A NEW FORMULA OF PRODUCTS OF THE APOSTOL-BERNOULLI AND APOSTOL-EULER POLYNOMIALS

YUAN HE

ABSTRACT. In this paper, a further investigation for the Apostol-Bernoulli and Apostol-Euler polynomials is performed, and a new formula of products of the Apostol-Bernoulli and Apostol-Euler polynomials is established by applying the generating function methods and some summation transform techniques. It turns out that some known results are obtained as special cases.

1. INTRODUCTION

The classical Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are usually defined by means of the following generating functions:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad(|t|<2\pi),\qquad\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\quad(|t|<\pi).\tag{1.1}$$

In particular, the rational numbers $B_n = B_n(0)$ and integers $E_n = 2^nE_n(1/2)$ are called the classical Bernoulli numbers and Euler numbers, respectively. These numbers and polynomials appear in many different areas of mathematics including combinatorics, number theory, special functions, and analysis. Numerous interesting properties for them can be found in many books; see, for example, [12, 41, 42].

Some analogues of the classical Bernoulli and Euler polynomials are the Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ and Apostol-Euler polynomials $\mathcal{E}_n(x;\lambda)$ given by means of the following generating functions (see, e.g., [25, 26, 27]):

$$\frac{te^{xt}}{\lambda e^t-1}=\sum_{n=0}^{\infty}\mathcal{B}_n(x;\lambda)\frac{t^n}{n!}\quad(|t|<2\pi\text{ if }\lambda=1;\ |t|<|\log\lambda|\text{ otherwise}),\tag{1.2}$$

$$\frac{2e^{xt}}{\lambda e^t+1}=\sum_{n=0}^{\infty}\mathcal{E}_n(x;\lambda)\frac{t^n}{n!}\quad(|t|<\pi\text{ if }\lambda=1;\ |t|<|\log(-\lambda)|\text{ otherwise}).\tag{1.3}$$

Moreover, $\mathcal{B}_n(\lambda) = \mathcal{B}_n(0;\lambda)$ and $\mathcal{E}_n(\lambda) = 2^n\mathcal{E}_n(1/2;\lambda)$ are called the Apostol-Bernoulli numbers and Apostol-Euler numbers, respectively. Obviously $\mathcal{B}_n(x;\lambda)$ and $\mathcal{E}_n(x;\lambda)$ reduce to $B_n(x)$ and $E_n(x)$ when $\lambda = 1$. It is worth of mentioning that the Apostol-Bernoulli polynomials were firstly introduced by Apostol [3] (see also Srivastava [46] for a systematic study) in order to evaluate the value of the Hurwitz-Lerch zeta function. Since the Apostol-Bernoulli and Apostol-Euler polynomials and numbers appeared, some arithmetic properties for them have been well investigated by many authors. For example, in 1998, Srivastava and Todorov [45] gave the close formula for the Apostol-Bernoulli polynomials in terms of the

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Gaussian hypergeometric function and the Stirling numbers of the second kind. Following the work of Srivastava and Todorov, Luo [27] obtained the close formula for the Apostol-Euler polynomials in a similar technique. After that, Srivastava [46] and Luo [31] found some elegant formulae between the Apostol-Bernoulli and Apostol-Euler polynomials and the Hurwitz-Lerch zeta function, respectively, and the later also obtained some series representations of the Apostol-Euler polynomials of higher order in terms of the generalized Hurwitz-Lerch zeta function. Further, Luo [29, 38] established some multiplication formulas for the Apostol-Bernoulli and Apostol-Euler polynomials. And Luo [30] also showed the Fourier expansions for the Apostol-Bernoulli and Apostol-Euler polynomials by applying the Lipschitz summation formula and derived some explicit formulae at rational arguments for these polynomials in terms of the Hurwitz zeta function; see also Bayad [5], Navas, Francisco and Varona [40] for further consideration on the work of Luo [30]. In [43], Özarslan presented and studied a unified family of polynomials which involves the Apostol-Bernoulli and Apostol-Euler polynomials. Tremblay, Gaboury and Fugere [52] introduced and investigated a new class of generalized Apostol-Bernoulli polynomials and obtained a generalization of the Srivastava-Pintér addition theorem. Choi, Jang and Srivastava [11] gave an explicit representation of the generalized Bernoulli polynomials in terms of a generalization for the Hurwitz-Lerch zeta function. Garg, Jain and Srivastava [13] investigated some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch zeta functions. Luo [33] introduced and investigated the λ-Stirling numbers of the second kind, in particular, he gave an explicit relationship between the generalized Apostol-Bernoulli and Apostol-Euler polynomials in terms of the λ-Stirling numbers of the second kind. Boyadzhiev [6] derived some relationships between the Apostol-Bernoulli polynomials, the classical Eulerian polynomials and the derivative polynomials for the cotangent functions. Kim and Hu [20] obtained the sums of products of any number of the Apostol-Bernoulli numbers which is a more generalization of the famous Euler's formula on the classical Bernoulli numbers. More recently, Srivastava and Choi [51] revised, enlarged and updated version of their earlier book entitled "Series Associated with the Zeta and Related Functions" (Kluwer Academic Publishers, Dordrecht, Boston and London, 2001). In this new book, the authors gave a systematic collection of various families of series associated with the Riemann and Hurwitz Zeta functions, as well as with many other higher transcendental functions, which are closely related to these functions. In particular, the historical account, vast new literatures and many fundamental properties for the Apostol-type polynomials and numbers have been well introduced. For another elegant results and nice methods related to the Apostol-Bernoulli and Apostol-Euler polynomials and numbers, one is referred to [4, 9, 19, 23, 24, 34, 35, 37, 39, 48, 49, 50, 54].

