

The (k, ℓ) -rainbow index of random graphs*

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Abstract

A tree in an edge-colored graph G is said to be a rainbow tree if no two edges on the tree share the same color. Given two positive integers k, ℓ with $k \geq 3$, the (k, ℓ) -rainbow index $rx_{k,\ell}(G)$ of G is the minimum number of colors needed in an edge-coloring of G such that for any set S of k vertices of G , there exist ℓ internally disjoint rainbow trees connecting S . This concept was introduced by Chartrand et. al., and there have been very few known results about it. In this paper, we establish a sharp threshold function for $rx_{k,\ell}(G_{n,p}) \leq k$ and $rx_{k,\ell}(G_{n,M}) \leq k$, respectively, where $G_{n,p}$ and $G_{n,M}$ are the usually defined random graphs.

Keywords: rainbow index, random graphs, threshold function

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1 Introduction

All graphs in this paper are undirected, finite and simple. We follow [3] for graph theoretical notation and terminology not defined here. Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \rightarrow \{1, 2, \dots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored the same. A path of G is said to be a *rainbow path* if no two edges on the path have the same color. An edge-colored graph G is called *rainbow connected* if for every pair of distinct vertices of G there exists a rainbow path connecting them. The *rainbow*

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connection number of a graph G , denoted by $rc(G)$, is defined as the minimum number of colors that are needed in order to make G rainbow connected. For any two vertices u and v of G , a rainbow $u - v$ geodesic in G is a rainbow $u - v$ path of length $d(u, v)$, where $d(u, v)$ is the distance between u and v . The graph G is *strongly rainbow connected* if there exists a rainbow $u - v$ geodesic for any pair of vertices u and v in G . Similarly, we define the *strong rainbow connection number* of a connected graph G , denoted by $src(G)$, as the smallest number of colors that are needed in order to make G strongly rainbow connected. Clearly, we have $diam(G) \leq rc(G) \leq src(G) \leq m$, where $diam(G)$ denotes the diameter of G and m is the number of edges of G . The *rainbow k -connectivity* of G , denoted by $rc_k(G)$, is defined as the minimum number of colors in an edge-coloring of G such that every two distinct vertices of G are connected by k internally disjoint rainbow paths. These concepts were introduced by Chartrand et al. in [7, 8]. Recently, a lot of relevant results have been published; see [5, 6, 10, 11, 12, 18]. The interested readers can see [16, 17] for a survey on this topic.

Here we recall the concept of generalized connectivity. Let G be a connected graph of order n and size m . For $S \subseteq V(G)$, an S -tree is a tree connecting the vertices of S . Suppose that $\{T_1, T_2, \dots, T_\ell\}$ is a set of S -trees. They are called *internally disjoint* if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$ for every pair of distinct integers i, j with $1 \leq i, j \leq \ell$ (note that the trees are vertex-disjoint in $G \setminus S$). For a set S of k vertices of G , let $\kappa(S)$ denote the maximum number of internally disjoint S -trees in G . The k -connectivity $\kappa_k(G)$ of G is defined by $\kappa_k(G) = \min\{\kappa(S)\}$, where the minimum is taken over all k -element subsets S of $V(G)$. We refer to [9, 13, 14, 15] for more details about the generalized connectivity.

A tree T in an edge-colored graph G is called a *rainbow tree* if no two edges of T have the same color. Given two positive integers k, ℓ with $2 \leq k \leq n$ and $1 \leq \ell \leq \kappa_k(G)$, the (k, ℓ) -rainbow index $rx_{k, \ell}(G)$ of G is the minimum number of colors needed in an edge-coloring of G such that for any set S of k vertices of G , there exist ℓ internally disjoint rainbow S -trees. In particular, for $\ell = 1$, we often write $rx_k(G)$ rather than $rx_{k, 1}(G)$ and call it the *k -rainbow index*. It is easy to see that $rx_{2, \ell}(G) = rc_\ell(G)$. So the (k, ℓ) -rainbow index can be viewed as a generalization of the rainbow connectivity. In the sequel, we always assume $k \geq 3$.

The concept of (k, ℓ) -rainbow index was also introduced by Chartrand et al.; see [9]. They determined the k -rainbow index of all unicyclic graphs and the $(3, \ell)$ -rainbow index of complete graphs for $\ell = 1, 2$. In [4], we investigated the (k, ℓ) -rainbow index of complete graphs for every pair of integers k, ℓ . We proved that for every pair of positive integers k, ℓ with $k \geq 3$, there exists a positive integer $N = N(k, \ell)$ such that $rx_{k, \ell}(K_n) = k$ for

every integer $n \geq N$, which settled down the two conjectures in [9].

