A new kind of nonlocal-integral fractional boundary value problems

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Abstract

In this paper, we discuss a new class of nonlocal boundary value problems of fractional differential equations and inclusions with a new integral boundary condition. This new boundary condition states that the value of the unknown function at an arbitrary (local) point ξ is proportional to the contribution due to a sub-strip of arbitrary length $(1-\eta)$, that is, $x(\xi) = a\int_\eta^1 x(s)ds$, where $0 < \xi < \eta < 1$ and a is constant of proportionality. The existence of solutions for the given problems is shown by means of contraction mapping principle, a fixed point theorem due to O'Regan and nonlinear alternative for multivalued maps. The results are well illustrated with the aid of examples.

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1 Introduction

Fractional calculus has emerged as an effective and powerful tool in the modelling of many physical and engineering phenomena. One of the reasons for the popularity of this subject is that fractional derivatives naturally describe memory and hereditary properties of several materials and processes. Specific applications of fractional calculus can be found in a variety of disciplines such as physics, chemistry, statistics, economics, biology, biophysics, blood flow phenomena, control theory, signal and image processing, etc. [9, 22, 28, 30]. For some recent development in the theory, methods and applications of fractional calculus, we refer the reader to [2, 4, 5, 7, 8, 10, 18, 19, 23, 32] and the references therein.

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The concept of nonlocal Cauchy problems introduced by Byszewski [12] is found to be more practical than the classical Cauchy problems with the initial conditions [13, 14]. In the last few decades, several kinds of nonlocal problems have been studied. More recently, the topic of nonlocal integral boundary conditions has attracted a considerable attention. In most of the work dealing with nonlocal boundary value problems, the contribution expressed in terms of the integral is related to the value of the unknown function at a fixed point (left/right end-point of the interval under consideration), for instance, see [1, 3, 25, 34] and references therein.

In the present study, we introduce a more general variant of nonlocal integral boundary conditions, which relates the integral contribution due to a strip of arbitrary length with the value of the unknown function at an arbitrary (nonlocal) point of the interval instead of its value at a fixed point. Precisely, we formulate this variant of nonlocal integral boundary conditions as follows: $x(\xi) \propto \int_\eta^1 x(s)ds,\ 0 < \xi < \eta < 1$ or $x(\xi) = a\int_\eta^1 x(s)ds$ (a is constant of proportionality). Thus we consider the following nonlocal boundary value problem of fractional differential equations with new integral boundary conditions:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad t \in [0,1], \quad 1 < q \le 2, \\ x(0) = x_0 + g(x), \quad x(\xi) = a\displaystyle\int_\eta^1 x(s)ds, \quad 0 < \xi < \eta < 1,\ x_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : C([0,1], \mathbb{R}) \to \mathbb{R}$ are given continuous functions, and a is real constant.

As a second problem, we study the multivalued analogue of problem (1.1) given by

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \quad t \in [0,1], \quad 1 < q \le 2, \\ x(0) = x_0 + g(x), \quad x(\xi) = a\displaystyle\int_\eta^1 x(s)ds, \quad 0 < \xi < \eta < 1,\ x_0 \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $g(x)$ and a are the same as defined in the problem (1.1). Moreover, $g(x)$ considered in the problems (1.1) and (1.2) may be understood as $g(x) = \sum_{j=1}^p \alpha_j x(t_j)$ where $\alpha_j, j = 1, \ldots, p$, are given constants and $0 < t_1 < \ldots < t_p \le 1$. For more details we refer to the work by Byszewski [12, 13]. For some real world problems and engineering applications involving the strip conditions similar to the ones considered in the present study , we refer the reader to the works [6, 15, 29, 31, 33].

The paper is organized as follows. In Section 2, we recall some basic definitions from fractional calculus and establish a lemma which plays a pivotal role in the sequel. Section 3 deals with the existence results for the problem (1.1) which are shown by applying Banach's contraction principle and a fixed point theorem due to D. O'Regan. In Section 4, we discuss the existence of solutions for the problem (1.2) by means of the nonlinear alternative for contractive maps and the combination of the nonlinear alternative of Leray-Schauder type for single-valued maps and a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values. Some examples are constructed for the illustration of main results.

2 Preliminaries

In this section, some basic definitions on fractional calculus and an auxiliary lemma are presented [22, 28].

Definition 2.1 *The Riemann-Liouville fractional integral of order q for a continuous function g is defined as*

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

Definition 2.2 *For at least n-times continuously differentiable function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q.

