_{a^{(m)}}\right)^{2^m} = h_a \quad (m = 0, 1, 2, \ldots).$$

Thus H is root-approximable.

If $\zeta \in \mathbb{S}$ then there is an element $b \in \partial\Omega$ with $\phi(\zeta) = b$. Since $\phi(-e_1) = 0$, $a = (-\bar{b_1}, -\acute{b})$ and we get $\phi^{-1} h_a \phi(\zeta) = -e_1$. □

We need another elementary lemma.

Lemma 3.6. *Let U be a unitary operator on $\mathbb{C}^n$ such that $Ue_1 = e_1$. Then there is a unitary operator U_1 such that $U_1^2 = U$ and $U_1 e_1 = e_1$.*

Proof. First, let $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ be a unitary matrix; i.e., $|\lambda_j| = 1$ for $1 \leqslant j \leqslant n$. Suppose that $a = (a_1, \ldots, a_n)$ is a fixed point of D. Then, clearly, $a_j \neq 0$ if and only if $\lambda_j = 1$. If $D_1 = \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$ then D_1 is unitary and $D_1 a = a$ as well.

Now, for any unitary matrix U we can find unitary matrices V and D such that D is diagonal and $U = V^{-1}DV$. If $Ue_1 = e_1$, setting $Ve_1 = a$, we have $Da = a$, $D_1 a = a$. Setting $U_1 = V^{-1} D_1 V$, we see that $U_1^2 = U$ and $U_1 e_1 = V^{-1} D_1 V e_1 = V^{-1} D_1 a = V^{-1} a = e_1$. □

Now we are in a position to prove the following result.

Proposition 3.7. *Every element in $\mathcal{H}$ is root-approximable.*

Proof. Let us remark two primary observations.

(i) For any $U \in \mathscr{U}$ with $U(e_1) = e_1$ and any $\zeta \in \mathbb{S}$, we have $U\psi_{r\zeta} = \psi_{r\zeta} U$ for all $-1 < r < 1$.

(ii) For every $r, t \in (-1,1)$, we have

$$\begin{aligned}\psi_{re_1}\psi_{te_1}(z) =&\psi_{re_1}\left(\frac{z_1-t}{1-tz_1}, \frac{\sqrt{1-t^2}}{1-tz_1}z'\right)\\ =&\left(\frac{z_1-q}{1-qz_1}, \frac{\sqrt{1-q^2}}{1-qz_1}z'\right) = \psi_{qe_1}(z)\end{aligned}$$

where $q = \frac{r+t}{1+rt}$.

Now we turn to the proof. Let $\psi \in \mathcal{H}$. First assume that $\pm e_1$ are the fixed points of ψ. By the proof of Lemma 3.4, there exist a unitary operator U and a real number $r \in (-1,1)$ such that $Ue_1 = e_1$ and $\psi = U\psi_{re_1} = \psi_{re_1}U$. Hence, by Lemma 3.6 and above observations, we obtain

$$\psi = U\psi_{re_1} = U_1^2\psi_{r_1e_1}^2 = (U_1\psi_{r_1e_1})^2$$

where

$$r_1 = \begin{cases}\frac{1-\sqrt{1-r^2}}{r}, & r \neq 0;\\ 0, & r = 0.\end{cases}$$

If real numbers $r_1, \ldots, r_m$ and unitary operators $U_1, \ldots, U_m$ have been chosen, we select r_{m+1} and unitary operator U_{m+1} such that $U_{m+1}^2 = U_m$ and $\psi_{r_{m+1}e_1}^2 = \psi_{r_me_1}$ with $U_{m+1}(e_1) = e_1$ by Lemma 3.6. Now, since U_m commutes with $\psi_{r_me_1}$, $\psi_m = (U_{m+1}\psi_{r_{m+1}e_1})^2$ and $\psi_{re_1} = (\psi_{r_me_1})^{2^m}$, $U_m^{2^m} = U$ for all m. Also, it can be easily checked that $\psi_{r_me_1} \to \mathrm{id}_{\mathbb{B}}$ and $U_m \to \mathrm{id}_{\mathbb{B}}$. Thus, setting $\psi_m := \psi_{r_me_1}U_m$, we have $(\psi_m)^{2^m} = \psi$ and $\psi_m \to \mathrm{id}_{\mathbb{B}}$ as $m \to \infty$.

Now let e_1 and ζ be the fixed points of ψ. By Lemma 3.5, there is an $a \in \partial\Omega$ such that $\phi^{-1}h_a\phi(\zeta) = -e_1$. Let $\varphi = \phi^{-1}h_a\phi$ and $\kappa = \varphi\psi\varphi^{-1}$. Then $\pm e_1$ are the fixed points of κ. So, by the first part, κ and hence ψ is root-approximable by Lemma 3.1.