In this paper, we study the (k, ℓ) -rainbow index of random graphs and establish a sharp threshold function for the property $rx_{k,\ell}(G_{n,p}) \leq k$ and $rx_{k,\ell}(G_{n,M}) \leq k$, respectively, where $G_{n,p}$ and $G_{n,M}$ are defined as usual; see [2].

2 Basic notation on random graphs

The two most frequently occurring probability models of random graphs are $\mathcal{G}(n, p)$ and $\mathcal{G}(n, M)$. The model $\mathcal{G}(n, p)$ consists of all graphs on n vertices, in which the edges are chosen independently and randomly with probability p ; whereas the model $\mathcal{G}(n, M)$ consists of all graphs on n vertices and M edges, in which each graph has the same probability. Let $G_{n,p}, G_{n,M}$ stand for random graphs from the models $\mathcal{G}(n, p)$ and $\mathcal{G}(n, M)$. We say that an event $E = E(n)$ happens *almost surely* (or *a.s.* for short) if $\lim_{n \rightarrow \infty} Pr[E(n)] = 1$. Let K, G, H be three graphs on n vertices. A property Q is said to be *monotone* if whenever $G \subseteq H$ and G satisfies Q , then H also satisfies Q . Moreover, We call a property Q *convex* if whenever $K \subset G \subset H$, and K satisfies Q and H satisfies Q , then G also satisfies Q . For a graph property Q , a function $p(n)$ is called a *threshold function* of Q if

- $\frac{p'(n)}{p(n)} \rightarrow 0$, then $G_{n,p'(n)}$ almost surely does not satisfy Q ; and
- $\frac{p''(n)}{p(n)} \rightarrow \infty$, then $G_{n,p''(n)}$ almost surely satisfies Q .

Furthermore, $p(n)$ is called a *sharp threshold function* of Q if there are two positive constants c and C such that

- for every $p'(n) \leq cp(n)$, $G_{n,p'(n)}$ almost surely does not satisfy Q ; and
- for every $p''(n) \geq Cp(n)$, $G_{n,p''(n)}$ almost surely satisfies Q .

Similarly, we can define $M(n)$ as a threshold function of Q in the model $\mathcal{G}(n, M)$; see [2].

It is well known that all monotone graph properties have a threshold function [2]. Obviously, for every pair of positive integers k, ℓ , the property that the (k, ℓ) -rainbow index is at most k is monotone, and thus has a threshold.

3 Main results

As Caro et al. pointed out, the random graph setting poses several intriguing questions. In [5], Caro et al. proved that $p = \sqrt{\log n/n}$ is a sharp threshold for the property $rc(G_{n,p}) \leq 2$. This was generalized by Fujita et al. [10], who obtained that $p = \sqrt{\log n/n}$

is a sharp threshold for the property $rc_k(G_{n,p}) \leq 2$ and $M = \sqrt{n^3 \log n}$ is a sharp threshold for the property $rc_k(G_{n,M}) \leq 2$ for all integer $k \geq 1$. In this section, we employ similar methods to study the (k, ℓ) -rainbow index of random graphs $G_{n,p}$ and $G_{n,M}$.

Theorem 1. *For every pair of positive integers k, ℓ with $k \geq 3$, $\sqrt[k]{\frac{\log_a n}{n}}$ is a sharp threshold function for the property $rx_{k,\ell}(G_{n,p}) \leq k$, where $a = \frac{k^k}{k^k - k!}$.*

Proof. The proof will be two-fold. For the first part, we show that there exists a positive constant c_1 such that for every $p \geq c_1 \sqrt[k]{\frac{\log_a n}{n}}$, almost surely $rx_{k,\ell}(G_{n,p}) \leq k$, which can be derived from the following two claims.

Claim 1: For any $c_1 \geq 3$, if $p \geq c_1 \sqrt[k]{\frac{\log_a n}{n}}$, then almost surely any k vertices in $G_{n,p}$ have at least $2k \log_a n$ common neighbors.