In his classical book [41], Nielsen presented three formulae of products of the classical Bernoulli and Euler polynomials $B_m(x)B_n(x)$, $E_m(x)E_n(x)$ and $B_m(x)E_n(x)$. In 1959, Carlitz [7] rediscovered the expression of $B_m(x)B_n(x)$, by virtue of which he established a reciprocity formula for Rademacher's Dedekind sums in [8]; see also [22] for another application to dealt with the discrete mean square of Dirichlet L-function at integral arguments. More recently, the author and Zhang [14] established three similar ones associated with the Nielsen's formulae on the classical Bernoulli and Euler polynomials to give the brief proofs of some extensions of the famous Miki's and Woodcock's identities on the Bernoulli numbers stated in [44].

We also mention [16, 17, 18, 53] for further discoveries of the Nielsen's formula on the classical Bernoulli polynomials following the work of Agoh and Dilcher [1, 2] on the classical Bernoulli numbers.

In the present paper, we will be concerned with some formulae of products of the Apostol-Bernoulli and Apostol-Euler polynomials. The idea stems from the expressions of $\mathcal{B}_m(x;\lambda)\mathcal{B}_n(y;\mu)$, $\mathcal{E}_m(x;\lambda)\mathcal{E}_n(y;\mu)$ and $\mathcal{E}_m(x;\lambda)\mathcal{B}_n(y;\mu)$ described in [15]. By making use of the generating function methods and some summation transform techniques, we establish a new formula of products of the Apostol-Bernoulli and Apostol-Euler polynomials, as follows.

Theorem 1.1. *Let m and n be any non-negative integers. Then*

$$\begin{aligned}\frac{1}{2}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\mathcal{E}_{m-k}(y-x;\frac{1}{\lambda})\frac{\mathcal{E}_{n+k+1}(y;\lambda\mu)}{n+k+1}\\
+\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}\mathcal{B}_{n+1-k}(y-x;\mu)\frac{\mathcal{E}_{m+k+1}(x;\lambda\mu)}{m+k+1}\\
=n!\sum_{k=0}^{m}\binom{m}{k}\mathcal{E}_{m-k}(x;\lambda)(-1)^k k!\frac{\mathcal{B}_{n+k+2}(y;\mu)}{(n+k+2)!}\\
+(-1)^{m+1}\frac{2}{1+\lambda\mu}\cdot\frac{m!\cdot n!\mathcal{B}_{m+n+2}(y-x;\mu)}{(m+n+2)!}.\end{aligned}\tag{1.4}$$

It is interesting to point out that the Theorem 1.1 is very analogous to the expressions of $\mathcal{B}_m(x;\lambda)\mathcal{B}_n(y;\mu)$ and $\mathcal{E}_m(x;\lambda)\mathcal{E}_n(y;\mu)$. Meanwhile, the Theorem 1.1 above can be also used to give the expression of $\mathcal{E}_m(x;\lambda)\mathcal{B}_n(y;\mu)$. For example, since the Apostol-Bernoulli and Apostol-Euler polynomials satisfy the differential relations $\frac{\partial}{\partial x}\mathcal{B}_n(x;\lambda)=n\mathcal{B}_{n-1}(x;\lambda)$ and $\frac{\partial}{\partial x}\mathcal{E}_n(x;\lambda)=n\mathcal{E}_{n-1}(x;\lambda)$ for positive integer n (see, e.g., [26]), by substituting $x+y$ for x in Theorem 1.1 and then taking differences with respect to y, we obtain

