A Generalized Banach Fixed Point Theorem

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Abstract

Motivated by **the recent work of H. Liu and S.Y. Xu**, we prove a generalized Banach fixed point theorem for the setting of cone rectangular Banach algebra valued metric spaces without assuming the normality of the underlying cone. **Our work generalizes Some recent results in cone rectangular Banach algebra valued metric spaces**. An example to illustrate the main result is also presented.

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1 Introduction

The concept of metric spaces was introduced in 1906, by the French mathematician M. Fréchet [3]. A metric is just a function (which satisfies some properties) that takes values in the set of real numbers with its usual ordering. In 1934, the Serbian mathematician Đ. Kurepa in [7, 8], generalized the notion of a metric space, by allowing the metric to take values in more general ordered sets. These abstract spaces he introduced were also considered by several other authors (see [9,15,16] and the references therein). In [4] L.G. Huang and X. Zhang introduced and studied cone metric spaces. These are spaces where the metric function takes values in a normal cone of a Real Banach space. They proved the Banach fixed point theorem for such spaces. Very recently in [10], H. Liu and S. Xu have considered cone metric spaces where the underlying cone is contained in a real unital Banach algebra and proved a generalized Banach fixed point theorem on such spaces with the assumption that the underlying cone is normal.

Branciari in [2], introduced a class of generalized metric spaces, called rectangular metric spaces, which are obtained by replacing the standard triangular inequality in a metric space by a rectangular inequality, i.e. an inequality that involves four (distinct) points, and proved the Banach fixed point theorem for such spaces. Azam et al. in [1], introduced the notion of cone rectangular metric spaces and proved the Banach fixed point theorem for such spaces.

In this article, we consider cone rectangular Banach algebra valued metric spaces, i.e., spaces where the metric function actually takes values in a cone contained in a real unital Banach algebra. We prove a generalized Banach fixed point theorem for such spaces without assuming the normality of cone. Our results generalize and improve the result of Branciari [2], Huang and Zhang [4], Azam et al. [1] and H. Liu and S. Xu [10]. An example is also given which illustrates the main result.

2 Preliminaries

First we recall some well-known definitions which will be needed in the sequel.

Definition 1. Let $\mathcal{A}$ be a real Banach algebra, i.e., $\mathcal{A}$ is a real Banach space with a product that satisfies

1. $x(yz) = (xy)z$;
2. $x(y + z) = xy + xz$;
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. $\|xy\| \leq \|x\|\|y\|$,

for all $x, y, z \in \mathcal{A}$, $\alpha \in \mathbb{R}$.

The Banach algebra $\mathcal{A}$ is said to be unital if there exists an element $e \in \mathcal{A}$ such that $ex = xe = x$ for all $x \in \mathcal{A}$. The element e is called the unit. An $x \in \mathcal{A}$ is said to be invertible if there is a $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x, if it exists, is unique and will be denoted by x^{-1}. For more details, see [14].

Proposition 1. [14] Let $\mathcal{A}$ be a Banach algebra with a unit e and $x \in \mathcal{A}$. If the spectral radius $\rho(x)$ of x is less than 1, i.e.,

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|x^n\|^{1/n} < 1,$$

then $e - x$ is invertible and

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Let $\mathcal{A}$ be a unital Banach algebra. A non-empty closed set $P \subset \mathcal{A}$ is said to be a cone (see [10, 11]) if

1. $e \in P$
2. $P + P \subset P$,
3. $\alpha P \subset P$ for all $\alpha \geq 0$,
4. $P^2 \subset P$
5. $P \cap (-P) = \{\theta\}$, where θ is the zero of the unital Banach algebra $\mathcal{A}$.

Clearly $\theta \in P$. Given a cone $P \subset \mathcal{A}$ one can define a partial order $\preceq$ on $\mathcal{A}$ by $x \preceq y$ if and only if $y - x \in P$. The notation $x \ll y$ will stand for $y - x \in P^0$, where P^0 denotes the interior of P.