For any $S \subseteq V(G_{n,p})$ with $|S| = k$, let $D(S)$ denote the event that the vertices in S have at least $2k \log_a n$ common neighbors. Then it suffices to prove that, for $p = c_1 \sqrt[k]{\frac{\log_a n}{n}}$, $Pr[\bigcap_S D(S)] \rightarrow 1$, as $n \rightarrow \infty$. Define X as the number of common neighbors of all the vertices in S . Then $X \sim \text{Bin}\left(n - k, (c_1 \sqrt[k]{\frac{\log_a n}{n}})^k\right)$ and $E(X) = \frac{n-k}{n} c_1^k \log_a n$. Assume that $n > \frac{c_1^k k}{c_1^k - 2k}$. Using the Chernoff Bound [1], we get that

$$\begin{aligned} Pr[\overline{D(S)}] &= Pr[X < 2k \log_a n] \\ &= Pr\left[X < \frac{c_1^k (n-k)}{n} \log_a n \left(1 - \frac{(c_1^k - 2k)n - c_1^k k}{c_1^k (n-k)}\right)\right] \\ &\leq e^{-\frac{c_1^k (n-k)}{2n} \log_a n \left(\frac{(c_1^k - 2k)n - c_1^k k}{c_1^k (n-k)}\right)^2} \\ &< n^{-\frac{c_1^k (n-k)}{2n} \left(\frac{(c_1^k - 2k)n - c_1^k k}{c_1^k (n-k)}\right)^2}. \end{aligned}$$

Note that the assumption $n > \frac{c_1^k k}{c_1^k - 2k}$ ensures $\frac{(c_1^k - 2k)n - c_1^k k}{c_1^k (n-k)} > 0$. So we can apply the Chernoff Bound to scaling the above inequalities. The last inequality holds, since $1 < a = \frac{k^k}{k^k - k!} < e$ and then $\log_a n > \ln n$.

It follows from the union bound that

$$\begin{aligned} Pr\left[\bigcap_S D(S)\right] &= 1 - Pr\left[\bigcup_S \overline{D(S)}\right] \\ &\geq 1 - \sum_S Pr[\overline{D(S)}] \end{aligned}$$

$$\begin{aligned}
&> 1 - \binom{n}{k} n^{-\frac{c_1^k(n-k)}{2n}} \left(\frac{(c_1^k-2k)n-c_1^k k}{c_1^k(n-k)} \right)^2 \\
&> 1 - n^{k-\frac{c_1^k(n-k)}{2n}} \left(\frac{(c_1^k-2k)n-c_1^k k}{c_1^k(n-k)} \right)^2.
\end{aligned}$$

It is not hard to see that $c_1 > 3$ can guarantee $k - \frac{c_1^k(n-k)}{2n} \left(\frac{(c_1^k-2k)n-c_1^k k}{c_1^k(n-k)} \right)^2 < 0$ for sufficiently large n . Then $\lim_{n \rightarrow \infty} 1 - n^{k-\frac{c_1^k(n-k)}{2n}} \left(\frac{(c_1^k-2k)n-c_1^k k}{c_1^k(n-k)} \right)^2 = 1$, which implies that $\lim_{n \rightarrow \infty} \Pr[\bigcap_S D(S)] = 1$ as desired.

Claim 2: If any k vertices in $G_{n,p}$ have at least $2k \log_a n$ common neighbors, then there exists a positive integer $N = N(k)$ such that $rx_{k,\ell}(G_{n,p}) \leq k$ for every integer $n \geq N$.

Let $C = \{1, 2, \dots, k\}$ be a set of k different colors. We color the edges of $G_{n,p}$ with the colors from C randomly and independently. For $S \subseteq V(G_{n,p})$ with $|S| = k$, define $F(S)$ as the event that there exist at least ℓ internally disjoint rainbow S -trees. It suffices to prove that $\Pr[\bigcap_S F(S)] > 0$.

Suppose $S = \{v_1, v_2, \dots, v_k\}$. For any common neighbor u of the vertices in S , let $T(u)$ denote the star with $V(T(u)) = \{u, v_1, v_2, \dots, v_k\}$ and $E(T(u)) = \{uv_1, uv_2, \dots, uv_k\}$. Set $\mathcal{T} = \{T(u) | u \text{ is a common neighbor of the vertices in } S\}$. Then \mathcal{T} is a set of at least $2k \log_a n$ internally disjoint S -trees. It is easy to see that $q := \Pr[T \in \mathcal{T} \text{ is a rainbow tree}] = \frac{k!}{k^k} < \frac{1}{2}$. So $1 - q > q$. Define Y as the number of rainbow S -trees in \mathcal{T} . Then we have

$$\begin{aligned}
\Pr[\overline{F(S)}] &\leq \Pr[Y \leq \ell - 1] \\
&\leq \sum_{i=0}^{\ell-1} \binom{2k \log_a n}{i} q^i (1-q)^{2k \log_a n - i} \\
&\leq (1-q)^{2k \log_a n} \sum_{i=0}^{\ell-1} \binom{2k \log_a n}{i} \\
&\leq (1-q)^{2k \log_a n} (1 + 2k \log_a n)^{\ell-1} \\
&= \frac{(1 + 2k \log_a n)^{\ell-1}}{n^{2k}}.
\end{aligned}$$

