CHENG-YAU OPERATOR AND GAUSS MAP OF SURFACES OF REVOLUTION

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Abstract. We study the Gauss map G of surfaces of revolution in the 3-dimensional Euclidean space $\mathbb{E}^3$ with respect to the so called Cheng-Yau operator $\Box$ acting on the functions defined on the surfaces. As a result, we establish the classification theorem that the only surfaces of revolution with Gauss map G satisfying $\Box G = AG$ for some 3×3 matrix A are the planes, right circular cones, circular cylinders and spheres.

1. Introduction

The theory of Gauss map of a surface in a Euclidean space and a pseudo-Euclidean space is always one of interesting topics and it has been investigated from the various viewpoints by many differential geometers ([2, 3, 8, 9, 10, 6, 11, 13, 14, 15, 16, 18, 19, 20, 21]).

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Let M be a surface of the Euclidean 3-space $\mathbb{E}^3$. The map $G : M \to S^2 \subset \mathbb{E}^3$ which sends each point of M to the unit normal vector to M at the point is called the *Gauss map* of the surface M, where S^2 is the unit sphere in $\mathbb{E}^3$ centered at the origin. It is well known that M has constant mean curvature if and only if $\Delta G = ||dG||^2 G$, where Δ is the Laplace operator on M corresponding to the induced metric on M from $\mathbb{E}^3$ ([23]). Surfaces whose Gauss map is an eigenfunction of Laplacian, that is, $\Delta G = \lambda G$ for some constant $\lambda \in R$, are the planes, circular cylinders and spheres ([6]).

Generalizing this equation, F. Dillen, J. Pas and L. Verstraelen ([11]) studied surfaces of revolution in a Euclidean 3-space $\mathbb{E}^3$ such that its Gauss map G satisfies the condition

$$\Delta G = AG, \quad A \in R^{3\times 3}. \tag{1.1}$$

As a result, they proved ([11])

Proposition 1.1. *Among the surfaces of revolution in* $\mathbb{E}^3$*, the only ones whose Gauss map satisfies (1.1) are the planes, the spheres and the circular cylinders.*

Baikoussis and Blair also studied ruled surfaces in $\mathbb{E}^3$ and proved ([2])

Proposition 1.2. *Among the ruled surfaces in* $\mathbb{E}^3$*, the only ones whose Gauss map satisfies* (1.1) *are the planes and the circular cylinders.*

Generalized slant cylindrical surfaces (GSCS's) are natural extended notion of surfaces of revolution ([17]). Surfaces of revolution, cylindrical surfaces and tubes along a plane curve are special cases of GSCS's. In [19], the first author and B. Song proved that among the GSCS's in $\mathbb{E}^3$, the only ones whose Gauss map satisfies (1.1) are the planes, the spheres and the circular cylinders.

The so-called Cheng-Yau operator $\Box$ (or, L_1) is a natural extension of the Laplace operator Δ (cf. [1], [7]). Hence, following the condition (1.1), it is natural to ask as follows.

Question 1.3. Among the surfaces of revolution in a Euclidean 3-space $\mathbb{E}^3$, which one satisfy the following condition?

$$\Box G = AG, \quad A \in R^{3\times 3}. \tag{1.2}$$

In this paper, we give a complete answer to the above question.

Throughout this paper, we assume that all objects are smooth and connected, unless otherwise mentioned.

2. Cheng-Yau operator and Lemmas

Let M be an oriented surface in E^3 with Gauss map G. We denote by S the shape operator of M with respect to the Gauss map G. For each $k = 0, 1$, we put $P_0 = I, P_1 = tr(S)I - S$, where I is the identity operator acting on the tangent bundle of M. Let us define an operator $L_k : C^\infty(M) \to C^\infty(M)$ by $L_k(f) = -tr(P_k \circ \nabla^2 f)$, where $\nabla^2 f : \chi(M) \to \chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the hessian of f. Then, up to signature, L_k is the linearized operator of the first variation of the $(k+1)$-th mean curvature arising from normal variations of the surface. Note that the operator L_0 is nothing but the Laplace operator acting on M, i.e., $L_0 = \Delta$ and $L_1 = \Box$ is called the Cheng-Yau operator introduced in [7].