The cone P is called normal if there exists a number $K > 0$ such that for all $a, b \in \mathcal{A}$,

$$a \preceq b \quad \text{implies} \quad \|a\| \leq K\|b\|.$$

The least positive value of K satisfying the above inequality is called the normal constant (see [4]). Note that, for any normal cone P we have $K \geq 1$ (see [13]).

Henceforth, we will assume that the real Banach algebra $\mathcal{A}$ is unital and that the cone $P \subset \mathcal{A}$ is a solid cone. i.e. $P^0 \neq \emptyset$.

Lemma 1. *[6] Let $P \subset \mathcal{A}$ be a solid cone and $a, b, c \in P$.*

(a) If $a \preceq b$ and $b \ll c$ then $a \ll c$.

(b) If $a \ll b$ and $b \ll c$ then $a \ll c$.

(c) If $\theta \preceq u \ll c$ for each $c \in P^0$ then $u = \theta$.

(d) If $c \in P^0$ and $a_n \to \theta$ then there exist $n_0 \in \mathbb{N}$ such that, for all $n > n_0$ we have $a_n \ll c$.

(e) If $\theta \preceq a_n \preceq b_n$ for each n and $a_n \to a,\ b_n \to b$ then $a \preceq b$.

Lemma 2. *Let $P \subset \mathcal{A}$ be a cone.*

(a) If $a, b \in \mathcal{A}, c \in P$ and $a \preceq b$, then $ca \preceq cb$.

(b) If $a, k \in P$ are such that $\rho(k) < 1$ and $a \preceq ka$ then, $a = \theta$.

(c) If $k \in P$ and $\rho(k) < 1$, then for any fixed $m \in \mathbb{N}$ we have $\rho(k^m) < 1$.

Proof. (a) Since $b - a \in P$, it follows that $c(b - a) \in P$. i.e. $ca \preceq cb$.

(b) Since $a, k \in P$ and $a \preceq ka$ we have $\theta \preceq (k - e)a$, i.e., $(k - e)a \in P$. Since $\rho(k) < 1$, by Proposition 1, $(e - k)^{-1}$ exists and $(e - k)^{-1} \in P$. Multiplication by $(e - k)^{-1}$ yields $-a \in P$. Since $P \cap (-P) = \{\theta\}$, it follows that $a = \theta$.

(c) Since $\rho(k) < 1$ and $m \in \mathbb{N}$ is fixed, it follows that $\rho(k^m) = \lim\limits_{n\to\infty} \|(k^m)^n\|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \|(k^n)^m\|^{\frac{1}{n}} \leq \lim\limits_{n\to\infty} \left(\|k^n\|^{\frac{1}{n}}\right)^m = \left(\lim\limits_{n\to\infty} \|k^n\|^{\frac{1}{n}}\right)^m = (\rho(k))^m < 1.$ □

Definition 2. [1] Let X be a nonempty set, E be a real Banach space. The map $d : X \times X \to E$, is called a cone rectangular metric on X if

(i) $\theta \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \preceq d(x, w) + d(w, z) + d(z, y)$ for all $x, y \in X$ and for all distinct points $w, z \in X \setminus \{x, y\}$ (The Rectangular Property).

The pair (X, d) is called a cone rectangular metric space. If E is a real unital Banach algebra and $P \subset E$ is a cone, then we say that the pair (X, d) is a cone rectangular Banach algebra valued metric space.

The following are some examples of cone rectangular Banach algebra valued metric spaces.