It yields that

$$\begin{aligned}
Pr\left[\bigcap_S F(S)\right] &= 1 - Pr\left[\bigcup_S \overline{F(S)}\right] \\
&\geq 1 - \sum_S Pr\left[\overline{F(S)}\right] \\
&\geq 1 - \binom{n}{k} \frac{(1 + 2k \log_a n)^{\ell-1}}{n^{2k}} \\
&> 1 - \frac{(1 + 2k \log_a n)^{\ell-1}}{n^k}.
\end{aligned}$$

Obviously, $\lim_{n \rightarrow \infty} 1 - \frac{(1+2k \log_a n)^{\ell-1}}{n^k} = 1$, and then $\lim_{n \rightarrow \infty} Pr\left[\bigcap_S F(S)\right] = 1$. Thus there exists a positive integer $N = N(k)$ such that $Pr\left[\bigcap_S F(S)\right] > 0$ for every integer $n \geq N$.

For the other direction, we show that there exists a positive constant c_2 such that for every $p \leq c_2 \sqrt[k]{\frac{\log_a n}{n}}$, almost surely $rx_{k,\ell}(G_{n,p}) \geq k + 1$.

It suffices to prove that for a sufficiently small constant c_2 , the random graph $G_{n,p}$ with $p = c_2 \sqrt[k]{\frac{\log_a n}{n}}$ almost surely contains a set S of k vertices satisfying

- (i) S is an independent set;
- (ii) the vertices in S have no common neighbors.

Clearly, for such S there exists no rainbow S -trees in any k -edge-coloring, which implies that $rx_{k,\ell}(G_{n,p}) \geq k + 1$.

Fix a set H of $n^{1/(2k+1)}$ vertices in $G_{n,p}$ (we may and will assume that $n^{1/(2k+1)}/k$ is an integer). Let E_1 be the event that H is an independent set. Then

$$Pr[E_1] = (1 - c_2^k \sqrt[k]{\frac{\log_a n}{n}})^{\binom{n^{1/(2k+1)}}{2}} = 1 - o(1),$$

where $o(1)$ denotes a function tending to 0 as n tends to infinity.

Partition H into t subsets H_1, H_2, \dots, H_t arbitrarily, where $t = n^{1/(2k+1)}/k$ and $|H_1| = |H_2| = \dots = |H_t| = k$. Let E_2 be the event that there exists some H_i without common neighbors in $V(G_{n,p}) \setminus H$. Then, for sufficiently small c_2 ,

$$Pr[E_2] = 1 - \left(1 - (1 - c_2^k \frac{\log_a n}{n})^{n - n^{1/(2k+1)}}\right)^{n^{1/(2k+1)}/k} = 1 - o(1).$$

So, almost surely there exists some set H_i of k vertices satisfying properties (i) and (ii). Thus, for sufficiently small c_2 and every $p \leq c_2 \sqrt[k]{\frac{\log_a n}{n}}$, almost surely $rx_{k,\ell}(G_{n,p}) \geq k + 1$. The proof is thus complete. \square

Next we will turn to another well-known random graph model $\mathcal{G}(n, M)$. We start with a useful lemma which reveals the relationship between $\mathcal{G}(n, p)$ and $\mathcal{G}(n, M)$. Set $N = \binom{n}{2}$.

Lemma 2. [2] *If Q is a convex property and $p(1-p)N \rightarrow \infty$, then $G_{n,p}$ almost surely has Q if and only if for every fixed x , $G_{n,M}$ almost surely has Q , where $M = \lfloor pN + x(p(1-p)N)^{1/2} \rfloor$.*

Clearly, the property that the (k, ℓ) -rainbow index of a given graph is at most k , is a convex property. By Theorem 1 and Lemma 2, we get the following result.

Corollary 3. *For every pair of positive integers k, ℓ with $k \geq 3$, $M(n) = \sqrt[k]{n^{2k-1} \log_a n}$ is a sharp threshold function for the property $rx_{k,\ell}(G_{n,M}) \leq k$, where $a = \frac{k^k}{k^k - k!}$.*

Remark: If p is a threshold function for a given property Q , then so is λp for any positive constant λ . It follows that $p(n) = \sqrt[k]{\frac{\log n}{n}}$ ($M(n) = \sqrt[k]{n^{2k-1} \log n}$) is also a sharp threshold function for the property $rx_{k,\ell}(G_{n,p}) \leq k$ ($rx_{k,\ell}(G_{n,M}) \leq k$), which corresponds to the results in [10].

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