Now, we state a useful lemma as follows ([1]).

Lemma 2.1. *Let M be an oriented surface in E^3 with Gaussian curvature K and mean curvature H. Then, the Gauss map G of M satisfies*

$$\square G = \nabla K + 2HKG, \tag{2.1}$$

where ∇K denotes the gradient of K.

Now, using Lemma 2.1 we give some examples of surfaces with Gauss map satisfying (1.2).

Examples.

(1) Flat surfaces. In this case, we have $\square G = 0$, and hence flat surfaces satisfy $\square G = AG$ for some 3×3 matrix A. Note that the matrix A must be singular.

(2) Spheres: $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$. In this case, we have $G = \frac{1}{r}(x-a, y-b, z-c)$ so the sphere satisfies $\square G = AG$ with $A = \frac{-1}{r^3}I$, where I denotes the identity matrix.

3. Gauss map of surfaces of revolution

We consider a unit speed plane curve $C : (x(s), 0, z(s))$ with $x(s) > 0$ in the xz plane which is defined on an interval I. By rotating the curve C around z-axis, we get a surface of revolution M, which is parametrized by

$$X(s,t) = (x(s)\cos t, x(s)\sin t, z(s)). \tag{3.1}$$

The adapted frame field $\{e_1, e_2, G\}$ on the surface of revolution M are given by

$$\begin{aligned} e_1 &= X_s = (x'(s)\cos t, x'(s)\sin t, z'(s)), \\ e_2 &= \frac{1}{x}X_t = (-\sin t, \cos t, 0), \\ G &= e_1 \times e_2 = (-z'\cos t, -z'\sin t, x'). \end{aligned} \tag{3.2}$$

The principal curvatures k_1, k_2 of M with respect to the Gauss map G are respectively ([12])

$$\begin{aligned} k_1 &= \langle S(e_1), e_1\rangle = x'z'' - x''z' = \kappa, \\ k_2 &= \langle S(e_2), e_2\rangle = \frac{z'}{x}, \end{aligned} \tag{3.3}$$

where S and κ denote the shape operator of M and the plane curvature of the plane curve C, respectively.

Since the parametrization $(x(s), 0, z(s))$ of the plane curve C is of unit speed, there exists a smooth function $\theta = \theta(s)$ such that $x' = \cos\theta$ and $z' = \sin\theta$. Then, the Gaussian curvature K and the mean curvature H of M are, respectively, given by

$$\begin{aligned} K &= k_1k_2 = \frac{\theta'(s)\sin\theta}{x}, \\ 2H &= k_1 + k_2 = \theta'(s) + \frac{\sin\theta}{x}. \end{aligned} \tag{3.4}$$

Hence, the gradient ∇K of the Gaussian curvature K of M is given by

$$\nabla K = K'(s)e_1, \tag{3.5}$$

where

$$K'(s) = \frac{1}{x^2}\{x\theta''(s)\sin\theta + x\theta'(s)^2\cos\theta - \theta'(s)\cos\theta\sin\theta\}. \tag{3.6}$$

We now suppose that the Gauss map G of the surface of revolution M satisfies for a 3×3 matrix $A = (a_{ij})$

$$\Box G = AG. \tag{3.7}$$

Recall that the Gauss map G is given by

$$G(s,t) = (-\sin\theta\cos t, -\sin\theta\sin t, \cos\theta). \tag{3.8}$$

Then, it follows from (2.1), (3.4) and (3.5) that

$$\begin{aligned}&\{K'(s)\cos\theta - 2KH\sin\theta\}\cos t\\ &= -a_{11}\sin\theta\cos t - a_{12}\sin\theta\sin t + a_{13}\cos\theta,\end{aligned} \tag{3.9}$$

$$\begin{aligned}&\{K'(s)\cos\theta - 2KH\sin\theta\}\sin t\\ &= -a_{21}\sin\theta\cos t - a_{22}\sin\theta\sin t + a_{23}\cos\theta\end{aligned} \tag{3.10}$$

and

$$K'(s)\sin\theta + 2KH\cos\theta = -a_{31}\sin\theta\cos t - a_{32}\sin\theta\sin t + a_{33}\cos\theta. \tag{3.11}$$