Finally, if α and β are fixed points of ψ, then there is a unitary operator $U \in \mathscr{U}$ such that $Ue_1 = \alpha$. Let $\zeta = U^{-1}\beta$ and $\eta = U^{-1}\psi U$, then $\eta(e_1) = e_1$ and $\eta(\zeta) = \zeta$. Now the rest is clear.

□

We need to recall the following result from [2].

Proposition 3.8. *Let $\psi \in \mathcal{P}$. Then ψ is conjugate in* $\mathrm{Aut}(\mathbb{B})$, *either to*

$$\psi_1(z) = \frac{((1-it)z_1 + it, e^{i\theta_2}z_2, \ldots, e^{i\theta_n}z_n)}{-itz_1 + 1 + it}$$

where $t \in \mathbb{R} \setminus \{0\}$ *and* $\theta_j \in \mathbb{R}$ *for* $j = 2, \ldots, n$, *or to*

$$\psi_2(z) = \frac{((1-\beta)z_1 - sz_2 + \beta, sz_1 + z_2 - s, e^{i\theta_3} z_3, \ldots, e^{i\theta_n} z_n)}{-\beta z_1 - sz_2 + \beta + 1}$$

where $\mathrm{Re}\ \beta > 0$, $s = \sqrt{2\mathrm{Re}\ \beta}$ *and* $\theta_j \in \mathbb{R}$ *for* $j = 3, \ldots, n$.

Now we can deal with the last class of automorphisms.

Proposition 3.9. *Every element in* $\mathcal{P}$ *is root-approximable.*

Proof. Let $\psi \in \mathcal{P}$ and ψ_1, ψ_2 be as in Proposition 3.8. It is easily seen that $\kappa_j = \phi \psi_j \phi^{-1}$, where $j = 1, 2$ and ϕ is Cayley transformation, are given by

$$\kappa_1(w) = (w_1 - 2t, e^{i\theta_2} w_2, \ldots, e^{i\theta_n} w_n)$$

and

$$\kappa_2(w) = \Big(w_1 - 2\sqrt{\mathrm{Re}\ 2\beta}\, w_2 + 2i\beta, w_2 - i\sqrt{\mathrm{Re}\ 2\beta}, e^{i\theta_3} w_3, \ldots, e^{i\theta_n} w_n\Big)$$

with $\mathrm{Re}\ \beta > 0$. Conjugating κ_1 by

$$\varphi_1(w) = (\lambda_1^2 w_1, \lambda_1 w_2, \ldots . \lambda_1 w_n)$$

where $\lambda_1 = \sqrt{r}$ and κ_2 by

$$\varphi_2(w) = (\lambda_2^2(w_1 - 2iw_2\alpha + i\alpha^2), \lambda_2(w_2 - \alpha), \lambda_2 w_3, \ldots, \lambda_2 w_n)$$

where $\lambda_2 = \sqrt{\mathrm{Re}\ 2\beta}$ and $\alpha = \mathrm{Im}\ \beta / 4\mathrm{Re}\ \beta$. If $\eta_j = \varphi_j^{-1} \kappa_j \varphi_j$, $j = 1, 2$, one has

$$\eta_1(w) = (w_1 + 1, \gamma_2 w_2, \ldots, \gamma_n w_n)$$

and

$$\eta_2(w) = (w_1 - 2w_2 + i, w_2 - i, \gamma_3 w_3, \ldots, \gamma_n w_n),$$

where $\gamma_j = e^{i\theta_j}$, $j = 2, 3, \ldots, n$.

Now by Lemma 3.1, we only need to prove that η_1 and η_2 are root-approximable. To this end, let, for $m \geq 1$,

$$\rho_m(w) = (w_1 + \frac{1}{2^m}, \gamma_2^{(m)} w_2, \ldots, \gamma_n^{(m)} w_n),$$

and

$$\varrho_m(w) = (w_1 - \frac{1}{2^{m-1}} w_2 + \frac{i}{4^m}, w_2 - \frac{i}{2^m}, \gamma_3^{(m)} w_3, \ldots, \gamma_n^{(m)} w_n),$$

where $\gamma_j^{(m)} = e^{\frac{i}{2^m}\theta_j}$ for $j = 2, \ldots, n$. Then, by induction, it is easy to see that $\rho_m^{2^m} = \eta_1$ and $\varrho_m^{2^m} = \eta_2$ for all m and $\rho_m \longrightarrow \mathrm{id}_{\mathbb{B}}$ and $\varrho_m \longrightarrow \mathrm{id}_{\mathbb{B}}$. □

Finally we can state our main result.

Theorem 3.10. *For $n \geq 1$, the automorphism group of the unit ball in $\mathbb{C}^n$ is root-approximable.*

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