Lemma 2.1 *For any $y \in C([0,1], \mathbb{R})$ the unique solution of the linear fractional boundary value problem*

$$\begin{cases} {}^cD^q x(t) = y(t), & 1 < q \leq 2, \\ x(0) = x_0 + g(x), \quad x(\xi) = a \displaystyle\int_\eta^1 x(s) ds, & t \in [0,1] \end{cases} \tag{2.1}$$

is

$$\begin{aligned} x(t) \;=\; & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s) ds + \frac{t}{A} \left\{ a \int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} y(s) ds \right. \\ & \left. - a \int_0^\eta \frac{(\eta - s)^q}{\Gamma(q+1)} y(s) ds - \int_0^\xi \frac{(\xi - s)^{q-1}}{\Gamma(q)} y(s) ds \right\} \\ & + \left[1 + \frac{t}{A} \Big(a(1-\eta) - 1 \Big) \right] (x_0 + g(x)), \end{aligned} \tag{2.2}$$

where

$$A = \xi - \frac{a}{2}(1 - \eta^2) \neq 0. \tag{2.3}$$

Proof. It is well known that the general solution of the fractional differential equation in (2.1) can be written as

$$x(t) = c_0 + c_1 t + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds, \tag{2.4}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants.
Applying the given boundary conditions, we find that $c_0 = x_0 + g(x)$, and

$$\begin{aligned} c_1 &= \frac{1}{A}\Big(a(1-\eta)-1\Big)(x_0+g(x)) + \frac{1}{A}\Bigg\{a\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} y(s)ds \\ &- a\int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} y(s)ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} y(s)ds\Bigg\}. \end{aligned} \tag{2.5}$$

Substituting the values of c_0, c_1 in (2.4), we get (2.2). This completes the proof. □

3 Existence results for single-valued problem (1.1)

We denote by $\mathcal{C} = C([0,1],\mathbb{R})$ the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Also by $L^1([0,1],\mathbb{R})$ we denote the Banach space of measurable functions $x : [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|dt$.

In view of Lemma 2.1, we define an operator $\mathcal{P} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned} (\mathcal{P}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \frac{t}{A}\Bigg\{a\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f(s,x(s))ds \\ &-a\int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} f(s,x(s))ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\Bigg\} \\ &+\Big[1+\frac{t}{A}\Big(a(1-\eta)-1\Big)\Big](x_0+g(x)), \quad t\in[0,1]. \end{aligned} \tag{3.1}$$

Let us define $\mathcal{P}_{1,2} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned} (\mathcal{P}_1x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \frac{t}{A}\Bigg\{a\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f(s,x(s))ds \\ &-a\int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} f(s,x(s))ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\Bigg\}, \end{aligned} \tag{3.2}$$

and

$$(\mathcal{P}_2x)(t) = \Big[1+\frac{t}{A}\Big(a(1-\eta)-1\Big)\Big](x_0+g(x)). \tag{3.3}$$

Clearly

$$(\mathcal{P}x)(t) = (\mathcal{P}_1x)(t) + (\mathcal{P}_2x)(t), \quad t \in [0,1]. \tag{3.4}$$

For convenience we introduce the notations:

$$p_0 := \frac{1}{\Gamma(q+1)}\left[1 + \frac{1}{|A|}\left\{\frac{|a|(1+\eta^{q+1})}{q+1} + \xi^q\right\}\right], \tag{3.5}$$

and

$$k_0 := \left|1 + \frac{1}{A}\Big(a(1-\eta) - 1\Big)\right|. \tag{3.6}$$

Theorem 3.1 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(A_1) $|f(t,x) - f(t,y)| \leq L\|x-y\|, \forall t \in [0,1],\ L > 0,\ x, y \in \mathbb{R}$;

(A_2) $g : C([0,1],\mathbb{R}) \to \mathbb{R}$ *is a continuous function satisfying the condition:*

$$|g(u) - g(v)| \leq \ell\|u - v\|,\ \ell < k_0^{-1}, \quad \forall\ u, v \in C([0,1],\mathbb{R}),\ \ell > 0;$$

(A_3) $\gamma := Lp_0 + k_0\ell < 1$.

Then the boundary value problem (1.1) has a unique solution.

Proof. For $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, from the definition of $\mathcal{P}$ and assumptions (A_1) and (A_2), we obtain

$$\begin{aligned}
|(\mathcal{P}x)(t) - (\mathcal{P}y)(t)| \ \leq\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s)) - f(s,y(s))|ds \\
& + \frac{1}{|A|}\left\{|a|\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)}|f(s,x(s)) - f(s,y(s))|ds\right. \\
& + |a|\int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)}|f(s,x(s)) - f(s,y(s))|ds \\
& \left. + \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)}|f(s,x(s)) - f(s,y(s))|ds\right\} \\
& + \left|1 + \frac{1}{A}\Big(a(1-\eta) - 1\Big)\right||g(x) - g(y)| \\
\leq\ & L\|x-y\|\left[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds + \frac{1}{|A|}\left\{|a|\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)}ds\right.\right. \\
& \left.\left. + |a|\int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)}ds + \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)}ds\right\}\right] \\
& + \left|1 + \frac{1}{A}\Big(a(1-\eta) - 1\Big)\right|\ell\|x-y\|
\end{aligned}$$

$$\begin{aligned}
&\leq L\|x-y\|\left[\frac{1}{\Gamma(q+1)}+\frac{1}{|A|}\left\{\frac{|a|(1+\eta^{q+1})}{(q+1)\Gamma(q+1)}+\frac{\xi^q}{\Gamma(q+1)}\right\}\right] \\
&\quad+\left|1+\frac{1}{A}\big(a(1-\eta)-1\big)\right|\ell\|x-y\| \\
&= \frac{L}{\Gamma(q+1)}\left(\left[1+\frac{1}{|A|}\left\{\frac{|a|(1+\eta^{q+1})}{q+1}+\xi^q\right\}\right]\right. \\
&\quad\left.+\left|1+\frac{1}{A}\Big(a(1-\eta)-1\Big)\right|\ell\right)\|x-y\| \\
&= (Lp_0+k_0\ell)\|x-y\|.
\end{aligned}$$