$$\begin{aligned}\frac{1}{2}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\mathcal{E}_{m-k}(-x;\frac{1}{\lambda})\mathcal{E}_{n+k}(y;\lambda\mu)\\
+\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}\mathcal{B}_{n+1-k}(-x;\mu)\mathcal{E}_{m+k}(x+y;\lambda\mu)\\
=\frac{1}{n+1}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+k+1}{n+1}}(-1)^k\Big(\mathcal{E}_{m-k}(x+y;\lambda)\mathcal{B}_{n+k+1}(y;\mu)\\
+(m-k)\mathcal{E}_{m-k-1}(x+y;\lambda)\frac{\mathcal{B}_{n+k+2}(y;\mu)}{n+k+2}\Big).\end{aligned}\tag{1.5}$$

It is easy to see that for any non-negative integers k,m,n,

$$\frac{m-k}{n+k+2}\cdot\frac{\binom{m}{k}}{\binom{n+k+1}{n+1}}=\frac{\binom{m}{k+1}}{\binom{n+k+2}{n+1}}.\tag{1.6}$$

Hence, the right hand side of (1.5) can be rewritten in the following way:

$$\frac{1}{n+1}\Bigg(\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+k+1}{n+1}}(-1)^k\mathcal{E}_{m-k}(x+y;\lambda)\mathcal{B}_{n+k+1}(y;\mu)-\sum_{k=1}^{m}\frac{\binom{m}{k}}{\binom{n+k+1}{n+1}}\times(-1)^k\mathcal{E}_{m-k}(x+y;\lambda)\mathcal{B}_{n+k+1}(y;\mu)\Bigg)=\frac{\mathcal{E}_m(x+y;\lambda)\mathcal{B}_{n+1}(y;\mu)}{n+1}. \quad (1.7)$$

Thus, combining (1.5) and (1.7) and then replacing x by $x-y$ gives that for non-negative integer m and positive integer n (see, e.g., [15, 16, 53]),

$$\mathcal{E}_m(x;\lambda)\mathcal{B}_n(y;\mu)=\frac{n}{2}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\mathcal{E}_{m-k}(y-x;\frac{1}{\lambda})\mathcal{E}_{n+k-1}(y;\lambda\mu)+\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}(y-x;\mu)\mathcal{E}_{m+k}(x;\lambda\mu). \quad (1.8)$$

It is worthy noticing that since $\mathcal{E}_0(x;\lambda)=2/(1+\lambda)$ (see, e.g., [26]) then the case $m=0$ in the formula (1.8) arises that for positive integer n,

$$2\mathcal{B}_n(y;\mu)=\lambda n\mathcal{E}_{n-1}(y;\lambda\mu)+(1+\lambda)\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k(y-x;\mu)\mathcal{E}_{n-k}(x;\lambda\mu). \quad (1.9)$$

So by setting $x=y$ and $\lambda=\mu=1$ in (1.9), in light of $B_1=-1/2$, we get

$$B_n(x)=\sum_{\substack{k=0\\(k\neq 1)}}^{n}\binom{n}{k}B_kE_{n-k}(x)\quad(n\geq 1), \quad (1.10)$$

which was derived by Cheon [10] who made use of some fairly standard techniques based upon series rearrangement. In other words, the formula (1.9) can be regarded as a generalization of the Cheon's formula. For the generalization of (1.10) in the other directions, one is referred to [21, 28, 47]. On the other hand, setting $m=0$ in Theorem 1.1, we obtain that for non-negative integer n,

$$2\frac{\mathcal{B}_{n+2}(y;\mu)}{n+2}-2\frac{1+\lambda}{1+\lambda\mu}\cdot\frac{\mathcal{B}_{n+2}(y-x;\mu)}{n+2}=\lambda\mathcal{E}_{n+1}(y;\lambda\mu)+(1+\lambda)\sum_{k=0}^{n+1}\binom{n+1}{k}\mathcal{B}_{n+1-k}(y-x;\mu)\frac{\mathcal{E}_{k+1}(x;\lambda\mu)}{k+1}. \quad (1.11)$$

If substituting $x+y$ for x in (1.11) and then taking differences with respect to y, one can reobtain the formula (1.9) by replacing x by $x-y$.