First, we suppose that the set $J = \{s \in I | \theta'(s) \neq 0\}$ is nonempty. Then $\theta(I)$ contains an interval, hence we get from (3.9)-(3.11) that $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$ and $a_{11} = a_{22}$. Thus we obtain $A = diag(\lambda, \lambda, \mu)$,

$$K'(s)\cos\theta - 2KH\sin\theta = -\lambda\sin\theta, \tag{3.12}$$

and

$$K'(s)\sin\theta + 2KH\cos\theta = \mu\cos\theta. \tag{3.13}$$

Note that (3.12) and (3.13) are equivalent to the following:

$$K'(s) = a\cos\theta\sin\theta, \tag{3.14}$$

and

$$2KH = -a\sin^2\theta + \mu, \tag{3.15}$$

where we put $a = \mu - \lambda$.

We prove the following lemma, which plays a crucial role in the proof of our main theorem.

Lemma 3.1. Let M be a surface of revolution given by (3.1) with nonempty set $J = \{s \in I | \theta'(s) \neq 0\}$. Suppose that the Gauss map G of M satisfies $\Box G = AG$ for some 3×3 matrix A. Then A is of the form λI, where I is an identity matrix.

Proof. The above discussions show that A is a diagonal matrix of the form $A = diag(\lambda, \lambda, \mu)$ for some constants λ and μ. We put $a = \mu - \lambda$. Then, it follows from (3.4), (3.6), (3.14) and (3.15) that

$$x\theta''(s) \sin\theta + x\theta'(s)^2 \cos\theta - \theta'(s) \cos\theta \sin\theta = ax^2 \cos\theta \sin\theta \tag{3.16}$$

and

$$x\theta'(s)^2 \sin\theta + \theta'(s) \sin^2\theta = (-a \sin^2\theta + \mu)x^2. \tag{3.17}$$

By differentiating the both sides of (3.17) with respect to s, we get

$$\begin{aligned} &\theta''(s) \sin^2\theta + 2x\theta'(s)\theta''(s) \sin\theta + x\theta'(s)^3 \cos\theta + 3\theta'(s)^2 \sin\theta \cos\theta \\ &+ 2ax^2\theta'(s) \sin\theta \cos\theta = 2x \cos\theta(-a \sin^2\theta + \mu). \end{aligned} \tag{3.18}$$

If we substitute $\theta''(s)$ in (3.16) into (3.18), then we have

(3.19)

$$\begin{aligned} &- x^2\theta'(s)^3 \cos\theta + 4x\theta'(s)^2 \cos\theta \sin\theta + \{\cos\theta \sin^2\theta + 4ax^3 \cos\theta \sin\theta\}\theta'(s) \\ &+ 3ax^2 \cos\theta \sin^2\theta - 2\mu x^2 \cos\theta = 0. \end{aligned}$$

Let us substitute $\theta'(s)^2$ in (3.17) into (3.19). Then we obtain

$$\begin{aligned} &5x\theta'(s)^2 \cos\theta \sin\theta + \{\cos\theta \sin^2\theta + 5ax^3 \cos\theta \sin\theta - \mu x^3 \cot\theta\}\theta'(s) \\ &+ 3ax^2 \cos\theta \sin^2\theta - 2\mu x^2 \cos\theta = 0. \end{aligned} \tag{3.20}$$

Once more, we substitute $\theta'(s)^2$ in (3.17) into (3.20). Then we get

$$\theta'(s) = \frac{\gamma x^2}{\alpha x^3 + \beta}, \tag{3.21}$$

where we put

$$\begin{aligned} &\alpha(s) = -5a\sin^2\theta(s) + \mu, \beta(s) = 4\sin^3\theta(s), \\ &\gamma(s) = -2a\sin^3\theta(s) + 3\mu\sin\theta(s). \end{aligned} \tag{3.22}$$