Hence

$$\|(\mathcal{P}x)-(\mathcal{P}y)\|\leq\gamma\|x-y\|.$$

As $\gamma<1$ by (A_3), the operator $\mathcal{P}$ is a contraction map from the Banach space $\mathcal{C}$ into itself. Hence the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Example 3.1 *Consider the following fractional boundary value problem*

$$\begin{cases} {}^cD^{3/2}x(t)=\dfrac{1}{t+16}(\sin x+x)+1, \quad t\in[0,1], \\ x(0)=\dfrac{1}{2}+\dfrac{1}{12}\tan^{-1}(x(1/8)), \quad x\left(\dfrac{1}{4}\right)=\dfrac{1}{4}\displaystyle\int_{1/3}^{1}x(s)ds. \end{cases} \tag{3.7}$$

Here, $q=3/2, x_0=1/2,\ a=1/4,\ \xi=1/4,\ \eta=1/3,\ f(t,x)=\dfrac{1}{t+16}(\sin x+x)+1$, and $g(x)=\frac{1}{12}\tan^{-1}(x(1/8))$. Since $|f(t,x)-f(t,y)|\leq\dfrac{1}{8}\|x-y\|, |g(x)-g(y)|\leq\dfrac{1}{12}\|x-y\|$, therefore, (A_1) and (A_2) are respectively satisfied with $L=1/8$ and $\ell=1/12$. Using the given values, it is found that $A=5/36,\ p_0\approx 2.00565, k_0=5$. Clearly $\gamma=Lp_0+k_0\ell\approx 0.667373<1$. Thus, the conclusion of Theorem 3.1 applies and the boundary value problem (3.7) has a solution on $[0,1]$.

Our next existence result relies on a fixed point theorem due to O'Regan in [26].

Lemma 3.1 *Denote by U an open set in a closed, convex set C of a Banach space E. Assume $0\in U$. Also assume that $F(\bar U)$ is bounded and that $F:\bar U\to C$ is given by $F=F_1+F_2$, in which $F_1:\bar U\to E$ is continuous and completely continuous and $F_2:\bar U\to E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi:[0,\infty)\to[0,\infty)$ satisfying $\phi(z)<z$ for $z>0$, such that $\|F_2(x)-F_2(y)\|\leq\phi(\|x-y\|)$ for all $x,y\in\bar U$). Then, either*

(C1) F has a fixed point $u\in\bar U$; or

(C2) there exist a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$, where $\bar{U}$ and ∂U, respectively, represent the closure and boundary of U.

Let

$$\Omega_r = \{x \in C([0,1],\mathbb{R}) : \|x\| < r\},$$

and denote the maximum number by

$$M_r = \max\{|f(t,x)| : (t,x) \in [0,1] \times [-r,r]\}.$$

Theorem 3.2 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that $(A_1), (A_2)$ hold. In addition we assume that*

(A_4) $g(0) = 0$;

(A_5) *there exists a nonnegative function $p \in C([0,1],\mathbb{R})$ and a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that*

$$|f(t,u)| \le p(t)\psi(|u|) \quad \textit{for any} \quad (t,u) \in [0,1] \times \mathbb{R};$$

(A_6) $\sup_{r \in (0,\infty)} \dfrac{r}{k_0|x_0| + p_0\psi(r)\|p\|} > \dfrac{1}{1-k_0\ell}$, *where p_0 and k_0 are defined in (3.5) and (3.6) respectively.*

Then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. By the assumption (A_6), there exists a number $r_0 > 0$ such that

$$\frac{r_0}{k_0|x_0| + p_0\psi(r_0)\|p\|} > \frac{1}{1-k_0\ell}. \tag{3.8}$$

We shall show that the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ defined by (3.2) and (3.3) respectively, satisfy all the conditions of Lemma 3.1.