Example 1. Let $\mathcal{A} = \mathbb{R}^2$ and define a norm on $\mathcal{A}$ by $\|(x_1, x_2)\| = |x_1| + |x_2|$. Let the multiplication be defined by

$$(x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1).$$

Then $\mathcal{A}$ is a real unital Banach algebra with unit $e = (1, 0)$. Let $P = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1, x_2 \geq 0\}$. Then P is a normal cone in $\mathcal{A}$, with normal constant $K = 1$.
Let $X = \Delta$, where Δ is the diagonal of $\mathbb{N}_0 \times \mathbb{N}_0$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Define $d \colon X \times X \to \mathcal{A}$ by

$$d(x,y) = \begin{cases} (0,0) & \text{if} \quad x = y; \\ (1/n, 1/n) & \text{if} \quad x, y \in \{(0,0), (n,n)\}, x \neq y \\ (1,1) & \text{if} \quad x, y \in \Delta \setminus \{(0,0)\}, x \neq y. \end{cases}$$

Then (X, d) is a cone rectangular Banach algebra valued metric spaces.

Example 2. [12] Let $\mathcal{A} = C^1_{\mathbb{R}}[0, 1]$ and define a norm on $\mathcal{A}$ by $\|x\| = \|x\|_\infty + \|x'\|_\infty$ for $x \in \mathcal{A}$. Define multiplication in $\mathcal{A}$ as just pointwise multiplication. Then $\mathcal{A}$ is a real unital Banach algebra with unit $e = 1$. The set $P = \{x \in \mathcal{A} \colon x \geq 0\}$ is a cone in $\mathcal{A}$. Moreover P is not normal (See [13]).
Let $X = \mathbb{N}$. Define $d \colon X \times X \to \mathcal{A}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y; \\ 3e^t & \text{if } x, y \in \{1, 2\} \text{ and } x \neq y. \\ e^t & \text{otherwise.} \end{cases}$$

Then (X, d) is a cone rectangular Banach algebra valued metric spaces.

For $x \in X$ and $c \gg \theta$, define $B(x, c) = \{y : d(x, y) \ll c\} \subset X$. The collection $\mathcal{B} = \{B(x, c) : x \in X, c \gg \theta\}$ being a subbasis generates a topology on X, say Γ. Clearly $\mathcal{B} \subset \Gamma$. We will henceforth view (X, Γ) as a topological space.

The following definitions are adapted from [1].

Definition 3. A sequence $\{x_n\}$ in the cone rectangular metric space X is said to be Cauchy, if given $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ which is independent of p, such that $d(x_n, x_{n+p}) \ll c$ for all $n \geq n_0$.

Definition 4. A sequence $\{x_n\}$ in the cone rectangular metric space X is said to *converge weakly* to $x \in X$, if given $c \gg \theta$, there exists an $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq n_0$, i.e., $x_n \in B(x, c)$ for all $n \geq n_0$. We will denote $\{x_n\}$ converging weakly to x by $x_n \twoheadrightarrow x$ as $n \to \infty$.

Definition 5. A cone rectangular metric space X is said to be *weakly complete* if every Cauchy sequence in the space converges weakly.

Remark 1. For a general topological space Y, one says that a sequence $\{x_n\}$ in Y converges to x if and only if given any open set U containing x, there exists an $n_0 \in \mathbb{N}$ such that $x_n \in U$, for all $n \geq n_0$. Observe that for our purposes, we only consider a weaker form of convergence. This is because we demand the existence of an $n_0 \in \mathbb{N}$ not for all, but only for certain open sets containing x, namely, sets of the form $B(x, c)$.

Note that, the limit of a weakly convergent sequence may not be unique (see [5]). The following lemma is a minor variant of Lemma 1.10 from [5].

Lemma 3. *Let (x_n) be a Cauchy sequence in a cone rectangular metric space X such that $x_n \neq x_m$ whenever $n \neq m$. If $x, y \notin \{x_n : n \in \mathbb{N}\}$ and (x_n) converges weakly to both x and y, then $x = y$.*

Proof. Since $x_n \twoheadrightarrow x, y$, given $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that $d(x, x_n) \ll \frac{c}{3}$, $d(x_m, x_{m+1}) \ll \frac{c}{3}$ and $d(x_k, y) \ll \frac{c}{3}$, for all $m, n, k \geq n_0$. Thus

$$d(x, y) \leq d(x, x_{n_0}) + d(x_{n_0}, x_{n_0+1}) + d(x_{n_0+1}, y) \ll c.$$

An application of part (c) of Lemma 1 completes the proof. □

3 The Main Result

In this section, we prove a generalized Banach fixed point theorem for weakly complete cone rectangular Banach algebra valued metric spaces.