2. The proof of Theorem 1.1

In this section, we shall give the detailed proof of Theorem 1.1. Our proof depends on the following identity:

$$\frac{1}{\lambda e^u+1}\cdot\frac{1}{\mu e^v-1}=\left(\frac{\lambda e^u}{\lambda e^u+1}+\frac{1}{\mu e^v-1}\right)\frac{1}{\lambda\mu e^{u+v}+1}. \quad (2.1)$$

Multiplying the two sides of the above identity by $2e^{(x-y)u+y(u+v)}/(u+v)$, we obtain

$$\begin{aligned}&\frac{1}{u+v}\cdot\frac{2e^{xu}}{\lambda e^u+1}\cdot\frac{e^{yv}}{\mu e^v-1}\\&\qquad=\frac{1}{u+v}\cdot\frac{\lambda e^{(1+x-y)u}}{\lambda e^u+1}\cdot\frac{2e^{y(u+v)}}{\lambda\mu e^{u+v}+1}+\frac{1}{u+v}\cdot\frac{e^{(y-x)v}}{\mu e^v-1}\cdot\frac{2e^{x(u+v)}}{\lambda\mu e^{u+v}+1}.\end{aligned}\tag{2.2}$$

Since $\mathcal{B}_0(x;\lambda)=1$ when $\lambda=1$ and $\mathcal{B}_0(x;\lambda)=0$ when $\lambda\neq 1$ (see, e.g., [26]), so we denote by $\delta_{1,\lambda}$ the Kronecker symbol given by $\delta_{1,\lambda}=0$ or 1 according to $\lambda=1$ or $\lambda\neq 1$, we have $\mathcal{B}_0(x;\lambda)=\delta_{1,\lambda}$. It follows from (2.2) and $\mathcal{E}_0(x;\lambda)=2/(1+\lambda)$ that

$$\begin{aligned}&\frac{1}{u+v}\cdot\frac{2e^{xu}}{\lambda e^u+1}\cdot\frac{e^{yv}}{\mu e^v-1}\\&=\frac{1}{2}\cdot\frac{1}{u+v}\cdot\frac{2\lambda e^{(1+x-y)u}}{\lambda e^u+1}\left(\frac{2e^{y(u+v)}}{\lambda\mu e^{u+v}+1}-\frac{2}{1+\lambda\mu}\right)+\frac{2}{1+\lambda\mu}\cdot\frac{1}{u+v}\cdot\frac{\lambda e^{(1+x-y)u}}{\lambda e^u+1}\\&+\frac{1}{u+v}\left(\frac{e^{(y-x)v}}{\mu e^v-1}-\frac{\delta_{1,\mu}}{v}\right)\left(\frac{2e^{x(u+v)}}{\lambda\mu e^{u+v}+1}-\frac{2}{1+\lambda\mu}\right)+\frac{2}{1+\lambda\mu}\cdot\frac{1}{u+v}\cdot\frac{e^{(y-x)v}}{\mu e^v-1}\\&\qquad+\frac{1}{u+v}\cdot\frac{\delta_{1,\mu}}{v}\left(\frac{2e^{x(u+v)}}{\lambda\mu e^{u+v}+1}-\frac{2}{1+\lambda\mu}\right).\end{aligned}\tag{2.3}$$

Noticing that the generating function of the Apostol-Bernoulli and Apostol-Euler polynomials arises

$$\frac{e^{xu}}{\lambda e^u-1}-\frac{\delta_{1,\lambda}}{u}=\sum_{m=0}^{\infty}\frac{\mathcal{B}_{m+1}(x;\lambda)}{m+1}\cdot\frac{u^m}{m!},\tag{2.4}$$

and

$$\frac{1}{u}\left(\frac{2e^{xu}}{\lambda e^u+1}-\frac{2}{1+\lambda}\right)=\sum_{m=0}^{\infty}\frac{\mathcal{E}_{m+1}(x;\lambda)}{m+1}\cdot\frac{u^m}{m!},\tag{2.5}$$

respectively. More generally, the Taylor's theorem gives

$$\begin{aligned}\frac{1}{u+v}\left(\frac{2e^{x(u+v)}}{\lambda e^{u+v}+1}-\frac{2}{1+\lambda}\right)&=\sum_{n=0}^{\infty}\frac{\partial^n}{\partial u^n}\left\{\frac{1}{u}\left(\frac{2e^{xu}}{\lambda e^u+1}-\frac{2}{1+\lambda}\right)\right\}\frac{v^n}{n!}\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathcal{E}_{m+n+1}(x;\lambda)}{m+n+1}\cdot\frac{u^m}{m!}\cdot\frac{v^n}{n!}.\end{aligned}\tag{2.6}$$