Step 1. *The operator $\mathcal{P}_1$ is continuous and completely continuous.* We first show that $\mathcal{P}_1(\bar{\Omega}_{r_0})$ is bounded. For any $x \in \bar{\Omega}_{r_0}$, we have

$$\begin{aligned}
\|\mathcal{P}_1 x\| &\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds + \frac{1}{|A|}\Bigg\{|a|\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)}|f(s,x(s))|ds \\
&\quad + |a|\int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)}|f(s,x(s))|ds + \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\Bigg\} \\
&\le \frac{M_r}{\Gamma(q+1)}\left[1 + \frac{1}{|A|}\left\{\frac{|a|(1+\eta^{q+1})}{q+1} + \xi^q\right\}\right]\|p\| = M_r p_0 \|p\|.
\end{aligned}$$

Thus the operator $\mathcal{P}_1(\bar{\Omega}_{r_0})$ is uniformly bounded. For any $t_1, t_2 \in [0,1], t_1 < t_2$, we have

$$
\begin{aligned}
& |(\mathcal{P}_1 x)(t_2) - (\mathcal{P}_1 x)(t_1)| \\
\leq\ & \frac{1}{\Gamma(q)} \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] |f(s, x(s))| ds \\
& + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} |f(s, x(s))| ds \\
& + \frac{|t_2 - t_1|}{|A|} \Bigg\{ |a| \int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} |f(s, x(s))| ds \\
& + |a| \int_0^\eta \frac{(\eta - s)^q}{\Gamma(q+1)} |f(s, x(s))| ds + \int_0^\xi \frac{(\xi - s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \Bigg\} \\
\leq\ & \frac{M_r}{\Gamma(q)} \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] ds + \frac{M_r}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \\
& + \frac{M_r |t_2 - t_1|}{|A|} \Bigg\{ |a| \int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} ds + |a| \int_0^\eta \frac{(\eta - s)^q}{\Gamma(q+1)} ds + \int_0^\xi \frac{(\xi - s)^{q-1}}{\Gamma(q)} ds \Bigg\} \\
\leq\ & \frac{M_r}{\Gamma(q)} \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] ds + \frac{M_r}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \\
& + \frac{M_r |t_2 - t_1|}{|A| \Gamma(q+1)} \left\{ \frac{|a|(1 + \eta^{q+1})}{q+1} + \xi^q \right\},
\end{aligned}
$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{P}_1$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}_1(\bar{\Omega}_{r_0})$ is a relatively compact set. Now, let $x_n \subset \bar{\Omega}_{r_0}$ with $\|x_n - x\| \to 0$. Then the limit $\|x_n(t) - x(t)\| \to 0$ is uniformly valid on $[0,1]$. From the uniform continuity of $f(t,x)$ on the compact set $[0,1] \times [-r_0, r_0]$, it follows that $\|f(t, x_n(t)) - f(t, x(t))\| \to 0$ is uniformly valid on $[0,1]$. Hence $\|\mathcal{P}_1 x_n - \mathcal{P}_1 x\| \to 0$ as $n \to \infty$ which proves the continuity of $\mathcal{P}_1$. This completes the proof of Step 1.

Step 2. *The operator $\mathcal{P}_2 : \bar{\Omega}_{r_0} \to C([0,1], \mathbb{R})$ is contractive.* This is a consequence of (A_2).

Step 3. *The set $\mathcal{P}(\bar{\Omega}_{r_0})$ is bounded.* The assumptions (A_2) and (A_4) imply that

$$\|\mathcal{P}_2(x)\| \leq k_0(|x_0| + \ell r_0),$$

for any $x \in \bar{\Omega}_{r_0}$. This, with the boundedness of the set $\mathcal{P}_1(\bar{\Omega}_{r_0})$ implies that the set $\mathcal{P}(\bar{\Omega}_{r_0})$ is bounded.

Step 4. *Finally, it will be shown that the case (C2) in Lemma 3.1 does not hold.* On the contrary, we suppose that (C2) holds. Then, we have that there exist $\lambda \in (0,1)$

and $x \in \partial\Omega_{r_0}$ such that $x = \lambda \mathcal{P}x$. So, we have $\|x\| = r_0$ and

$$\begin{aligned} x(t) &= \lambda\Big[1 + \frac{t}{A}\Big(a(1-\eta) - 1\Big)\Big](x_0 + g(x)) + \lambda \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds \\ &\quad + \lambda \frac{t}{A}\Bigg\{ a\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f(s, x(s))ds - a\int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} f(s, x(s))ds \\ &\quad - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds \Bigg\}, \quad t \in [0,1]. \end{aligned}$$

Using the assumptions $(A_4) - (A_6)$, we get

$$\begin{aligned} r_0 &\leq \psi(r_0)\Bigg[\frac{1}{\Gamma(q)}\int_0^1 (t-s)^{q-1} p(s)ds + \frac{1}{|A|}\Bigg\{\frac{|a|}{\Gamma(q+1)}\int_0^1 (1-s)^q p(s)ds \\ &\quad + \frac{|a|}{\Gamma(q+1)}\int_0^\eta (\eta-s)^q p(s)ds + \frac{1}{\Gamma(q)}\int_0^\xi (\xi-s)^q p(s)ds\Bigg\}\Bigg] \\ &\quad + \Big|1 + \frac{1}{A}\Big(a(1-\eta)-1\Big)\Big|(|x_0| + \ell r_0), \end{aligned}$$

which yields

$$r_0 \leq k_0|x_0| + p_0\psi(r_0)\|p\| + k_0\ell r_0.$$

Thus, we get a contradiction:

$$\frac{r_0}{k_0|x_0| + p_0\psi(r_0)\|p\|} \leq \frac{1}{1 - k_0\ell}.$$

Thus the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy all the conditions of Lemma 3.1. Hence, the operator $\mathcal{P}$ has at least one fixed point $x \in \bar{\Omega}_{r_0}$, which is the solution of the problem (1.1). This completes the proof. □

