# WEIGHTED $p$-ADIC HARDY OPERATORS AND THEIR COMMUTATORS ON $p$-ADIC CENTRAL MORREY SPACES

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Abstract. In this paper, we establish necessary and sufficient conditions for boundedness of weighted $p$-adic Hardy operators on $p$-adic Morrey spaces, $p$-adic central Morrey spaces and $p$-adic $\lambda$-central BMO spaces, respectively, and obtain their sharp bounds. We also give the characterization of weight functions for which the commutators generated by weighted $p$-adic Hardy operators and $\lambda$-central BMO functions are bounded on the $p$-adic central Morrey spaces. This result is different from that on Euclidean spaces due to the special structure of $p$-adic integers.

## 1. Introduction

Let $\omega : [0, 1] \to [0, \infty)$ be a function. The weighted Hardy operator $H_\omega$ [6] is defined by

$$H_\omega f(x) := \int_0^1 f(tx)\omega(t)dt,$$

for all measurable complex-valued functions $f$ on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. Xiao [32] gave the characterization of $\omega$ for which $H_\omega$ is bounded on either $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, or $\mathrm{BMO}(\mathbb{R}^n)$. Meanwhile, the corresponding operator norms were worked out. In [12], Fu, Lu and Yuan gave the characterization of $\omega$ to ensure that $H_\omega$ is bounded on central Morrey spaces and $\lambda$-central BMO spaces, they also got the corresponding operator norms.

It is clear that if $\omega \equiv 1$ and $n = 1$, then $H_\omega$ is precisely reduced to the classical Hardy operator $H$ defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x \neq 0,$$

which is one of the fundamental integral averaging operators in real analysis. A celebrated operator norm estimate states that, for $1 < q < \infty$, the sharp norm of $H$ from $L^q(\mathbb{R})$ to $L^q(\mathbb{R})$ is $q/(q-1)$, see [14]. If $n = 1$ and $\omega(t) = (1-t)^{\alpha-1}/\Gamma(\alpha)$, $0 < \alpha < 1$, then for all $x > 0$,

$$H_\omega f(x) = x^{-\alpha} I_\alpha f(x),$$

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where $I_\alpha$ is Riemann-Liouville fractional integral defined by

$$I_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > 0,$$

for all locally integrable functions $f$ on $(0, \infty)$. For $n \geq 2$, if $\omega(t) = nt^{n-1}$ and $f$ is a radical function, then $H_\omega$ is just reduced to the $n$-dimensional Hardy operator $\mathcal{H}$ defined by

$$\mathcal{H} f(x) = \frac{1}{v_n |x|^n} \int_{|y|<|x|} f(y) dy,$$

where $v_n$ is the volume of the unit sphere $S^{n-1}$. See [33] for more details. In 1995, Christ and Grafakos [9] got that the precise norm of $\mathcal{H}$ from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ is also $q/(q-1)$, $1 < q < \infty$. More recently, Fu, Grafakos, Lu and Zhao [11] obtained the precise norm of $m$-linear Hardy operators on weighted Lebesgue spaces and central Morrey spaces.

In recent years, the field of $p$-adic numbers has been widely used in theoretical and mathematical physics (cf. [3], [5], [15–17], [20], [26–30]). And harmonic analysis on $p$-adic field has been drawn more and more concern ( [4], [7,8], [18,19], [22–25] and references therein).

For a prime number $p$, let $\mathbb{Q}_p$ be the field of $p$-adic numbers. It is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean $p$-adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$; if any non-zero rational number $x$ is represented as $x = p^\gamma \frac{m}{n}$, where $\gamma$ is an integer and integers $m$, $n$ are indivisible by $p$, then $|x|_p = p^{-\gamma}$. It is easy to see that the norm satisfies the following properties:

$$|xy|_p = |x|_p |y|_p, \quad |x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

Moreover, if $|x|_p \neq |y|_p$, then $|x \pm y|_p = \max\{|x|_p, |y|_p\}$. It is well-known that $\mathbb{Q}_p$ is a typical model of non-Archimedean local fields. From the standard $p$-adic analysis [28], we see that any non-zero $p$-adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the canonical series

$$x = p^\gamma \sum_{j=0}^{\infty} a_j p^j, \quad \gamma = \gamma(x) \in \mathbb{Z}, \tag{1.1}$$

where $a_j$ are integers, $0 \leq a_j \leq p-1$, $a_0 \neq 0$.

The space $\mathbb{Q}_p^n$ consists of points $x = (x_1, x_2, \cdots, x_n)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2, \cdots, n$. The $p$-adic norm on $\mathbb{Q}_p^n$ is

$$|x|_p := \max_{1 \leq j \leq n} |x_j|_p, \quad x \in \mathbb{Q}_p^n.$$

Denote by $B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^\gamma\}$, the ball with center at $a \in \mathbb{Q}_p^n$ and radius $p^\gamma$, and by $S_\gamma(a) := \{x \in \mathbb{Q}_p^n : |x - a|_p = p^\gamma\}$ the sphere with center at $a \in \mathbb{Q}_p^n$ and radius $p^\gamma$, $\gamma \in \mathbb{Z}$. It is clear that $S_\gamma(a) = B_\gamma(a) \setminus B_{\gamma-1}(a)$, and

$$B_\gamma(a) = \bigcup_{k \leq \gamma} S_k(a). \tag{1.2}$$

We set $B_\gamma(0) = B_\gamma$ and $S_\gamma(0) = S_\gamma$. Let $\mathbb{Z}_p = \{x \in |x|_p \leq 1\}$ be the class of all $p$-adic integers in $\mathbb{Q}_p$ and denote $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, it follows from the standard analysis that there exists a Haar measure $dx$ on $\mathbb{Q}_p^n$, which is unique up to a positive constant factor and is translation invariant. We normalize the measure $dx$ by the equality

$$\int_{B_0(0)} dx = |B_0(0)|_H = 1,$$

where $|E|_H$ denotes the Haar measure of a measurable subset $E$ of $\mathbb{Q}_p^n$. By simple calculation, we can obtain that

$$|B_\gamma(a)|_H = p^{\gamma n}, \qquad |S_\gamma(a)|_H = p^{\gamma n}(1-p^{-n})$$

for any $a \in \mathbb{Q}_p^n$. For a more complete introduction to the $p$-adic field, one can refer to [25] or [28].

On $p$-adic field, Rim and Lee [22] defined the weighted $p$-adic Hardy operator $\mathcal{H}_\psi^p$ by

$$\mathcal{H}_\psi^p f(x) = \int_{\mathbb{Z}_p^*} f(tx)\psi(t)dt, \tag{1.3}$$

where $\psi$ is a non-negative function defined on $\mathbb{Z}_p^*$, and gave the characterization of functions $\psi$ for which $\mathcal{H}_\psi^p$ are bounded on $L^r(\mathbb{Q}_n^p)$, $1 \le r \le \infty$, and on $\mathrm{BMO}(\mathbb{Q}_n^p)$. Also, in each cases, they got the corresponding operator norms.