Hence, applying (2.4) and (2.6) to (2.3) yields

$$\begin{aligned}&\frac{1}{u+v}\left\{\frac{2e^{xu}}{\lambda e^u+1}\cdot\frac{e^{yv}}{\mu e^v-1}-\frac{2}{1+\lambda\mu}\left(\frac{\lambda e^{(1+x-y)u}}{\lambda e^u+1}+\frac{e^{(y-x)v}}{\mu e^v-1}\right)\right\}\\&\quad=\frac{1}{2}\left(\sum_{m=0}^{\infty}\lambda\mathcal{E}_m(1+x-y;\lambda)\frac{u^m}{m!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathcal{E}_{m+n+1}(y;\lambda\mu)}{m+n+1}\cdot\frac{u^m}{m!}\cdot\frac{v^n}{n!}\right)\\&\qquad+\left(\sum_{n=0}^{\infty}\frac{\mathcal{B}_{n+1}(y-x;\mu)}{n+1}\cdot\frac{v^n}{n!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathcal{E}_{m+n+1}(x;\lambda\mu)}{m+n+1}\cdot\frac{u^m}{m!}\cdot\frac{v^n}{n!}\right)\\&\qquad\qquad+\frac{\delta_{1,\mu}}{v}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathcal{E}_{m+n+1}(x;\lambda\mu)}{m+n+1}\cdot\frac{u^m}{m!}\cdot\frac{v^n}{n!}\right).\end{aligned}\tag{2.7}$$

Since the Apostol-Euler polynomials satisfy the symmetric distribution $\lambda\mathcal{E}_n(1-x;\lambda)=(-1)^n\mathcal{E}_n(x;\frac{1}{\lambda})$ for non-negative integer n (see, e.g., [26]), so by applying the familiar Cauchy product to (2.7), we obtain

$$\begin{aligned}&\frac{1}{u+v}\left\{\frac{2e^{xu}}{\lambda e^u+1}\cdot\frac{e^{yv}}{\mu e^v-1}-\frac{2}{1+\lambda\mu}\left(\frac{\lambda e^{(1+x-y)u}}{\lambda e^u+1}+\frac{e^{(y-x)v}}{\mu e^v-1}\right)\right\}\\&=\frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\mathcal{E}_{m-k}(y-x;\frac{1}{\lambda})\frac{\mathcal{E}_{n+k+1}(y;\lambda\mu)}{n+k+1}\right]\frac{u^m}{m!}\cdot\frac{v^n}{n!}\\&\quad+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\frac{\mathcal{B}_{n+1-k}(y-x;\mu)}{n+1-k}\cdot\frac{\mathcal{E}_{m+k+1}(x;\lambda\mu)}{m+k+1}\right]\frac{u^m}{m!}\cdot\frac{v^n}{n!}\\&\quad+\frac{\delta_{1,\mu}}{v}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathcal{E}_{m+n+1}(x;\lambda\mu)}{m+n+1}\cdot\frac{u^m}{m!}\cdot\frac{v^n}{n!}.\end{aligned}\tag{2.8}$$

We are now in the position to deal with the last summation of the right hand side of (2.8). We have

$$\begin{aligned}&\frac{\delta_{1,\mu}}{v}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathcal{E}_{m+n+1}(x;\lambda\mu)}{m+n+1}\cdot\frac{u^m}{m!}\cdot\frac{v^n}{n!}\\&=\delta_{1,\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathcal{E}_{m+n+2}(x;\lambda\mu)}{m+n+2}\cdot\frac{u^m}{m!}\cdot\frac{v^n}{(n+1)!}+\frac{\delta_{1,\mu}}{v}\sum_{m=0}^{\infty}\frac{\mathcal{E}_{m+1}(x;\lambda\mu)}{m+1}\cdot\frac{u^m}{m!}\\&=\delta_{1,\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathcal{E}_{m+n+2}(x;\lambda\mu)}{(m+n+2)(n+1)}\cdot\frac{u^m}{m!}\cdot\frac{v^n}{n!}+\frac{\delta_{1,\mu}}{uv}\left(\frac{2e^{xu}}{\lambda\mu e^u+1}-\frac{2}{1+\lambda\mu}\right)\\&=\delta_{1,\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathcal{E}_{m+n+2}(x;\lambda\mu)}{(m+n+2)(n+1)}\cdot\frac{u^m}{m!}\cdot\frac{v^n}{n!}\\&\quad+\frac{\delta_{1,\mu}}{u+v}\left\{\frac{u+v}{uv}\cdot\frac{2e^{xu}}{\lambda\mu e^u+1}-\frac{2}{1+\lambda\mu}\left(\frac{1}{u}+\frac{1}{v}\right)\right\}.\end{aligned}\tag{2.9}$$