Now, we replace $\theta'(s)$ in (3.17) with that in (3.21). Then we have

$$a_6x^6 + a_3x^3 + a_0 = 0, \tag{3.23}$$

where we use the following notations:

$$\begin{aligned} a_6(\theta) &= 25a^3\cos^6\theta + 5a^2(15\lambda - 8\mu)\cos^4\theta \\ &\quad + a(5a-\mu)(4\mu - 15\lambda)\cos^2\theta + \lambda(5a-\mu)^2, \end{aligned} \tag{3.24}$$

$$a_3(\theta) = 26a^2\sin^7\theta - 19a\mu\sin^5\theta - 4\mu^2\sin^3\theta \tag{3.25}$$

and

$$a_0(\theta) = -8a\sin^8\theta + 4\mu\sin^6\theta. \tag{3.26}$$

Let us differentiate (3.23) with respect to s. Here, we denote by $\dot{a}_i(\theta)$ the derivative of $a_i(\theta)$ with respect to $\theta, i = 0, 3, 6$. Using $x' = \cos\theta$ and $\theta'(s)$ given by (3.21), we get

$$b_6x^6 + b_3x^3 + b_0 = 0, \tag{3.27}$$

where we denote

$$b_6(\theta) = 6\alpha\cos\theta a_6(\theta) + \gamma\dot{a}_6(\theta), \tag{3.28}$$

$$b_3(\theta) = 3\alpha\cos\theta a_3(\theta) + 6\beta\cos\theta a_6(\theta) + \gamma\dot{a}_3(\theta) \tag{3.29}$$

and

$$b_0(\theta) = 3\beta\cos\theta a_3(\theta) + \gamma\dot{a}_0(\theta). \tag{3.30}$$

If we compute $b_i(\theta)$ for $i = 0, 3, 6$, then we have

$$b_6(\theta) = \{1050a^4 \sin^8 \theta + \sum_{i=0}^{6} p_i(\lambda, \mu) \sin^i \theta\} \cos \theta, \tag{3.31}$$

$$b_3(\theta) = \{-214a^3 \sin^9 \theta + \sum_{i=0}^{7} q_i(\lambda, \mu) \sin^i \theta\} \cos \theta \tag{3.32}$$

and

$$b_0(\theta) = \{440a^2 \sin^{10} \theta + \sum_{i=0}^{8} r_i(\lambda, \mu) \sin^i \theta\} \cos \theta, \tag{3.33}$$

where $p_i(\lambda, \mu), q_i(\lambda, \mu)$ and $r_i(\lambda, \mu)$ are respectively some polynomials in λ and μ.

Eliminating x^6, it follows from (3.23) and (3.27) that

$$c_3 x^3 + c_0 = 0, \tag{3.34}$$

where

$$c_3 = a_3 b_6 - b_3 a_6, c_0 = a_0 b_6 - b_0 a_6. \tag{3.35}$$

Due to (3.24)-(3.26) and (3.31)-(3.33)3), we may compute c_3 and c_0 as follows:

$$c_3 = \{32650a^6 \sin^{15} \theta + \sum_{j=0}^{13} p_{3j}(\lambda, \mu) \sin^j \theta\} \cos \theta, \tag{3.36}$$

and

$$c_0 = \{-19400a^5 \sin^{16} \theta + \sum_{j=0}^{14} p_{0j}(\lambda, \mu) \sin^j \theta\} \cos \theta, \tag{3.37}$$

where each $p_{ij}(\lambda, \mu)(i = 0, 3)$ is a polynomial in λ and μ.