Obviously, if $\psi \equiv 1$ and $n = 1$, then $H_\omega$ is just reduced to the $p$-adic Hardy operator $H^p$ on $\mathbb{Q}_p$, which is defined by

$$H^p f(x) = \frac{1}{|x|_p} \int_{|t|_p \le |x|_p} f(t)dt, \quad x \ne 0.$$

Let $0 < \alpha < 1$. We define the $p$-adic Riemann-Liouville fractional integral $R_\alpha^p$ by

$$R_\alpha^p f(x) = \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} \int_{|y|_p < |x|_p} \frac{f(y)}{|x-y|_p^{1-\alpha}} dy.$$

For $n = 1$, if we take $\psi(t) = (1-p^{-\alpha})|1-t|_p^{\alpha-1}\chi_{B_0\setminus S_0}(t)/(1-p^{\alpha-1})$, then

$$\mathcal{H}_\psi^p f(x) = |x|_p^{-\alpha} R_\alpha^p f(x).$$

For $n \ge 2$, if we take $\psi(t) = (1-p^{-n})|t|_p^{n-1}/(1-p^{-1})$, and $f$ satisfies $f(x) = f(|x|_p^{-1})$, then

$$\mathcal{H}_\psi^p f(x) = \mathcal{H}^p f(x),$$

where $\mathcal{H}^p$ is the $p$-adic Hardy operator on $\mathbb{Q}_n^p$ defined by

$$\mathcal{H}^p f(x) = \frac{1}{|x|_p^n} \int_{B(0,|x|_p)} f(t)dt, \quad x \in \mathbb{Q}_p^n \setminus \{0\},$$

here $B(0,|x|_p)$ is a ball in $\mathbb{Q}_p^n$ with center at $0\in\mathbb{Q}_p^n$ and radius $|x|_p$. In fact, by definition, we have

$$\begin{aligned}\mathcal{H}^p f(x) &= \frac{1}{|x|_p^n}\int_{B(0,|x|_p)} f(t)dt = \int_{B(0,1)} f(|x|_p^{-1}y)dy\\ &= \int_{B(0,1)} f(x|y|_p^{-1})dy = \sum_{k=-\infty}^{0}\int_{S_k} f(x|y|_p^{-1})dy\\ &= \sum_{k=-\infty}^{0} f(p^{-k}x)p^{kn}(1-p^{-n}) \qquad (1.4)\\ &= \frac{1-p^{-n}}{1-p^{-1}}\sum_{k=-\infty}^{0}\int_{|t|_p=p^k} f(|t|_p^{-1}x)|t|_p^{n-1}dt\\ &= \int_{\mathbb{Z}_p^*} f(tx)\frac{1-p^{-n}}{1-p^{-1}}|t|_p^{n-1}dt\end{aligned}$$

Fu, Wu and Lu [13] established the sharp estimates of $p$-adic Hardy operators on $p$-adic weighted Lebesgue spaces. Wu, Mi and Fu [31] obtained the sharp bounds of $p$-adic Hardy operators on $p$-adic central Morrey spaces and $p$-adic $\lambda$-central BMO spaces. They also got the boundedness for commutators of $p$-adic Hardy operators on these spaces.

The main purpose of this paper is to make clear the mapping properties of weighted $p$-adic Hardy operators as well as their commutators on $p$-adic Morrey, central Morrey and $\lambda$-central BMO spaces.

Morrey [21] introduced the $L^{q,\lambda}(\mathbb{R}^n)$ spaces to study the local behavior of solutions to second order elliptic partial differential equations. The $p$-adic Morrey space is defined as follows.

**Definition 1.1.** Let $1\le q<\infty$ and $\lambda\ge -\frac{1}{q}$. The $p$-adic Morrey space $L^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$L^{q,\lambda}(\mathbb{Q}_p^n)=\left\{f\in L^q_{loc}(\mathbb{Q}_p^n):\|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}<\infty\right\},$$

where

$$\|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n)}=\sup_{a\in\mathbb{Q}_p^n,\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma(a)|_p^{1+\lambda q}}\int_{B_\gamma(a)}|f(x)|^q dx\right)^{\frac{1}{q}}.$$

*Remark* 1.2. It's clear that $L^{q,-1/q}(\mathbb{Q}_p^n)=L^q(\mathbb{Q}_p^n)$, $L^{q,0}(\mathbb{Q}_p^n)=L^\infty(\mathbb{Q}_p^n)$.

Alvarez, Guzmán-Partida and Lakey [1] studied the relationship between central BMO spaces and Morrey spaces. Furthermore, they introduced $\lambda$-central BMO spaces and central Morrey spaces, respectively. Next, we introduce their $p$-adic versions.

**Definition 1.3.** Let $\lambda\in\mathbb{R}$ and $1<q<\infty$. The $p$-adic central Morrey space $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by

$$\|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}:=\sup_{\gamma\in\mathbb{Z}}\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}|f(x)|^q dx\right)^{\frac{1}{q}}<\infty, \qquad (1.5)$$

where $B_\gamma=B_\gamma(0)$.

*Remark* 1.4. It's clear that

$$L^{q,\lambda}(\mathbb{Q}_p^n) \subset \dot{B}^{q,\lambda}(\mathbb{Q}_p^n), \quad \dot{B}^{q,-\frac{1}{q}}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n).$$

When $\lambda < -1/q$, the space $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ reduces to $\{0\}$, therefore, we can only consider the case $\lambda \geq -1/q$. If $1 \leq q_1 < q_2 < \infty$, by Hölder's inequality

$$\dot{B}^{q_2,\lambda}(\mathbb{Q}_p^n) \subset \dot{B}^{q_1,\lambda}(\mathbb{Q}_p^n)$$

for $\lambda \in \mathbb{R}$.

**Definition 1.5.** Let $\lambda < \frac{1}{n}$ and $1 < q < \infty$. The space $\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)$ is defined by the condition

$$\|f\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)} := \sup_{\gamma\in\mathbb{Z}} \left( \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(x) - f_{B_\gamma}|^q dx \right)^{\frac{1}{q}} < \infty. \tag{1.6}$$

*Remark* 1.6. When $\lambda = 0$, the space $\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)$ is just $\mathrm{CMO}^q(\mathbb{Q}_p^n)$, which is defined in [13]. If $1 \leq q_1 < q_2 < \infty$, by Hölder's inequality,

$$\mathrm{CMO}^{q_2,\lambda}(\mathbb{Q}_p^n) \subset \mathrm{CMO}^{q_1,\lambda}(\mathbb{Q}_p^n)$$

for $\lambda \in \mathbb{R}$. By the standard proof as that in $\mathbb{R}^n$, we can see that

$$\|f\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)} \sim \sup_{\gamma\in\mathbb{Z}} \inf_{c\in\mathbb{C}} \left( \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(x) - c|^q dx \right)^{\frac{1}{q}}.$$

*Remark* 1.7. The formulas (1.5) and (1.6) yield that $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ is a Banach space continuously included in $\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)$.