Theorem 4. *Let (X, d) be a weakly complete cone rectangular Banach algebra valued metric space and $T\colon X \to X$ be a map such that:*

$$d(Tx, Ty) \preceq kM(x, y) \quad \text{for all} \quad x, y \in X, \tag{1}$$

where $M(x, y) \in \{d(x, y), d(x, Tx), d(y, Ty)\}$ and $k \in P$ with $\rho(k) < 1$. Then T has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$ the iterative sequence $\{T^n x_0\}$ converges to x^*.

Proof. First we show that the mapping T has at most one fixed point. Suppose u and v are two fixed points of T, i.e., $Tu = u, Tv = v$. Then by (1) we have

$$d(u, v) = d(Tu, Tv) \preceq kM(u, v), \tag{2}$$

where $M(u, v) \in \{d(u, v), d(u, Tu), d(v, Tv)\} = \{d(u, v)\}$. Therefore, by (2) we have $d(u, v) \preceq kd(u, v)$ which together with part (b) of Lemma 2 yields $d(u, v) = \theta$, i.e., $u = v$. Thus, the fixed point of T, if exists, is unique.

For proving the existence of fixed point, fix an element $x_0 \in X$ and consider the sequence $\{x_n\}$ in X defined by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. For notational convenience, let $d_n = d(x_n, x_{n+1})$ for all $n \geq 0$. It follows from inequality (1) that for $n \geq 1$,

$$\begin{aligned} d_n &= d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \\ &\preceq kM(x_{n-1}, x_n), \end{aligned} \tag{3}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &\in \{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ &= \{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= \{d_{n-1}, d_n\}. \end{aligned}$$

If $M(x_{n-1}, x_n) = d_n$ then from (3) we have $d_n \preceq kd_n$. Since $\rho(k) < 1$, it follows from part (b) of Lemma 2 that $x_n = x_{n+1}$, i.e., $T^n x_0$ is a fixed point of T. Thus, we can assume that for each $n \in \mathbb{N}$, $M(x_{n-1}, x_n) = d_{n-1}$. Therefore, by (3) we have

$$d_n \preceq kd_{n-1} \preceq \cdots \preceq k^n d_0. \tag{4}$$

Now we can assume that the terms of the sequence $\{x_n\}$ are distinct. For otherwise, i.e., suppose $x_m = x_n$ for some $m > n$, it follows that

$$d(x_n, x_{n+1}) = d(x_m, x_{m+1}) \preceq k^{m-n} d(x_n, x_{n+1}).$$

From parts (b) and (c) of Lemma 2, it follows that $d(x_n, x_{n+1}) = 0$, i.e. $T^n x_0$ is a fixed point of T. Thus, in further discussion we assume that $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ with $n \neq m$.

Consider $d(x_n, x_{n+2})$ for all $n \in \mathbb{N}$. We have

$$d(x_n, x_{n+2}) \;=\; d(Tx_{n-1}, Tx_{n+1}) \preceq kM(x_{n-1}, x_{n+1}), \tag{5}$$

where

$$\begin{aligned} M(x_{n-1}, x_{n+1}) &\in \{d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1})\} \\ &= \{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\} \\ &= \{d(x_{n-1}, x_{n+1}), d_{n-1}, d_{n+1}\}. \end{aligned}$$

If $M(x_{n-1}, x_{n+1}) = d(x_{n-1}, x_{n+1})$ then $M(x_{n-1}, x_{n+1}) \preceq d(x_{n-1}, x_n) + d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+1})$ and by (5) we have

$$\begin{aligned} d(x_n, x_{n+2}) &\preceq k[d(x_{n-1}, x_n) + d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+1})] \\ &= k[d_{n-1} + d(x_n, x_{n+2}) + d_{n+1}]. \end{aligned}$$