Example 3.2 *Consider the following boundary value problem*

$$\begin{cases} {}^cD^{3/2}x(t) = \dfrac{1}{3}\Big(\sqrt{t} + \dfrac{1}{18}\Big)\sin x, \quad 0 < t < 1, \\ x(0) = \dfrac{1}{18}x\left(\dfrac{1}{9}\right), \quad x\left(\dfrac{1}{4}\right) = \dfrac{1}{4}\displaystyle\int_{1/3}^1 x(s)ds. \end{cases} \tag{3.9}$$

Observe that $q = 3/2, f(t,x) = \frac{1}{3}\Big(\sqrt{t} + \frac{1}{18}\Big)\sin x,\ x_0 = 0, g(x) = \frac{x(1/9)}{12},\ \ell = 1/18, a = 1/4, \xi = 1/4, \eta = 1/3$. Clearly $(A_1), (A_2)$ and (A_4) are satisfied. Further $|f(t,x)| \leq \frac{1}{3}\Big(\sqrt{t} + \frac{1}{18}\Big)x$, for any $(t,x) \in [0,1] \times \mathbb{R}$, and hence $p(t) = \sqrt{t} + \frac{1}{18}$

and $\psi(x) = \dfrac{1}{3}x$. Thus the condition (A_5) is satisfied. With the given data, $p_0 \approx 2.00565, k_0 = 5$,

$$\sup_{r\in(0,\infty)} \frac{r}{k_0|x_0| + p_0\psi(r)\|p\|} \approx 1.417049, \quad \frac{1}{1-k_0\ell} \approx 1.38461,$$

it is found that (A_6) holds. Therefore, all the conditions of Theorem 3.2 are satisfied and hence by its conclusion, the problem (3.9) has at least one solution on $[0,1]$.

4 Existence results for multi-valued problem (1.2)

Let us recall some basic definitions on multi-valued maps [16, 20].

For a normed space $(X, \|\cdot\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0;1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 4.1 *A function $x \in AC^1([0,1],\mathbb{R})$ is a solution of the problem (1.2) if $x(0) = x_0 + g(x)$, $x(\xi) = a\int_\eta^1 x(s)\,ds$, and there exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t) \in F(t, x(t))$ a.e. on $[0,1]$ and*

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \frac{t}{A}\Bigg\{ a\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f(s)ds \\ &\quad - a\int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} f(s)ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f(s)ds \Bigg\} \\ &\quad + \Big[1 + \frac{t}{A}\big(a(1-\eta)-1\big)\Big](x_0 + g(x)). \end{aligned} \tag{4.1}$$

Here $AC^1([0,1],\mathbb{R})$ *will denote the space of functions* $x:[0,1]\to\mathbb{R}$ *that are absolutely continuous and whose first derivative is absolutely continuous.*

4.1 The Carathéodory case

Definition 4.2 *A multivalued map* $F:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ *is said to be Carathéodory if*

(i) $t\longmapsto F(t,x)$ *is measurable for each* $x\in\mathbb{R}$*;*

(ii) $x\longmapsto F(t,x)$ *is upper semicontinuous for almost all* $t\in[0,1]$*;*

Further a Carathéodory function F *is called* L^1-*Carathéodory if*

(iii) for each $\alpha>0$*, there exists* $\varphi_\alpha\in L^1([0,1],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\| = \sup\{|v| : v\in F(t,x)\} \le \varphi_\alpha(t)$$

for all $\|x\|\le\alpha$ *and for a. e.* $t\in[0,1]$.

For each $y\in C([0,1],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v\in L^1([0,1],\mathbb{R}) : v(t)\in F(t,y(t)) \text{ for a.e. } t\in[0,1]\}.$$

The following lemma will be used in the sequel.

Lemma 4.1 *([24]) Let* X *be a Banach space. Let* $F:[0,1]\times\mathbb{R}\to\mathcal{P}_{cp,c}(X)$ *be an* L^1- *Carathéodory multivalued map and let* Θ *be a linear continuous mapping from* $L^1([0,1],X)$ *to* $C([0,1],X)$*. Then the operator*

$$\Theta\circ S_F : C([0,1],X)\to\mathcal{P}_{cp,c}(C([0,1],X)),\quad x\mapsto(\Theta\circ S_F)(x)=\Theta(S_{F,x})$$

is a closed graph operator in $C([0,1],X)\times C([0,1],X)$.

To prove our main result in this section, we use the following form of the Nonlinear Alternative for contractive maps [27, Corollary 3.8].