Observe that

$$\begin{aligned}&\frac{2e^{xu}}{\lambda e^u+1}\cdot\frac{e^{yv}}{\mu e^v-1}-\frac{2}{1+\lambda\mu}\left(\frac{\lambda e^{(1+x-y)u}}{\lambda e^u+1}+\frac{e^{(y-x)v}}{\mu e^v-1}\right)\\&\qquad-\delta_{1,\mu}\left\{\frac{u+v}{uv}\cdot\frac{2e^{xu}}{\lambda\mu e^u+1}-\frac{2}{1+\lambda\mu}\left(\frac{1}{u}+\frac{1}{v}\right)\right\}\\&=\frac{2e^{xu}}{\lambda e^u+1}\cdot\frac{e^{yv}}{\mu e^v-1}-\frac{2}{1+\lambda\mu}\cdot\frac{\lambda e^{(1+x-y)u}}{\lambda e^u+1}-\delta_{1,\mu}\frac{u+v}{uv}\cdot\frac{2e^{xu}}{\lambda\mu e^u+1}\\&\quad+\frac{2}{1+\lambda\mu}\cdot\frac{\frac{1}{\mu}e^{(1+x-y)u}}{\frac{1}{\mu}e^u-1}-\frac{2}{1+\lambda\mu}\\&\quad\times\left(\frac{\frac{1}{\mu}e^{(1+x-y)u}}{\frac{1}{\mu}e^u-1}-\frac{\delta_{1,\frac{1}{\mu}}}{u}+\frac{e^{(y-x)v}}{\mu e^v-1}-\frac{\delta_{1,\mu}}{v}\right),\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}\frac{2e^{(1+x-y)u}}{1+\lambda\mu}\left(\frac{\frac{1}{\mu}}{\frac{1}{\mu}e^u-1}-\frac{\lambda}{\lambda e^u+1}\right) &= \frac{1}{1+\lambda\mu}\cdot\frac{2e^{(1+x-y)u}}{\lambda e^u+1}\left(\frac{\frac{1}{\mu}(\lambda e^u+1)}{\frac{1}{\mu}e^u-1}-\lambda\right)\\ &= \frac{2e^{xu}}{\lambda e^u+1}\cdot\frac{\frac{1}{\mu}e^{(1-y)u}}{\frac{1}{\mu}e^u-1}. \qquad (2.11)\end{aligned}$$

Applying (2.11) to (2.10) gives

$$\begin{aligned}&\frac{2e^{xu}}{\lambda e^u+1}\cdot\frac{e^{yv}}{\mu e^v-1}-\frac{2}{1+\lambda\mu}\left(\frac{\lambda e^{(1+x-y)u}}{\lambda e^u+1}+\frac{e^{(y-x)v}}{\mu e^v-1}\right)\\ &\qquad -\delta_{1,\mu}\left\{\frac{u+v}{uv}\cdot\frac{2e^{xu}}{\lambda\mu e^u+1}-\frac{2}{1+\lambda\mu}\left(\frac{1}{u}+\frac{1}{v}\right)\right\}\\ &\qquad = \frac{2e^{xu}}{\lambda e^u+1}\left(\frac{\frac{1}{\mu}e^{(1-y)u}}{\frac{1}{\mu}e^u-1}-\frac{\delta_{1,\frac{1}{\mu}}}{u}+\frac{e^{yv}}{\mu e^v-1}-\frac{\delta_{1,\mu}}{v}\right)\\ &\qquad\quad -\frac{2}{1+\lambda\mu}\left(\frac{\frac{1}{\mu}e^{(1+x-y)u}}{\frac{1}{\mu}e^u-1}-\frac{\delta_{1,\frac{1}{\mu}}}{u}+\frac{e^{(y-x)v}}{\mu e^v-1}-\frac{\delta_{1,\mu}}{v}\right), \qquad (2.12)\end{aligned}$$