The outline of this paper is as follows. In Section 2, we establish the necessary and sufficient conditions for boundedness of $p$-adic Hardy operators on $p$-adic Morrey spaces, $p$-adic central Morrey spaces and $p$-adic $\lambda$-central BMO spaces, respectively, and obtain their corresponding operator norms. In Section 3, we give the characterization of weight functions for which the commutators generated by weighted $p$-adic Hardy operators and $p$-adic central BMO functions are bounded on $p$-adic central Morrey spaces.

Throughout this paper the letter $C$ will be used to denote various constants, and the various uses of the letter do not, however, denote the same constant.

## 2. Sharp estimates of weighted $p$-adic Hardy operator

We get the following sufficient and necessary conditions of weight functions, for which the weighted $p$-adic Hardy operators are bounded on $p$-adic Morrey, central Morrey and $\lambda$-central BMO spaces.

**Theorem 2.1.** *Let $1 < q < \infty$ and $-1/q < \lambda \leq 0$. Then $\mathcal{H}_\psi^p$ is bounded on $L^{q,\lambda}(\mathbb{Q}_p^n)$ if and only if*

$$\int_{\mathbb{Z}_p^*} |t|_p^{n\lambda} \psi(t) dt < \infty. \tag{2.1}$$

*Moreover,*

$$\|\mathcal{H}_\psi^p\|_{L^{q,\lambda}(\mathbb{Q}_p^n)\to L^{q,\lambda}(\mathbb{Q}_p^n)} = \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda} \psi(t) dt.$$

**Corollary 2.2.** Let $1<q<\infty$, $-1/q<\lambda\leq 0$ and $0<\alpha<1$. Then

$$\|H^p\|_{L^{q,\lambda}(\mathbb{Q}_p)\to L^{q,\lambda}(\mathbb{Q}_p)}=\frac{1-p^{-1}}{1-p^{-(1+\lambda)}},$$

$$\|R^p_\alpha\|_{L^{q,\lambda}(\mathbb{Q}_p)\to L^{q,\lambda}(|x|_p^{-\alpha q}dx)}=\frac{(1-p^{-\alpha})(1-p^{-1})}{(1-p^{\alpha-1})(p^{1+\lambda}-1)}.$$

Moreover, write $\mathcal{L}^{q,\lambda}(\mathbb{Q}_p^n)=\{f: f\in L^{q,\lambda}(\mathbb{Q}_p^n)$ and satisfies$f(x)=f(|x|_p^{-1})\}$. Then

$$\|\mathcal{H}^p\|_{\mathcal{L}^{q,\lambda}(\mathbb{Q}_p^n)\to L^{q,\lambda}(\mathbb{Q}_p^n)}=\frac{1-p^{-n}}{1-p^{-n(1+\lambda)}}.$$

**Theorem 2.3.** *Let $1<q<\infty$ and $-1/q<\lambda\leq 0$. Then $\mathcal{H}^p_\psi$ is bounded on $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ if and only if (2.1) holds. Moreover,*

$$\|\mathcal{H}^p_\psi\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)\to\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}=\int_{\mathbb{Z}_p^*}|t|_p^{n\lambda}\psi(t)dt.$$

**Corollary 2.4.** Let $1<q<\infty$, $-1/q<\lambda\leq 0$ and $0<\alpha<1$. Then

$$\|H^p\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p)\to\dot{B}^{q,\lambda}(\mathbb{Q}_p)}=\frac{1-p^{-1}}{1-p^{-(1+\lambda)}},$$

$$\|R^p_\alpha\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p)\to\dot{B}^{q,\lambda}(|x|_p^{-\alpha q}dx)}=\frac{(1-p^{-\alpha})(1-p^{-1})}{(1-p^{\alpha-1})(p^{1+\lambda}-1)}.$$

Moreover, set $\dot{\mathcal{B}}^{q,\lambda}(\mathbb{Q}_p^n)=\{f: f\in\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ and satisfies $f(x)=f(|x|_p^{-1})\}$. Then

$$\|\mathcal{H}^p\|_{\dot{\mathcal{B}}^{q,\lambda}(\mathbb{Q}_p^n)\to\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}=\frac{1-p^{-n}}{1-p^{-n(1+\lambda)}}.$$

**Theorem 2.5.** *Let $1<q<\infty$ and $0\leq\lambda<1/n$. Then $\mathcal{H}^p_\psi$ is bounded on $\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)$ if and only if (2.1) holds. Moreover,*

$$\|\mathcal{H}^p_\psi\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)\to\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)}=\int_{\mathbb{Z}_p^*}|t|_p^{n\lambda}\psi(t)dt.$$

**Corollary 2.6.** Let $1<q<\infty$.
(I). If $0\leq\lambda<1$, then

$$\|\mathcal{H}^p\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p)\to\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p)}=\frac{1-p^{-1}}{1-p^{-(1+\lambda)}},$$

$$\|R^p_\alpha\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p)\to\mathrm{CMO}^{q,\lambda}(|x|_p^{-\alpha q}dx)}=\frac{(1-p^{-\alpha})(1-p^{-1})}{(1-p^{\alpha-1})(p^{1+\lambda}-1)}.$$

(II). If $0\leq\lambda<1/n$ and set $\mathcal{CMO}^{q,\lambda}(\mathbb{Q}_p^n)=\{f: f\in\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)$ and satisfies $f(x)=f(|x|_p^{-1})\}$, then

$$\|\mathcal{H}^p\|_{\mathcal{CMO}^{q,\lambda}(\mathbb{Q}_p^n)\to\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)}=\frac{1-p^{-n}}{1-p^{-n(1+\lambda)}}.$$