Theorem 4.1 *Let* X *be a Banach space, and* D *a bounded neighborhood of* $0\in X$*. Let* $Z_1 : X\to\mathcal{P}_{cp,c}(X)$ *and* $Z_2:\bar{D}\to\mathcal{P}_{cp,c}(X)$ *two multi-valued operators satisfying*

(a) Z_1 *is contraction, and*

(b) Z_2 *is u.s.c and compact.*

Then, if $G = Z_1+Z_2$*, either*

(i) G has a fixed point in $\bar{D}$ or

(ii) there is a point $u \in \partial D$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$.

Theorem 4.2 *Assume that (A_2) holds. In addition we suppose that:*

(H_1) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory multivalued map;*

(H_2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,1],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \ \text{ for each } (t,x) \in [0,1] \times \mathbb{R};$$

(H_3) *there exists a number $M > 0$ such that*

$$\frac{(1-k_0\ell)M}{\psi(M)p_0\|p\| + k_0|x_0|} > 1, \tag{4.2}$$

where p_0, k_0 are defined in (3.5) and (3.6) respectively.

Then the boundary value problem (1.2) has at least one solution on $[0,1]$.

Proof. To transform the problem (1.2) to a fixed point, we introduce an operator $\mathcal{N} : C([0,1],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,1],\mathbb{R}))$ defined by

$$\mathcal{N}(x) = \left\{ \begin{array}{l} h \in C([0,1],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \frac{t}{A}\Bigg\{ a\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f(s)ds \\ \displaystyle -a\int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} f(s)ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f(s)ds \Bigg\} \\ \displaystyle +\Big[1 + \frac{t}{A}\Big(a(1-\eta) - 1\Big)\Big](x_0 + g(x)), \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$.

Now, we define two operators $\mathcal{A} : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ by

$$\mathcal{A}x(t) = \Big[1 + \frac{t}{A}\Big(a(1-\eta) - 1\Big)\Big](x_0 + g(x)), \tag{4.3}$$

and a multi-valued operator $\mathcal{B} : C([0,1],\mathbb{R}) \longrightarrow \mathcal{P}(C([0,1],\mathbb{R}))$ by

$$\mathcal{B}(x) = \left\{ \begin{array}{l} h \in C([0,1],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \frac{t}{A}\Bigg\{ a\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f(s)ds \\ \displaystyle -a\int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} f(s)ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f(s)ds \Bigg\}. \end{array} \right. \end{array} \right\} \tag{4.4}$$

Observe that $\mathcal{N} = \mathcal{A} + \mathcal{B}$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.1 on $[0, 1]$. The proof consists of several steps and claims.

Step 1: We show that $\mathcal{A}$ *is a contraction on* $C([0, 1], \mathbb{R})$. For $x, y \in C([0, 1], \mathbb{R})$, we have

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &= \Big|1 + \frac{t}{A}\big(a(1-\eta) - 1\big)\Big| |g(x) - g(y)| \\ &\leq \Big|1 + \frac{1}{A}\big(a(1-\eta) - 1\big)\Big| |g(x) - g(y)|, \\ &\leq k_0 \ell \|x - y\|, \end{aligned}$$

which, on taking supremum over $t \in [0, 1]$, yields

$$\|\mathcal{A}x - \mathcal{A}y\| \leq L_0 \|x - y\|, \quad L_0 = k_0 \ell < 1.$$

This shows that $\mathcal{A}$ is a contraction as $L_0 < 1$.

Step 2: $\mathcal{B}$ *is compact and convex valued and it is completely continuous.* This will be established in several claims.

Claim I: $\mathcal{B}$ *maps bounded sets into bounded sets in* $C([0, 1], \mathbb{R})$. Let $B_r = \{x \in C([0, 1], \mathbb{R}) : \|x\| \leq r\}$ be a bounded set in $C([0, 1], \mathbb{R})$. Then, for each $h \in \mathcal{B}(x), x \in B_\rho$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \frac{t}{A}\Bigg\{ a \int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f(s)ds \\ &\quad - a \int_0^\eta \frac{(\eta - s)^q}{\Gamma(q+1)} f(s)ds - \int_0^\xi \frac{(\xi - s)^{q-1}}{\Gamma(q)} f(s)ds \Bigg\}. \end{aligned}$$

Then, for $t \in [0, 1]$, we have

$$\begin{aligned} |h(t)| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s)|ds + \frac{1}{|A|}\Bigg\{ |a| \int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} |f(s)|ds \\ &\quad + |a| \int_0^\eta \frac{(\eta - s)^q}{\Gamma(q+1)} |f(s)|ds + \int_0^\xi \frac{(\xi - s)^{q-1}}{\Gamma(q)} |f(s)|ds \Bigg\} \\ &\leq \frac{\psi(\|x\|)}{\Gamma(q+1)} \left[1 + \frac{1}{|A|} \left\{ \frac{|a|(1 + \eta^{q+1})}{q+1} + \xi^q \right\} \right] \|p\|. \end{aligned}$$