which together with (2.8) and (2.9) yields

$$\begin{aligned}&\frac{1}{u+v}\cdot\frac{2e^{xu}}{\lambda e^u+1}\left(\frac{\frac{1}{\mu}e^{(1-y)u}}{\frac{1}{\mu}e^u-1}-\frac{\delta_{1,\frac{1}{\mu}}}{u}+\frac{e^{yv}}{\mu e^v-1}-\frac{\delta_{1,\mu}}{v}\right)\\ &\qquad -\frac{1}{u+v}\cdot\frac{2}{1+\lambda\mu}\left(\frac{\frac{1}{\mu}e^{(1+x-y)u}}{\frac{1}{\mu}e^u-1}-\frac{\delta_{1,\frac{1}{\mu}}}{u}+\frac{e^{(y-x)v}}{\mu e^v-1}-\frac{\delta_{1,\mu}}{v}\right)\\ &\quad = \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\mathcal{E}_{m-k}(y-x;\frac{1}{\lambda})\frac{\mathcal{E}_{n+k+1}(y;\lambda\mu)}{n+k+1}\right]\frac{u^m}{m!}\cdot\frac{v^n}{n!}\\ &\qquad +\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\frac{\mathcal{B}_{n+1-k}(y-x;\mu)}{n+1-k}\cdot\frac{\mathcal{E}_{m+k+1}(x;\lambda\mu)}{m+k+1}\right]\frac{u^m}{m!}\cdot\frac{v^n}{n!}\\ &\qquad +\delta_{1,\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathcal{E}_{m+n+2}(x;\lambda\mu)}{(m+n+2)(n+1)}\cdot\frac{u^m}{m!}\cdot\frac{v^n}{n!}. \qquad (2.13)\end{aligned}$$

We next consider the left hand side of (2.13). Applying $u^m=\sum_{k=0}^{m}\binom{m}{k}(u+v)^k(-v)^{m-k}$ and the symmetric distribution of the Apostol-Bernoulli polynomials $\lambda\mathcal{B}_n(1-x;\lambda)=(-1)^n\mathcal{B}_n(x;\frac{1}{\lambda})$ for non-negative integer n (see, e.g., [26]) gives

$$\begin{aligned}\frac{\lambda e^{(1-x)u}}{\lambda e^u-1}-\frac{\delta_{1,\lambda}}{u} &= \sum_{m=0}^{\infty}(-1)^{m+1}\frac{\mathcal{B}_{m+1}(x;\frac{1}{\lambda})}{(m+1)!}\sum_{k=0}^{m}\binom{m}{k}(u+v)^k(-v)^{m-k}\\ &= \sum_{k=0}^{\infty}\sum_{m=k}^{\infty}(-1)^{m+1}\frac{\mathcal{B}_{m+1}(x;\frac{1}{\lambda})}{(m+1)!}\binom{m}{k}(u+v)^k(-v)^{m-k}\\ &= \sum_{k=0}^{\infty}\sum_{m=k+1}^{\infty}(-1)^{m+1}\frac{\mathcal{B}_{m+1}(x;\frac{1}{\lambda})}{(m+1)!}\binom{m}{k+1}(u+v)^{k+1}(-v)^{m-(k+1)}\\ &\qquad +\sum_{m=0}^{\infty}(-1)^{m+1}\frac{\mathcal{B}_{m+1}(x;\frac{1}{\lambda})}{m+1}\cdot\frac{(-v)^m}{m!}. \qquad (2.14)\end{aligned}$$

Combining (2.4) and (2.14) arises

$$\begin{aligned}
&\frac{1}{u+v}\left(\frac{\lambda e^{(1-x)u}}{\lambda e^{u}-1}-\frac{\delta_{1,\lambda}}{u}+\frac{e^{xv}}{\frac{1}{\lambda}e^{v}-1}-\frac{\delta_{1,\frac{1}{\lambda}}}{v}\right)\\
&\quad=\sum_{k=0}^{\infty}\sum_{m=k+1}^{\infty}(-1)^{k}\frac{\mathcal{B}_{m+1}(x;\frac{1}{\lambda})}{(m+1)!}\binom{m}{k+1}(u+v)^{k}v^{m-(k+1)}\\
&\quad=\sum_{k=0}^{\infty}\sum_{m=k+1}^{\infty}(-1)^{k}\frac{\mathcal{B}_{m+1}(x;\frac{1}{\lambda})}{(m+1)!}\binom{m}{k+1}\sum_{n=0}^{k}\binom{k}{n}u^{n}v^{m-(n+1)}\\
&\quad=\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\sum_{m=k+1}^{\infty}(-1)^{k}\frac{\mathcal{B}_{m+1}(x;\frac{1}{\lambda})}{(m+1)!}\binom{m}{k+1}\binom{k}{n}u^{n}v^{m-(n+1)}\\
&\quad=\sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}(-1)^{n}\frac{\mathcal{B}_{m+1}(x;\frac{1}{\lambda})}{(m+1)!}u^{n}v^{m-(n+1)}\\
&\quad=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n}\frac{m!\cdot n!\mathcal{B}_{m+n+2}(x;\frac{1}{\lambda})}{(m+n+2)!}\cdot\frac{u^{n}}{n!}\cdot\frac{v^{m}}{m!},
\end{aligned}\tag{2.15}$$

which means

$$\begin{aligned}
&\frac{1}{u+v}\left(\frac{\lambda e^{(1-x)u}}{\lambda e^{u}-1}-\frac{\delta_{1,\lambda}}{u}+\frac{e^{xv}}{\frac{1}{\lambda}e^{v}-1}-\frac{\delta_{1,\frac{1}{\lambda}}}{v}\right)\\
&\qquad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m}\frac{m!\cdot n!\mathcal{B}_{m+n+2}(x;\frac{1}{\lambda})}{(m+n+2)!}\cdot\frac{u^{m}}{m!}\cdot\frac{v^{n}}{n!}.
\end{aligned}\tag{2.16}$$