*Proof of Theorem 2.1.* Suppose that (2.1) holds. Let $\gamma\in\mathbb{Z}$ and denote $tB_\gamma(a)=B(ta,|t|_pp^\gamma)$. Using Minkowski's inequality and change of variable, we have

$$\left(\frac{1}{|B_\gamma(a)|_H^{1+\lambda q}}\int_{B_\gamma(a)}|\mathcal{H}^p_\psi f(x)|^q dx\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&\leq \int_{\mathbb{Z}_p^*} \left( \frac{1}{|B_\gamma(a)|_H^{1+\lambda q}} \int_{B_\gamma(a)} |f(tx)|^q dx \right)^{\frac{1}{q}} \psi(t) dt \\
&= \int_{\mathbb{Z}_p^*} \left( \frac{1}{|tB_\gamma(a)|_H^{1+\lambda q}} \int_{tB_\gamma(a)} |f(y)|^q dy \right)^{\frac{1}{q}} |t|_p^{\lambda n} \psi(t) dt \\
&\leq \|f\|_{L^{q,\lambda}(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda} \psi(t) dt.
\end{aligned}
$$

Thus, $\mathcal{H}_\psi^p$ is bounded on $L^{q,\lambda}(\mathbb{Q}_p^n)$ and

$$
\|\mathcal{H}_\psi^p\|_{L^{q,\lambda}(\mathbb{Q}_p^n) \to L^{q,\lambda}(\mathbb{Q}_p^n)} \leq \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda} \psi(t) dt. \tag{2.2}
$$

On the other hand, assume that $\mathcal{H}_\psi^p$ is bounded on $L^{q,\lambda}(\mathbb{Q}_p^n)$. Take

$$
f_0(x) = |x|_p^{n\lambda}, \quad x \in \mathbb{Q}_p^n.
$$

Now we show that $f_0 \in L^{q,\lambda}(\mathbb{Q}_p^n)$.

(I). If $|a|_p > p^\gamma$ and $x \in B_\gamma(a)$, then $|x|_p = \max\{|x-a|_p, |a|_p\} > p^\gamma$. Since $-\frac{1}{q} \leq \lambda < 0$, we have

$$
\frac{1}{|B_\gamma(a)|_H^{1+\lambda q}} \int_{B_\gamma(a)} |x|_p^{n\lambda q} dx < \frac{1}{|B_\gamma(a)|_H^{1+\lambda q}} \int_{B_\gamma(a)} p^{\gamma n\lambda q} dx = 1.
$$

(II). If $|a|_p \leq p^\gamma$ and $x \in B_\gamma(a)$, then $|x|_p \leq \max\{|x-a|_p, |a|_p\} \leq p^\gamma$, therefore, $x \in B_\gamma$. Recall that two balls in $\mathbb{Q}_p^n$ are either disjoint or one is contained in the other (cf. P.21 in [2]). So we have $B_\gamma(a) = B_\gamma$, thus

$$
\begin{aligned}
\frac{1}{|B_\gamma(a)|_H^{1+\lambda q}} \int_{B_\gamma(a)} |x|_p^{n\lambda q} dx &= \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |x|_p^{n\lambda q} dx \\
&= p^{-\gamma n(1+\lambda q)} \sum_{k=-\infty}^{\gamma} \int_{S_k} p^{kn\lambda q} dx \\
&= \frac{1-p^{-n}}{1-p^{-n(1+\lambda q)}}.
\end{aligned}
$$

From the above discussion, we can see that $f_0 \in L^{q,\lambda}(\mathbb{Q}_p^n)$. It's clear that

$$
\begin{aligned}
\mathcal{H}_\psi^p f_0(x) &= \int_{\mathbb{Z}_p^*} |tx|_p^{n\lambda} \psi(t) dt = |x|_p^{n\lambda} \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda} \psi(t) dt \\
&= f_0(x) \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda} \psi(t) dt.
\end{aligned} \tag{2.3}
$$

Therefore,

$$
\|\mathcal{H}_\psi^p\|_{L^{q,\lambda}(\mathbb{Q}_p^n) \to L^{q,\lambda}(\mathbb{Q}_p^n)} \geq \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda} \psi(t) dt. \tag{2.4}
$$

Consequently,

$$
\int_{\mathbb{Z}_p^*} |t|_p^{n\lambda} \psi(t) dt < \infty.
$$

And (2.2) and (2.4) yield the desired result. $\square$

*Proof of Theorem 2.3.* Suppose that (2.1) holds. For any $\gamma \in \mathbb{Z}$, by Minkowski's inequality and change of variable, we have

$$\begin{aligned}&\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}|\mathcal{H}_\psi^p f(x)|^q dx\right)^{\frac{1}{q}} \le \int_{\mathbb{Z}_p^*}\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}|f(tx)|^q dx\right)^{\frac{1}{q}}\psi(t)dt\\ &= \int_{\mathbb{Z}_p^*}\left(\frac{1}{|B(0,p^\gamma|t|_p)|_H^{1+\lambda q}}\int_{B(0,p^\gamma|t|_p)}|f(y)|^q dy\right)^{\frac{1}{q}}|t|_p^{\lambda n}\psi(t)dt\\ &\le \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}\int_{\mathbb{Z}_p^*}|t|_p^{n\lambda}\psi(t)dt.\end{aligned}$$

Therefore, $\mathcal{H}_\psi^p$ is bounded on $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ and

$$\|\mathcal{H}_\psi^p\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)\to\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \le \int_{\mathbb{Z}_p^*}|t|_p^{n\lambda}\psi(t)dt. \tag{2.5}$$

On the other hand, suppose that $\mathcal{H}_\psi^p$ is bounded on $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$. Take $f_0(x) = |x|_p^{n\lambda}$, then

$$\begin{aligned}\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}|f_0(x)|^q dx &= p^{-n\gamma(1+\lambda q)}\sum_{k=-\infty}^{\gamma}\int_{S_k}p^{nk\lambda q}dx\\ &= (1-p^{-n})p^{-n\gamma(1+\lambda q)}\sum_{k=-\infty}^{\gamma}p^{nk(1+\lambda q)}\\ &= \frac{1-p^{-n}}{1-p^{-n(1+\lambda q)}},\end{aligned} \tag{2.6}$$

where the series converge due to $\lambda > -1/q$. Thus $f_0 \in \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$. Then by (2.3), we can see that

$$\int_{\mathbb{Z}_p^*}|t|_p^{n\lambda}\psi(t)dt \le \|\mathcal{H}_\psi^p\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)\to\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} < \infty. \tag{2.7}$$

And (2.5) and (2.7) yield the desired result. □

*Proof of Theorem 2.5.* Suppose that (2.1) holds and $f \in \mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)$. Let $\gamma \in \mathbb{Z}$ and denote $tB_\gamma = B(0,|t|_p p^\gamma)$, by Fubini theorem and change of variable, we have

$$\begin{aligned}(\mathcal{H}_\psi^p f)_{B_\gamma} &= \frac{1}{|B_\gamma|_H}\int_{B_\gamma}\left(\int_{\mathbb{Z}_p^*}f(tx)\psi(t)dt\right)dx\\ &= \int_{\mathbb{Z}_p^*}\left(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}f(tx)dx\right)\psi(t)dt\\ &= \int_{\mathbb{Z}_p^*}f_{tB_\gamma}\psi(t)dt.\end{aligned} \tag{2.8}$$