Thus,

$$\|h\| \leq \frac{\psi(\|x\|)}{\Gamma(q+1)} \left[1 + \frac{1}{|A|} \left\{ \frac{|a|(1 + \eta^{q+1})}{q+1} + \xi^q \right\} \right] \|p\|.$$

Claim II: $\mathcal{B}$ *maps bounded sets into equi-continuous sets.* Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_\rho$. Then, for each $h \in \mathcal{B}(x)$, we obtain

$$\begin{aligned}
&|h(t_2) - h(t_1)| \\
\leq\ & \frac{\psi(\|x\|)\|p\|}{\Gamma(q)} \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}]ds + \frac{\psi(\|x\|)\|p\|}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \\
& + \frac{\psi(\|x\|)\|p\||t_2 - t_1|}{|A|} \Bigg\{ |a| \int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} ds \\
& + |a| \int_0^\eta \frac{(\eta - s)^q}{\Gamma(q+1)} ds + \int_0^\xi \frac{(\xi - s)^{q-1}}{\Gamma(q)} ds \Bigg\}.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $t_2 - t_1 \to 0$. Therefore it follows by the Ascoli-Arzelá theorem that $\mathcal{B} : C([0,1], \mathbb{R}) \to \mathcal{P}(C([0,1], \mathbb{R}))$ is completely continuous.
Claim III: $\mathcal{B}$ *has a closed graph.* Let $x_n \to x_*, h_n \in \mathcal{B}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{B}(x_*)$. Associated with $h_n \in \mathcal{B}(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$\begin{aligned}
h_n(t) \ =\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_n(s) ds + \frac{t}{A} \Bigg\{ a \int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f_n(s) ds \\
& - a \int_0^\eta \frac{(\eta - s)^q}{\Gamma(q+1)} f_n(s) ds - \int_0^\xi \frac{(\xi - s)^{q-1}}{\Gamma(q)} f_n(s) ds \Bigg\}.
\end{aligned}$$

Thus it suffices to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$\begin{aligned}
h_*(t) \ =\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s) ds + \frac{t}{A} \Bigg\{ a \int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f_*(s) ds \\
& - a \int_0^\eta \frac{(\eta - s)^q}{\Gamma(q+1)} f_*(s) ds - \int_0^\xi \frac{(\xi - s)^{q-1}}{\Gamma(q)} f_*(s) ds \Bigg\}.
\end{aligned}$$

Let us consider the linear operator $\Theta : L^1([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ given by

$$\begin{aligned}
f \mapsto \Theta(f)(t) \ =\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds + \frac{t}{A} \Bigg\{ a \int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f(s) ds \\
& - a \int_0^\eta \frac{(\eta - s)^q}{\Gamma(q+1)} f(s) ds - \int_0^\xi \frac{(\xi - s)^{q-1}}{\Gamma(q)} f(s) ds \Bigg\}.
\end{aligned}$$

Observe that

$$\|h_n(t) - h_*(t)\|$$

$$= \left\| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}(f_n(s)-f_*(s))ds + \frac{t}{A}\left\{ a\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)}(f_n(s)-f_*(s))ds \right.\right.$$
$$\left.\left. -a\int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)}(f_n(s)-f_*(s))ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)}(f_n(s)-f_*(s))ds \right\}\right\| \to 0,$$

as $n \to \infty$.

Thus, it follows by Lemma 4.1 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned} h_*(t) \ &= \ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds + \frac{t}{A}\left\{ a\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f_*(s)ds \right. \\ &\quad \left. -a\int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} f_*(s)ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f_*(s)ds \right\}, \end{aligned}$$

for some $f_* \in S_{F,x_*}$. Hence $\mathcal{B}$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathcal{B}$ is compact valued.

Thus the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.1 and hence its conclusion implies either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x \in \lambda\mathcal{A}(x) + \lambda\mathcal{B}(x)$ for $\lambda \in (0,1)$, then there exists $f \in S_{F,x}$ such that

$$\begin{aligned} x(t) \ &= \ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds + \frac{t}{A}\left\{ a\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f(s)ds \right. \\ &\quad \left. -a\int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} f(s)ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f(s)ds \right\} \\ &\quad +\left[1 + \frac{t}{A}\Big(a(1-\eta)-1\Big)\right](x_0 + g(x)), \quad t \in [0,1]. \end{aligned}$$

Following the method for proof of Claim I, we can obtain

$$\begin{aligned} |x(t)| \ &\leq \ \frac{\psi(\|x\|)}{\Gamma(q+1)}\left[1 + \frac{1}{|A|}\left\{ \frac{|a|(1+\eta^{q+1})}{q+1} + \xi^q \right\}\right]\|p\| \\ &\quad +\left|1 + \frac{1}{A}\Big(a(1-\eta)-1\Big)\right|[|x_0| + \ell\|x\|]. \end{aligned}$$