Using (4) we obtain $d(x_n, x_{n+2}) \preceq (e-k)^{-1} k^n [e + k^2] d_0$.
If $M(x_{n-1}, x_{n+1}) = d_{n-1}$ then using (4) and (5) we obtain

$$d(x_n, x_{n+2}) \preceq k^n d_0 \preceq (e-k)^{-1} k^n [e + k^2] d_0.$$

If $M(x_{n-1}, x_{n+1}) = d_{n+1}$ then using (4) and (5) we obtain

$$d(x_n, x_{n+2}) \preceq k^{n+2} d_0 \preceq (e-k)^{-1} k^n [e + k^2] d_0.$$

Thus

$$d(x_n, x_{n+2}) \preceq (e-k)^{-1} k^n [e + k^2] d_0 \text{ for all } n \in \mathbb{N}. \tag{6}$$

Now we shall show that the sequence $\{x_n\}$ is a Cauchy sequence. Then we consider the value $d(x_n, x_{n+p})$ in two cases.
If p is odd, say $2m + 1$ then using (4) we have

$$\begin{aligned} d(x_n, x_{n+2m+1}) &\preceq d(x_{n+2m}, x_{n+2m+1}) + d(x_{n+2m-1}, x_{n+2m}) + d(x_n, x_{n+2m-1}) \\ &= d_{n+2m} + d_{n+2m-1} + d(x_n, x_{n+2m-1}) \\ &\preceq d_{n+2m} + d_{n+2m-1} + d_{n+2m-2} + d_{n+2m-3} + \cdots + d_n \\ &\preceq k^{n+2m} d_0 + k^{n+2m-1} d_0 + k^{n+2m-2} d_0 + \cdots + k^n d_0 \\ &\preceq \left(\sum_{i=0}^{\infty} k^i \right) k^n d_0 = (e-k)^{-1} k^n d_0 \\ &\preceq (e-k)^{-1} (2k^2 + e) k^n d_0. \end{aligned} \tag{7}$$

If p is even, say $2m$, then using (4) and (6) we obtain

$$\begin{aligned} d(x_n, x_{n+2m}) &\preceq d(x_{n+2m-1}, x_{n+2m}) + d(x_{n+2m-1}, x_{n+2m-2}) + d(x_n, x_{n+2m-2}) \\ &= d_{n+2m-1} + d_{n+2m-2} + d(x_n, x_{n+2m-2}) \\ &\preceq d_{n+2m-1} + d_{n+2m-2} + d_{n+2m-3} + \cdots + d_{n+2} + d(x_n, x_{n+2}) \\ &\preceq k^{n+2m-1}d_0 + k^{n+2m-2}d_0 + k^{n+2m-3}d_0 + \cdots + k^{n+2}d_0 \\ &\quad +(e-k)^{-1}k^n[e+k^2]d_0 \\ &\preceq \left(\sum_{i=0}^{\infty} k^i\right) k^{n+2}d_0 + (e-k)^{-1}k^n[e+k^2]d_0 \\ &= (e-k)^{-1}(2k^2+e)k^n d_0. \end{aligned} \tag{8}$$

Since $\rho(k) < 1$ we have $\|k^n\| \to 0$, i.e., $k^n \to \theta$, as $n \to \infty$. Hence $(e-k)^{-1}(2k^2+e)k^n d_0 \to \theta$, as $n \to \infty$. Now by parts (a) and (d) of Lemma 1 and inequalities (6), (7) and (8), it follows that there exists $n_0 \in \mathbb{N}$ which is independent of p such that $d(x_n, x_{n+p}) \ll c$ for all $n > n_0$, and all $p \in \mathbb{N}$. Thus $\{x_n\}$ is a Cauchy sequence in X. By the weak completeness of X and Lemma 3, there exists a unique $x^* \in X$ such that $x_n \twoheadrightarrow x^*$ as $n \to \infty$.