Using Minkowski's inequality, we get

$$\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}|\mathcal{H}_\psi^p f(x) - (\mathcal{H}_\psi^p f)_{B_\gamma}|^q dx\right)^{\frac{1}{q}}$$

$$= \left( \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} \left| \int_{\mathbb{Z}_p^*} (f(tx) - f_{tB_\gamma}) \psi(t) dt \right|^q dx \right)^{\frac{1}{q}}$$
$$\leq \int_{\mathbb{Z}_p^*} \left( \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} \left| f(tx) - f_{tB_\gamma} \right|^q dx \right)^{\frac{1}{q}} \psi(t) dt$$
$$= \int_{\mathbb{Z}_p^*} \left( \frac{1}{|tB_\gamma|_H^{1+\lambda q}} \int_{tB_\gamma} \left| f(y) - f_{tB_\gamma} \right|^q dy \right)^{\frac{1}{q}} |t|_p^{n\lambda} \psi(t) dt$$
$$\leq \|f\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda} \psi(t) dt.$$

Therefore, $\mathcal{H}_\psi^p$ is bounded on $\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)$ and

$$\|\mathcal{H}_\psi^p\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n) \to \mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)} \leq \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda} \psi(t) dt. \tag{2.9}$$

Conversely, if $\mathcal{H}_\psi^p$ is bounded on $\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)$, take $f_0(x) = |x|_p^{n\lambda}$, from (2.6) we can see that $f_0 \in \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$. Recall that $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ is continuously embedded in $\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)$. Therefore, $f_0 \in \mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)$. By (2.3) and (2.8), we get

$$\left( \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |\mathcal{H}_\psi^p f_0(x) - (\mathcal{H}_\psi^p f_0)_{B_\gamma}|^q dx \right)^{\frac{1}{q}}$$
$$= \left( \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} \left| f_0 - (f_0)_{B_\gamma} \right|^q dx \right)^{\frac{1}{q}} \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda} \psi(t) dt.$$

Therefore,

$$\|\mathcal{H}_\psi^p f_0\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)} = \|f_0\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda} \psi(t) dt,$$

which implies that

$$\|\mathcal{H}_\psi^p\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n) \to \mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)} \geq \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda} \psi(t) dt, \tag{2.10}$$

and

$$\int_{\mathbb{Z}_p^*} |t|_p^{n\lambda} \psi(t) dt < \infty.$$

This completes the theorem. □

## 3. Characterizations of weight functions via commutators

Recently, commutators of operators have been paid much attention due to their important applications. For example, some function spaces can be characterized in terms of commutators [10]. In this section, we consider the boundedness for commutators generated by $\mathcal{H}^p$ and $\lambda$-central BMO functions on $p$-adic central Morrey spaces.

**Definition 3.1.** The commutator between a function $b$ that is locally integrable on $\mathbb{Q}_p^n$ and the weighted $p$-adic Hardy operator $\mathcal{H}_\psi^p$ is defined by

$$\mathcal{H}_\psi^{p,b} f = b\mathcal{H}_\psi^p f - \mathcal{H}_\psi^p (bf), \tag{3.1}$$

for some suitable functions $f$.

We establish the following sufficient and necessary condition for weight functions to ensure that the commutators generated by weighted $p$-adic Hardy operators and $p$-adic central BMO functions are bounded on $p$-adic central Morrey spaces.

**Theorem 3.2.** *Let $1 < q < q_1 < \infty$, $1/q = 1/q_1 + 1/q_2$ and $-1/q_1 \leq \lambda < 0$. Assume that $\psi$ is a positive integrable function on $\mathbb{Z}_p^*$. Then for any $b \in \mathrm{CMO}^{q_2}(\mathbb{Q}_p^n)$ the commutator $\mathcal{H}_\psi^{p,b}$ is bounded from $\dot{B}^{q_1,\lambda}(\mathbb{Q}_p^n)$ to $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ if and only if*

$$\int_{\mathbb{Z}_p^*} \psi(t)|t|_p^{n\lambda} \log_p \frac{1}{|t|_p} dt < \infty. \tag{3.2}$$

*Remark* 3.3. Since $\psi : \mathbb{Z}_p^* \to [0, +\infty)$ is integrable, and $\log_p \frac{1}{|t|_p} \geq 1$ for $|t|_p \leq p^{-1}$, we have

$$\begin{aligned}
\int_{\mathbb{Z}_p^*} \psi(t)|t|_p^{n\lambda} dt &= \int_{0<|t|_p \leq p^{-1}} \psi(t)|t|_p^{n\lambda} dt + \int_{|t|_p=1} \psi(t)|t|_p^{n\lambda} dt \\
&\leq \int_{0<|t|_p \leq p^{-1}} \psi(t)|t|_p^{n\lambda} \log_p \frac{1}{|t|_p} dt + \int_{\mathbb{Z}_p^*} \psi(t) dt \\
&= \int_{\mathbb{Z}_p^*} \psi(t)|t|_p^{n\lambda} \log_p \frac{1}{|t|_p} dt + \int_{\mathbb{Z}_p^*} \psi(t) dt < \infty,
\end{aligned}$$

if (3.2) holds.

**Corollary 3.4.** Let $1 < q < q_1 < \infty$, $1/q = 1/q_1 + 1/q_2$, $-1/q_1 \leq \lambda < 0$ and $0 < \alpha < 1$. Then for any $b \in \mathrm{CMO}^{q_2}(\mathbb{Q}_p)$,
(I) the commutator $H^{p,b}$ is bounded from $\dot{B}^{q_1,\lambda}(\mathbb{Q}_p)$ to $\dot{B}^{q,\lambda}(\mathbb{Q}_p)$,
(II) the commutator $R_\alpha^{p,b}$ is bounded from $\dot{B}^{q_1,\lambda}(\mathbb{Q}_p)$ to $\dot{B}^{q,\lambda}(|x|^{-\alpha q}dx)$,
and for any $b \in \mathrm{CMO}^{q_2}(\mathbb{Q}_p^n)$,
(III) the commutator $\mathcal{H}^{p,b}$ is bounded from $\dot{\mathcal{B}}^{q_1,\lambda}(\mathbb{Q}_p^n)$ to $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$, where $\dot{\mathcal{B}}^{q_1,\lambda}(\mathbb{Q}_p^n)$ is defined in Corollary 2.4.

When $b \in \mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)$ with $\lambda \neq 0$, we have the following conclusion.