Thus

$$\|x\| \leq \psi(\|x\|)p_0\|p\| + k_0(|x_0| + \ell\|x\|). \tag{4.5}$$

If condition (ii) of Theorem 4.1 holds, then there exists $\lambda \in (0,1)$ and $x \in \partial B_r$ with $x = \lambda\mathcal{N}(x)$. Then, x is a solution of (1.2) with $\|x\| = M$. Now, by the inequality (4.5), we get

$$\frac{(1-k_0\ell)M}{\psi(M)p_0\|p\| + k_0|x_0|} \leq 1$$

which contradicts (4.2). Hence, $\mathcal{N}$ has a fixed point in $[0,1]$ by Theorem 4.1, and consequently the problem (1.2) has a solution. This completes the proof. $\square$

Example 4.1 *Consider the following boundary value problem*

$$\begin{cases} {}^cD^{3/2}x(t) \in F(t,x), \quad 0 < t < 1, \\ x(0) = \dfrac{1}{15} + \dfrac{1}{12}x(1/9), \quad x\left(\dfrac{1}{4}\right) = \dfrac{1}{4}\displaystyle\int_{1/3}^{1} x(s)ds. \end{cases} \tag{4.6}$$

Here, $q = 3/2$, $x_0 = 1/15$, $g(x) = \frac{1}{12}x(1/9)$, $a = 1/4, \xi = 1/4, \eta = 1/3$, and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$x \to F(t,x) = \left[\frac{1}{9}\frac{x^3}{x^3+3} + \frac{1}{8}(t+1), \frac{1}{2}\cos x\right].$$

For $f \in F$, we have

$$|f| \le \max\left[\frac{1}{9}\frac{x^3}{x^3+3} + \frac{1}{8}(t+1), \frac{1}{2}\cos x\right] \le \frac{1}{2}.$$

Thus,

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|), \quad x \in \mathbb{R},$$

with $p(t) = 1$, $\psi(\|x\|) = 1/2$. Furthermore, $p_0 \approx 2.00565, k_0 = 5$ and by the condition (H_3), we have $\ell = 1/12$. Obviously $\ell < k_0^{-1}$. With the given data, it is found that $M > M_1$, $M_1 \approx 2.29056$. Clearly, all the conditions of Theorem 4.2 are satisfied and hence the problem (4.6) has at least one solution on $[0,1]$.

4.2 The lower semi-continuous case

In this section, we study the case when F is not necessarily convex valued by applying the nonlinear alternative of Leray-Schauder type and a selection theorem due to Bressan and Colombo [11] for lower semi-continuous maps with decomposable values.

Let us mention some auxiliary facts. Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 4.3 *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.*

Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,1] \times \mathbb{R}) \to \mathcal{P}(L^1([0,1], \mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([0,1], \mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,1]\},$$

which is called the Nemytskii operator associated with F.

Definition 4.4 *Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

Lemma 4.2 *([17]) Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1], \mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1], \mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 4.3 *Assume that $(A_2), (H_2), (H_3)$ and the following condition hold:*

(H_4) $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

- (a) $(t,x) \longmapsto F(t,x)$ *is $\mathcal{L} \otimes \mathcal{B}$ measurable;*
- (b) $x \longmapsto F(t,x)$ *is lower semicontinuous for each $t \in [0,1]$.*

Then the boundary value problem (1.2) has at least one solution on $[0,1]$.

Proof. It follows from (H_2) and (H_4) that F is of l.s.c. type. Then, by Lemma 4.2, there exists a continuous function $f : C([0,1], \mathbb{R}) \to L^1([0,1], \mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1], \mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)), \quad 0 < t < 1, \quad 1 < q \le 2, \\ x(0) = x_0 + g(x), \quad x(\xi) = a\displaystyle\int_\eta^1 x(s)ds, \quad 0 < \xi < \eta. \end{cases} \tag{4.7}$$

Observe that if $x \in AC^1([0,1], \mathbb{R})$ is a solution of (4.7), then x is a solution to the problem (1.2). Now, we define two operators, namely $\mathcal{A}' : C([0,1], \mathbb{R}) \longrightarrow C([0,1], \mathbb{R})$ by

$$\mathcal{A}'x(t) = \Big[1 + \frac{t}{A}\Big(a(1-\eta) - 1\Big)\Big](x_0 + g(x)), \tag{4.8}$$

and $\mathcal{B}' : C([0,1], \mathbb{R}) \longrightarrow C([0,1], \mathbb{R})$ by

$$\mathcal{B}'x(t) = \begin{cases} \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s))ds + \frac{t}{A}\Bigg\{ a\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f(x(s))ds \\ \displaystyle -a\int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} f(x(s))ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f(x(s))ds \Bigg\}. \end{cases} \tag{4.9}$$

Clearly $\mathcal{A}', \mathcal{B}' : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ are continuous. The arguments used in the proof of Theorem 4.2 apply and hence guarantee that $\mathcal{A}'$ and $\mathcal{B}'$ satisfy all the conditions of the Nonlinear Alternative for contractive maps in the single valued setting [21] and hence the problem (4.7) has a solution. $\square$

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