We shall now show that x^* is a fixed point of T. Note that, if $x_n = x^*, x_m = Tx^*$ for some $n, m \in \mathbb{N}$, then we can choose $\ell \in \mathbb{N}$ such that $x_n \neq x^*, Tx^*$ for all $n \geq \ell$. Observe that such an ℓ exists because the terms of the sequence $\{x_n\}$ are distinct. For all $n \geq \ell$ we have

$$\begin{aligned} d(x^*, Tx^*) &\preceq d(x^*, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tx^*) \\ &= d(x^*, x_n) + d_n + d(Tx_n, Tx^*) \\ &\preceq d(x^*, x_n) + k^n d_0 + kM(x_n, x^*) \\ &\preceq d(x^*, x_n) + k^n d_0 + k\{d(x_n, x^*) + d(x_n, Tx_n) + d(x^*, Tx^*)\} \\ &\preceq (e+k)d(x_n, x^*) + (e+k)k^n d_0 + kd(x^*, Tx^*). \end{aligned}$$

An application of part (a) of Lemma 2 yields, for all $n \geq \ell$,

$$\begin{aligned} d(x^*, Tx^*) &\preceq (e-k)^{-1}(e+k)d(x_n, x^*) + (e-k)^{-1}(e+k)k^n d_0. \\ &\preceq 2(e-k)^{-1}\{d(x_n, x^*) + k^n d_0\}. \end{aligned} \tag{9}$$

Fix $c \gg \theta$. Since $x_n \twoheadrightarrow x^*$ and $k^n d_0 \to \theta$ as $n \to \infty$, it follows that for each $m \in \mathbb{N}$, there exists $n_1 \geq \ell$ such that $d(x_{n_1}, x^*) + k^{n_1} d_0 \ll \dfrac{c}{2m}$. Letting $n = n_1$ in inequality (9) and an application of part (a) of Lemma 2 yields

$$d(x^*, Tx^*) \preceq \frac{(e-k)^{-1}c}{m}.$$

for all $m \in \mathbb{N}$. Since $\dfrac{(e-k)^{-1}c}{m} \to \theta$ as $m \to \infty$, it follows from part (e) of Lemma 1 that $d(x^*, Tx^*) = \theta$, i.e. x^* is a fixed point of T. □

Remark 2. *Note that the hypothesis $\rho(k) < 1$ in Theorem 4 is better than the assumption $\|k\| < 1$. Neither do we assume $k \prec e$ here.*

The following corollary is a generalization of the Banach fixed point theorem.

Corollary 5. *Let (X,d) be a weakly complete cone rectangular Banach algebra valued metric space and $T\colon X \to X$ be a mapping such that the following condition is satisfied:*

$$d(Tx,Ty) \preceq kd(x,y) \quad \textit{for all} \quad x,y \in X,$$

where $k \in P$ with $\rho(k) < 1$. Then T has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$ the iterative sequence $\{T^n x_0\}$ converges to x^*.

Proof. Letting $M(x,y) = d(x,y)$ for all $x,y \in X$ in Theorem 4 yields the desired result. □

Remark 3. Setting $\mathcal{A} = \mathbb{R}, P = [0,\infty)$ with usual norm in Corollary 5 we obtain the main result of [2], i.e., the theorem of Banach-Caccioppoli type on a class of rectangular metric spaces.

Corollary 6. *Let (X,d) be a weakly complete cone rectangular Banach algebra valued metric space. Suppose $T\colon X \to X$ be such that for some $n \in \mathbb{N}$,*

$$d(T^n x, T^n y) \preceq kM_n(x,y) \quad \textit{for all} \quad x,y \in X, \tag{10}$$

where $M_n(x,y) \in \{d(x,y), d(x,T^n x), d(y,T^n y)\}$ and $k \in P$ with $\rho(k) < 1$. Then T has a unique fixed point $x^* \in X$.