**Theorem 3.5.** *Let $1 < q < q_1 < \infty$, $1/q = 1/q_1 + 1/q_2$, $-1/q < \lambda < 0$, $-1/q_1 < \lambda_1 < 0$, $0 < \lambda_2 < \frac{1}{n}$ and $\lambda = \lambda_1 + \lambda_2$. If*

$$\int_{\mathbb{Z}_p^*} \psi(t)|t|_p^{n\lambda_1} dt < \infty, \tag{3.3}$$

*then for any $b \in \mathrm{CMO}^{q_2,\lambda_2}(\mathbb{Q}_p^n)$, the corresponding commutator $\mathcal{H}_\psi^{p,b}$ is bounded from $\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)$ to $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$.*

We have got the values of $\int_{\mathbb{Z}_p^*} \psi(t)|t|_p^{n\lambda} dt$ in Corollary 2.2, 2.4 and 2.6, Thus by Theorem 3.5, we obtain the following result.

**Corollary 3.6.** Let $1 < q < q_1 < \infty$, $1/q = 1/q_1 + 1/q_2$, $-1/q < \lambda < 0$, $-1/q_1 < \lambda_1 < 0$.

(I) If $0 < \lambda_2 < 1$, $\lambda = \lambda_1 + \lambda_2$ and $0 < \alpha < 1$, then for any $b \in \mathrm{CMO}^{q_2,\lambda_2}(\mathbb{Q}_p)$,
(i) the commutator $H^{p,b}$ is bounded from $\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p)$ to $\dot{B}^{q,\lambda}(\mathbb{Q}_p)$.
(ii) the commutator $R_\alpha^{p,b}$ is bounded from $\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p)$ to $\dot{B}^{q,\lambda}(|x|^{-\alpha q}dx)$.

(II) If $0<\lambda_2<\frac{1}{n}$ and $\lambda=\lambda_1+\lambda_2$, then for any $b\in \mathrm{CMO}^{q_2,\lambda_2}(\mathbb{Q}_p^n)$, the commutator $\mathcal{H}^{p,b}$ is bounded from $\dot{\mathcal{B}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)$ to $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$, where $\dot{\mathcal{B}}^{q_1,\lambda}(\mathbb{Q}_p^n)$ is defined in Corollary 2.4.

Before proving these theorems, we need the following result. One can refer to (Lemma 15 in [31]) for another version, here we give a more accurate estimation.

**Lemma 3.7.** *Suppose that $b\in \mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)$ and $j,k\in\mathbb{Z}$.*
*(I). If $\lambda>0$, then*

$$|b_{B_j}-b_{B_k}|\le \frac{p^n(1+p^{-|k-j|n\lambda})}{1-p^{-n\lambda}}\|b\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)}\max\left\{|B_j|_H^{\lambda},|B_k|_H^{\lambda}\right\}.$$

*(II). If $\lambda=0$, then*

$$|b_{B_j}-b_{B_k}|\le p^n|j-k|\|b\|_{\mathrm{CMO}^{q}(\mathbb{Q}_p^n)}.$$

*Proof.* Without loss of generality, we may assume that $k>j$. Recall that $b_{B_i}=\frac{1}{|B_i|_H}\int_{B_i}b(x)dx$. By Hölder's inequality, we have

$$\begin{aligned}
|b_{B_{i+1}}-b_{B_i}| &\le \frac{1}{|B_i|_H}\int_{B_i}|b(x)-b_{B_{i+1}}|dx\le \frac{1}{|B_i|_H}\int_{B_{i+1}}|b(x)-b_{B_{i+1}}|dx\\
&\le \frac{1}{|B_i|_H}\left(\int_{B_{i+1}}|b(x)-b_{B_{i+1}}|^q dx\right)^{\frac1q}|B_{i+1}|_H^{1-\frac1q}\\
&\le \frac{|B_{i+1}|_H^{1+\lambda}}{|B_i|_H}\|b\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)}=p^n|B_{i+1}|_H^{\lambda}\|b\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)}.
\end{aligned}$$

Therefore, if $\lambda>0$, then

$$\begin{aligned}
|b_{B_j}-b_{B_k}| &\le \sum_{i=j}^{k-1}|b_{B_{i+1}}-b_{B_i}|\le p^n\|b\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)}\sum_{i=j}^{k-1}|B_{i+1}|_H^{\lambda}\\
&=\frac{p^n\left(1+p^{-(k-j)n\lambda}\right)}{1-p^{-n\lambda}}\|b\|_{\mathrm{CMO}^{q,\lambda}(\mathbb{Q}_p^n)}|B_k|_H^{\lambda}.
\end{aligned}$$

If $\lambda=0$, then

$$|b_{B_j}-b_{B_k}|\le \sum_{i=j}^{k-1}|b_{B_{i+1}}-b_{B_i}|=(k-j)p^n\|b\|_{\mathrm{CMO}^{q}(\mathbb{Q}_p^n)}.$$

$\square$

*Proof of Theorem 3.2.* Let $\gamma\in\mathbb{Z}$ and denote $tB_\gamma=B(0,|t|_pp^{\gamma})$. Assume that (3.2) holds, by definition, we have

$$\begin{aligned}
&\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}|\mathcal{H}_\psi^{p,b}f(x)|^q dx\right)^{\frac1q}\\
&\le \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}\left(\int_{\mathbb{Z}_p^*}|(b(x)-b(tx))f(tx)|\psi(t)dt\right)^q dx\right)^{\frac1q}\\
&\le \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}\left(\int_{\mathbb{Z}_p^*}|(b(x)-b_{B_\gamma})f(tx)|\psi(t)dt\right)^q dx\right)^{\frac1q} \qquad (3.4)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}\left(\int_{\mathbb{Z}_p^*}|(b_{B_\gamma}-b_{tB_\gamma})f(tx)|\psi(t)dt\right)^q dx\right)^{\frac{1}{q}}\\
&+\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}\left(\int_{\mathbb{Z}_p^*}|(b(tx)-b_{tB_\gamma})f(tx)|\psi(t)dt\right)^q dx\right)^{\frac{1}{q}}\\
&:= I_1+I_2+I_3.
\end{aligned}
$$