Let us replace x^3 in (3.23) with $x^3 = -c_0/c_3$ given in (3.34). Then we have

$$a_6 c_0^2 - a_3 c_0 c_3 + a_0 c_3^2 = 0. \tag{3.38}$$

Using (3.31)-(3.33), (3.36) and (3.37), we may compute the leading terms of those in (3.38) as follows:

$$\begin{aligned} a_6c_0^2 &= -25(19400)^2a^{13}\sin^{40}\theta + \text{lower degree terms in} \sin\theta, \\ a_3c_0c_3 &= 26(19400)(32650)a^{13}\sin^{40}\theta + \text{lower degree terms in} \sin\theta, \\ a_0c_3^2 &= 8(32650)^2a^{13}\sin^{40}\theta + \text{lower degree terms in} \sin\theta. \end{aligned} \tag{3.39}$$

Hence we obtain

$$\begin{aligned} a_6c_0^2 - a_3c_0c_3 + a_0c_3^2 = &-17349480000a^{13}\sin^{40}\theta \\ &+ \text{lower degree terms in} \sin\theta. \end{aligned} \tag{3.40}$$

Since $\theta(I)$ contains an interval, together with (3.38), (3.40) shows that a must be zero. Thus we have $\mu = \lambda$ and hence $A = \lambda I$. This completes the proof. □

4. Main Theorems and Corollaries

Finally, we prove the main theorem as follows.

Theorem 4.1. *Let M be a surface of revolution. Then the Gauss map G of M satisfies $\square G = AG$ for some 3×3 matrix A if and only if M is an open part of the following surfaces:*

1) a plane,

2) a right circular cone,

3) a circular cylinder,

4) a sphere.

Proof. We consider a surface of revolution M obtained by rotating the unit speed plane curve $C : (x(s), 0, z(s))$ with $x(s) > 0$ around z-axis which is defined on an interval I.

Suppose that the Gauss map G of M satisfies $\Box G = AG$ for some 3×3 matrix A. For a function $\theta = \theta(s)$ satisfying $(x'(s), z'(s)) = (\cos\theta(s), \sin\theta(s))$, let us put $J = \{s \in I | \theta'(s) \neq 0\}$.

We divide by two cases.

Case 1. Suppose that J is nonempty. Then, as in the proof of Lemma 3.1 we have $A = diag(\lambda, \lambda, \mu)$ with $a = \lambda - \mu$. Furthermore, Lemma 3.1 shows that $a = 0$ that is, $\lambda = \mu$. Hence it follows from (3.14) and (3.15) that the Gaussian curvature K is constant and the mean curvature H satisfies $2KH = \lambda$.

If $\lambda \neq 0$, then both of K and H are nonzero constant. Hence it follows from a well-known theorem (cf. [22]) that M is an open part of a sphere. Using (3.17) and (3.21) with $a = 0$, it can be directly shown that $\theta'(s)$ is constant and $x(s) = r\sin\theta$ for a positive constant r. This shows that the profile curve C is an open part of a half circle centered on the rotation axis of M. Thus, M is an open portion of a round sphere.

If $\lambda = 0$, then K is constant with $2KH = 0$. Suppose that $K \neq 0$. Then we have $H = 0$. But catenoids are the only minimal nonflat surfaces of revolution, of which Gaussian curvature K are nonconstant. This contradiction shows that $K = 0$. Thus M is a flat surfaces of revolution. Therefore, M is an open part of a plane, a right circular cone or a circular cylinder.

Case 2. Suppose that J is empty. Then the profile curve C of M is a straight line. Thus, M is an open part of a plane, a right circular cone or a circular cylinder.

The converse is obvious from (2.1). $\Box$

Combining the results of [11, 19], the following characterization theorems can be obtained.

Corollary 4.2. *Let M be a surface of revolution. Then the following are equivalent.*

1) M is an open part of a round sphere.

2) The Gauss map G of M satisfies $\Box G = AG$ for some nonsingular 3×3 matrix A.

Corollary 4.3. *Let M be a surface of revolution. Then the following are equivalent.*

1) M is an open part of a right circular cone.

2) The Gauss map G of M satisfies $\Box G = AG$ for some 3×3 matrix A, but not satisfies $\Delta G = AG$ for any 3×3 matrix A.

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