Proof. From Theorem 4, T^n has a unique fixed point $x^* \in X$. Note that $T^n T x^* = TT^n x^* = Tx^*$. Thus Tx^* is also a fixed point of T^n. By uniqueness of the fixed point of T^n, it follows that $Tx^* = x^*$. i.e. x^* is a fixed point of T. Since any fixed point of T is also a fixed point of T^n, it follows that x^* is in fact the unique fixed point of T. □

In the following theorem the notion of weak completeness of X is replaced by another condition.

Theorem 7. *Let (X,d) be a cone rectangular Banach algebra valued metric space and $T\colon X \to X$ be a mapping such that the following condition is satisfied:*

$$d(Tx,Ty) \preceq kM(x,y) \quad \textit{for all} \quad x,y \in X, \tag{11}$$

where $M(x,y) \in \{d(x,y), d(x,Tx), d(y,Ty)\}$ and $k \in P$ with $\rho(k) < 1$. If there exists $x^* \in X$ such that $d(x^*,Tx^*) \preceq d(Tx^*,T^2x^*)$, then x^* is the unique fixed point $x^* \in X$.

Proof. An argument as in Theorem 4 yields the uniqueness of the fixed point. For the existence of a fixed point, let $F(x) = d(x,Tx)$ for all $x \in X$. It follows that $F(x) \in P$ and by hypothesis

$$F(x^*) \preceq F(Tx^*) \tag{12}$$

If $F(x^*) \neq \theta$, then by (11) we have

$$F(Tx^*) \quad = \quad d(Tx^*, TTx^*) \preceq kM(x^*, Tx^*), \tag{13}$$

where

$$M(x^*, Tx^*) \in \{d(x^*, Tx^*), d(x^*, Tx^*), d(Tx^*, TTx^*)\} = \{F(x^*), F(Tx^*)\}.$$

If $M(x^*, Tx^*) = F(Tx^*)$ then from (13) we have $F(Tx^*) \preceq kF(Tx^*)$, which together with part (b) of Lemma 2 implies that $F(Tx^*) = \theta$. It follows from (12) that $F(x^*) = \theta$, i.e. $Tx^* = x^*$.

If $M(x^*, Tx^*) = F(x^*)$ then from (13) we have $F(Tx^*) \preceq kF(x^*)$, which together with (12) implies that $F(x^*) \preceq kF(x^*)$. An application of part (b) of Lemma 2, yields $F(x^*) = \theta$, i.e., $Tx^* = x^*$. Thus, x^* is the unique fixed point of T. $\square$

We conclude with the following example which is motivated by [10].

Example 3. *Let the Banach algebra $\mathcal{A}$ with multiplication and the solid cone $P \subset \mathcal{A}$ be as in Example 1. Let $X = \{(1,1), (0,2), (0,3), (4,4)\}$. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X, we define a map $d\colon X \times X \to \mathcal{A}$ by*

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ \left(4, \frac{11}{2}\right), & \text{if } \ x, y \in \{(1,1), (0,2)\}, x \neq y; \\ \left(1 + |x_1 - y_1|, \frac{3}{2} + |x_2 - y_2|\right), & \text{otherwise.} \end{cases}$$

Then (X, d) is a cone rectangular Banach algebra valued metric space. Let the map $T\colon X \times X \to \mathcal{A}$ be defined by

$$T(1,1) = (0,2);\ T(0,2) = T(0,3) = T(4,4) = (0,3).$$

Let $k = (1/2, \alpha)$, where $\alpha > 1$. Clearly $k \in P$. It can be verified that

$$d(Tx, Ty) \preceq (1/2, \alpha)\, d(x,y) \quad \text{for all} \quad x, y \in X.$$

Moreover $\rho((1/2, \alpha)) < 1$. (see [10]). Thus the map T satisfies all the assumptions of Theorem 4. Hence it has a unique fixed point, namely, $(0,3)$. Also note that neither $\|k\| < 1$ nor $k \prec e$.

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