In the following, we will estimate $I_1,\ I_2$ and $I_3$, respectively. For $I_1$, by Minkowski's inequality and Hölder's inequality $(1/q = 1/q_1 + 1/q_2)$, we get

$$
\begin{aligned}
I_1 &\le \int_{\mathbb{Z}_p^*}\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}|(b(x)-b_{B_\gamma})f(tx)|^q dx\right)^{\frac{1}{q}}\psi(t)dt\\
&\le |B_\gamma|_H^{-\frac{1}{q}-\lambda}\int_{\mathbb{Z}_p^*}\left(\int_{B_\gamma}|b(x)-b_{B_\gamma}|^{q_2}dx\right)^{\frac{1}{q_2}}\left(\int_{B_\gamma}|f(tx)|^{q_1}dx\right)^{\frac{1}{q_1}}\psi(t)dt\\
&\le \|b\|_{CMO^{q_2}(\mathbb{Q}_p^n)}\int_{\mathbb{Z}_p^*}\left(\frac{1}{|tB_\gamma|_H^{1+\lambda q_1}}\int_{tB_\gamma}|f(y)|^{q_1}dy\right)^{\frac{1}{q_1}}|t|_p^{n\lambda}\psi(t)dt\\
&\le \|b\|_{CMO^{q_2}(\mathbb{Q}_p^n)}\|f\|_{\dot{B}^{q_1,\lambda}(\mathbb{Q}_p^n)}\int_{\mathbb{Z}_p^*}|t|_p^{n\lambda}\psi(t)dt.
\end{aligned}
\tag{3.5}
$$

Similarly, we have

$$
\begin{aligned}
I_3 &\le \int_{\mathbb{Z}_p^*}\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}|(b(tx)-b_{tB_\gamma})f(tx)|^q dx\right)^{\frac{1}{q}}\psi(t)dt\\
&\le \int_{\mathbb{Z}_p^*}\left(\frac{1}{|tB_\gamma|_H}\int_{tB_\gamma}|b(y)-b_{tB_\gamma}|^{q_2}dy\right)^{\frac{1}{q_2}}\\
&\quad\times\left(\frac{1}{|tB_\gamma|_H^{1+\lambda q_1}}\int_{tB_\gamma}|f(y)|^{q_1}dy\right)^{\frac{1}{q_1}}|t|_p^{n\lambda}\psi(t)dt\\
&\le \|b\|_{CMO^{q_2}(\mathbb{Q}_p^n)}\|f\|_{\dot{B}^{q_1,\lambda}(\mathbb{Q}_p^n)}\int_{\mathbb{Z}_p^*}|t|_p^{n\lambda}\psi(t)dt.
\end{aligned}
\tag{3.6}
$$

For $I_2$, by Minkowski's inequality and Hölder's inequality $(1/q = 1/q_1 + 1/q_2)$, we have

$$
\begin{aligned}
I_2 &\le \int_{\mathbb{Z}_p^*}\left(\frac{1}{|B_\gamma|_H^{1+\lambda q}}\int_{B_\gamma}|f(tx)|^q dx\right)^{\frac{1}{q}}|b_{B_\gamma}-b_{tB_\gamma}|\psi(t)dt\\
&\le \int_{\mathbb{Z}_p^*}\left(\frac{1}{|B_\gamma|_H^{1+\lambda q_1}}\int_{B_\gamma}|f(tx)|^{q_1}dx\right)^{\frac{1}{q_1}}|b_{B_\gamma}-b_{tB_\gamma}|\psi(t)dt\\
&= \int_{\mathbb{Z}_p^*}\left(\frac{1}{|tB_\gamma|_H^{1+\lambda q_1}}\int_{tB_\gamma}|f(y)|^{q_1}dy\right)^{\frac{1}{q_1}}|t|_p^{n\lambda}|b_{B_\gamma}-b_{tB_\gamma}|\psi(t)dt
\end{aligned}
\tag{3.7}
$$

$$\leq \|f\|_{\dot{B}^{q_1,\lambda}(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda} |b_{B_\gamma} - b_{tB_\gamma}| \psi(t) dt.$$

Note that $t \in \mathbb{Z}_p^*$, thus $|t|_p \leq 1$. By Lemma 3.7 for $\lambda = 0$, we get

$$|b_{B_\gamma} - b_{tB_\gamma}| = \sum_{k=\gamma+\log_p |t|_p}^{\gamma-1} |b_{B_{k+1}} - b_{B_k}| \leq p^n \|b\|_{\mathrm{CMO}^{q_2}(\mathbb{Q}_p^n)} \log_p \frac{1}{|t|_p}$$

Therefore,

$$I_2 \leq p^n \|b\|_{\mathrm{CMO}^{q_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1,\lambda}(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^*} \psi(t) |t|_p^{n\lambda} \log_p \frac{1}{|t|_p} dt. \tag{3.8}$$

By (3.2) and Remark 3.3 and then combine with the inequalities (3.5)-(3.8), we obtain

$$\begin{aligned} &\|\mathcal{H}_\psi^{p,b}\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n) \to \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \\ &\leq C \|b\|_{\mathrm{CMO}^{q_2}(\mathbb{Q}_p^n)} \left( \int_{\mathbb{Z}_p^*} \psi(t) |t|_p^{n\lambda} + \int_{\mathbb{Z}_p^*} \psi(t) |t|_p^{n\lambda} \log_p \frac{1}{|t|_p} dt \right) < \infty. \end{aligned}$$

On the other hand, suppose that $\mathcal{H}_\psi^{p,b}$ is bounded from $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ to $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ and $b \in \mathrm{CMO}^{q_2}(\mathbb{Q}_p^n)$, we will show that (3.2) holds.

In fact, take $b_0(x) = \log_p |x|_p$, $x \in \mathbb{Q}_p^n$. From Lemma 2.1 in [22], we can see that $b_0 \in \mathrm{BMO}(\mathbb{Q}_p^n)$. By Corollary 5.17 in [18], $\|\cdot\|_{\mathrm{BMO}(\mathbb{Q}_p^n)}$ and $\|\cdot\|_{\mathrm{BMO}^q(\mathbb{Q}_p^n)}$ are equivalent. Therefore, $b_0(x) \in \mathrm{BMO}^{q_2}(\mathbb{Q}_p^n) \subset \mathrm{CMO}^{q_2}(\mathbb{Q}_p^n)$. By assumption, we have

$$\|\mathcal{H}_\psi^{p,b_0}\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n) \to \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} < \infty. \tag{3.9}$$

We will also take $f_0(x) = |x|_p^{n\lambda}$, from (2.6) we can see that $f_0 \in \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$, and $\|f_0\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} = \frac{1-p^{-n}}{1-p^{-n(1+\lambda q)}}$. Since

$$\begin{aligned} \mathcal{H}_\psi^{p,b_0} f_0(x) &= \int_{\mathbb{Z}_p^*} (b_0(x) - b_0(tx)) f_0(tx) \psi(t) dt \\ &= \int_{\mathbb{Z}_p^*} (\log_p |x|_p - \log_p |tx|_p) |tx|_p^{n\lambda} \psi(t) dt \\ &= f_0(x) \int_{\mathbb{Z}_p^*} \psi(t) |t|_p^{n\lambda} \log_p \frac{1}{|t|_p} dt. \end{aligned}$$