It follows from (1.3), (2.16) and the Cauchy product that

$$\begin{aligned}
&\frac{1}{u+v}\cdot\frac{2e^{xu}}{\lambda e^{u}+1}\left(\frac{\frac{1}{\mu}e^{(1-y)u}}{\frac{1}{\mu}e^{u}-1}-\frac{\delta_{1,\frac{1}{\mu}}}{u}+\frac{e^{yv}}{\mu e^{v}-1}-\frac{\delta_{1,\mu}}{v}\right)\\
&\quad-\frac{1}{u+v}\cdot\frac{2}{1+\lambda\mu}\left(\frac{\frac{1}{\mu}e^{(1+x-y)u}}{\frac{1}{\mu}e^{u}-1}-\frac{\delta_{1,\frac{1}{\mu}}}{u}+\frac{e^{(y-x)v}}{\mu e^{v}-1}-\frac{\delta_{1,\mu}}{v}\right)\\
&=\left(\sum_{m=0}^{\infty}\mathcal{E}_{m}(x;\lambda)\frac{u^{m}}{m!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m}\frac{m!\cdot n!\mathcal{B}_{m+n+2}(y;\mu)}{(m+n+2)!}\cdot\frac{u^{m}}{m!}\cdot\frac{v^{n}}{n!}\right)\\
&\quad-\frac{2}{1+\lambda\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m}\frac{m!\cdot n!\mathcal{B}_{m+n+2}(y-x;\mu)}{(m+n+2)!}\cdot\frac{u^{m}}{m!}\cdot\frac{v^{n}}{n!}\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{m}\binom{m}{k}\mathcal{E}_{m-k}(x;\lambda)(-1)^{k}k!\cdot n!\frac{\mathcal{B}_{n+k+2}(y;\mu)}{(n+k+2)!}\right]\frac{u^{m}}{m!}\cdot\frac{v^{n}}{n!}\\
&\quad-\frac{2}{1+\lambda\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m}\frac{m!\cdot n!\mathcal{B}_{m+n+2}(y-x;\mu)}{(m+n+2)!}\cdot\frac{u^{m}}{m!}\cdot\frac{v^{n}}{n!}.
\end{aligned}\tag{2.17}$$

Equating (2.13) and (2.17) and comparing the coefficients of $u^m v^n/m!\cdot n!$ yields

$$\begin{aligned}\frac{1}{2}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\mathcal{E}_{m-k}(y-x;\frac{1}{\lambda})\frac{\mathcal{E}_{n+k+1}(y;\lambda\mu)}{n+k+1}\\ +\sum_{k=0}^{n}\binom{n}{k}\frac{\mathcal{B}_{n+1-k}(y-x;\mu)}{n+1-k}\cdot\frac{\mathcal{E}_{m+k+1}(x;\lambda\mu)}{m+k+1}+\delta_{1,\mu}\frac{\mathcal{E}_{m+n+2}(x;\lambda\mu)}{(m+n+2)(n+1)}\\ =n!\sum_{k=0}^{m}\binom{m}{k}\mathcal{E}_{m-k}(x;\lambda)(-1)^k k!\frac{\mathcal{B}_{n+k+2}(y;\mu)}{(n+k+2)!}\\ +(-1)^{m+1}\frac{2}{1+\lambda\mu}\cdot\frac{m!\cdot n!\mathcal{B}_{m+n+2}(y-x;\mu)}{(m+n+2)!},\end{aligned}\tag{2.18}$$

which together with $\mathcal{B}_0(y-x;\mu)=\delta_{1,\mu}$ and $\frac{1}{n+1-k}\binom{n}{k}=\frac{1}{n+1}\binom{n+1}{k}$ for non-negative integers k and n gives the desired result. This concludes the proof of Theorem 1.1.

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Faculty of Science, Kunming University of Science and Technology, Kunming, Yunnan 650500, People's Republic of China

E-mail address: hyyhe@aliyun.com,hyyhe@yahoo.com.cn