Using Hölder's inequality ($1 = q/q_1 + q/q_2$), we have

$$\begin{aligned} &\|\mathcal{H}_\psi^{p,b_0} f_0\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \\ &= \sup_{\gamma \in \mathbb{Z}} \left( \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |\mathcal{H}_\psi^{p,b_0} f_0(x)|^q \right)^{\frac{1}{q}} \\ &= \sup_{\gamma \in \mathbb{Z}} \left( \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f_0(x)|^q \right)^{\frac{1}{q}} \int_{\mathbb{Z}_p^*} \psi(t) |t|_p^{n\lambda} \log_p \frac{1}{|t|_p} dt \\ &= \frac{1-p^{-n}}{1-p^{-n(1+\lambda q)}} \int_{\mathbb{Z}_p^*} \psi(t) |t|_p^{n\lambda} \log_p \frac{1}{|t|_p} dt \end{aligned}$$

$$= \frac{1-p^{-n(1+\lambda q_1)}}{1-p^{-n(1+\lambda q)}} \cdot \frac{1-p^{-n}}{1-p^{-n(1+\lambda q_1)}} \int_{\mathbb{Z}_p^*} \psi(t)|t|_p^{n\lambda} \log_p \frac{1}{|t|_p} dt$$
$$= C_{q,q_1} \|f_0\|_{\dot{B}^{q_1,\lambda}(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^*} \psi(t)|t|_p^{n\lambda} \log_p \frac{1}{|t|_p} dt.$$

Therefore,

$$\|\mathcal{H}_\psi^{p,b_0}\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)\to\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \geq C_{q,q_1} \int_{\mathbb{Z}_p^*} \psi(t)|t|_p^{n\lambda} \log_p \frac{1}{|t|_p} dt.$$

Then by (3.9), we obtain

$$\int_{\mathbb{Z}_p^*} \psi(t)|t|_p^{n\lambda} \log_p \frac{1}{|t|_p} dt < \infty.$$

The proof is complete. □

*Proof of Corollary 3.4.* (1) When $\psi \equiv 1$ and $n = 1$, we have $H_\psi f = H^p f$. Since

$$\int_{\mathbb{Z}_p^*} |t|_p^{\lambda_1} \log_p \frac{1}{|t|_p} dt = \sum_{k=0}^{\infty} \int_{S_{-k}} |t|_p^{\lambda_1} \log_p \frac{1}{|t|_p} dt = (1-p^{-1}) \sum_{k=0}^{\infty} kp^{-k(1+\lambda_1)} < \infty.$$

We can get Corollary 3.4 (I) directly from Theorem 3.2.

(2) For $n = 1$, if we take $\psi(t) = (1-p^{-\alpha})|1-t|_p^{\alpha-1}\chi_{B_0\setminus S_0}(t)/(1-p^{\alpha-1})$, then $\mathcal{H}_\psi^p f(x) = |x|_p^{-\alpha} R_\alpha^p f(x)$. At this time

$$\int_{\mathbb{Z}_p^*} \psi(t)|t|_p^{\lambda_1} \log_p \frac{1}{|t|_p} dt = \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} \int_{0<|t|_p<1} |t|_p^{\lambda_1} |1-t|_p^{\alpha-1} \log_p \frac{1}{|t|_p} dt$$
$$= \frac{1-p^{-\alpha}}{1-p^{\alpha-1}} \int_{0<|t|_p<1} |t|_p^{\lambda_1} \log_p \frac{1}{|t|_p} dt$$
$$= \frac{(1-p^{-\alpha})(1-p^{-1})}{1-p^{\alpha-1}} \sum_{k=1}^{\infty} kp^{-k(1+\lambda_1)} < \infty.$$

Then Corollary 3.4 (II) follows from Theorem 3.2.

(3) For $n \geq 2$, if we take $\psi(t) = (1-p^{-n})|t|_p^{n-1}/(1-p^{-1})$, and $f$ satisfies $f(x) = f(|x|_p^{-1})$, then $\mathcal{H}_\psi^p f(x) = \mathcal{H}^p f(x)$, and

$$\int_{\mathbb{Z}_p^*} \psi(t)|t|_p^{\lambda_1} \log_p \frac{1}{|t|_p} dt = \frac{1-p^{-n}}{1-p^{-1}} \int_{\mathbb{Z}_p^*} |t|_p^{(1+\lambda_1)n-1} \log_p \frac{1}{|t|_p} dt$$
$$= (1-p^{-n}) \sum_{k=0}^{\infty} kp^{-k(1+\lambda_1)n} < \infty.$$

Therefore, Corollary 3.4 (III) holds. □

*Proof of Theorem 3.5.* As in the proof of Theorem 3.2, we can write

$$\left( \frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |\mathcal{H}_\psi^{p,b} f(x)|^q dx \right)^{\frac{1}{q}} = I_1 + I_2 + I_3,$$

where $I_1,\ I_2,\ I_3$ are the ones in (3.4).

By the similar estimates to (3.5) and (3.6), we have

$$I_1 \le \|b\|_{CMO^{q_2,\lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda_1} \psi(t)dt,$$

$$I_3 \le \|b\|_{CMO^{q_2,\lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda_1} \psi(t)dt.$$

For $I_2$, like (3.7), we have

$$I_2 \le \frac{1}{|B_\gamma|_H^{\lambda_2}} \|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^*} |t|_p^{n\lambda_1} |b_{B_\gamma} - b_{tB_\gamma}| \psi(t)dt.$$

By Lemma 3.7 and the fact that if $t \in \mathbb{Z}_p^*$ then $\log_p |t|_p$ is a nonnegative integer, we get

$$\begin{aligned} |b_{B_\gamma} - b_{tB_\gamma}| = |b_{B_\gamma} - b_{B_{\gamma+\log_p |t|_p}}| &\le \frac{p^n(1+p^{n\lambda_2 \log_p |t|_p})}{1-p^{-n\lambda_2}} \|b\|_{CMO^{q_2,\lambda_2}(\mathbb{Q}_p^n)} |B_\gamma|_H^{\lambda_2} \\ &\le \frac{2p^n}{1-p^{-n\lambda_2}} \|b\|_{CMO^{q_2,\lambda_2}(\mathbb{Q}_p^n)} |B_\gamma|_H^{\lambda_2} \end{aligned}$$

Consequently,

$$I_2 \le \frac{2p^n}{1-p^{-n\lambda_2}} \|b\|_{CMO^{q_2,\lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^*} \psi(t) |t|_p^{n\lambda_1} dt.$$

The estimates of $I_1,\ I_2,\ I_3$ imply that

$$\|\mathcal{H}_\psi^{p,b} f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \le C \|b\|_{CMO^{q_2,\lambda_2}(\mathbb{Q}_p^n)} \|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^*} \psi(t) |t|_p^{n\lambda_1} dt.$$

Theorem 3.5 is proved. □

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