COVERING PROBLEMS FOR FUNCTIONS n-FOLD SYMMETRIC AND CONVEX IN THE DIRECTION OF THE REAL AXIS II.

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Abstract

Let $\mathcal{F}$ denote the class of all functions univalent in the unit disk $\Delta \equiv \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and convex in the direction of the real axis. The paper deals with the subclass $\mathcal{F}^{(n)}$ of these functions f which satisfy the property $f(\varepsilon z) = \varepsilon f(z)$ for all $z \in \Delta$, where $\varepsilon = e^{2\pi i/n}$. The functions of this subclass are called n-fold symmetric. For $\mathcal{F}^{(n)}$, where n is odd positive integer, the following sets: $\bigcap_{f\in\mathcal{F}^{(n)}} f(\Delta)$ - the Koebe set and $\bigcup_{f\in\mathcal{F}^{(n)}} f(\Delta)$ - the covering set, are discussed. As corollaries, we derive the Koebe and the covering constants for $\mathcal{F}^{(n)}$.

1. Introduction

Let $\mathcal{F}$ denote the class of all functions f which are univalent in $\Delta \equiv \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, convex in the direction of the real axis and normalized by $f(0) = f'(0) - 1 = 0$. Recall that an analytic function f is said to be convex in the direction of the real axis if the intersection of $f(\Delta)$ with each horizontal line is either a connected set or empty.

For a given subclass A of $\mathcal{F}$ the following sets: $\bigcap_{f\in A} f(\Delta)$ and $\bigcup_{f\in A} f(\Delta)$ are called the Koebe set for A and the covering set for A. We denote them by K_A and L_A respectively. The radius of the largest disk with center at the origin contained in K_A is called the Koebe constant for A. Analogously, the radius of the smallest disk with center at the origin that contains L_A is called the covering constant for A.

In the class $\mathcal{F}$ we consider functions which satisfy the property of n-fold symmetry:

$$f(\varepsilon z) = \varepsilon f(z) \quad \text{for all} \quad z \in \Delta \ ,$$

where $\varepsilon = e^{2\pi i/n}$. The subclass of $\mathcal{F}$ consisting of n-fold symmetric functions is denoted by $\mathcal{F}^{(n)}$. By the definition, for every $f \in \mathcal{F}^{(n)}$ a set $f(\Delta)$ is n-fold symmetric, which means that $f(\Delta) = \varepsilon f(\Delta)$. In other words, $f(\Delta)$ may be obtained as the union of rotations about a multiple of $2\pi/n$ from a set $f(\Delta) \cap \{w : \arg w \in [0, 2\pi/n]\}$. From this reason the following notation is useful:

$$\Lambda_0 = \left\{ w : \arg w \in \left[0, \frac{2\pi}{n}\right] \right\} \quad , \quad \Lambda_j = \varepsilon^j \Lambda_0 \ , \ j = 1, 2, \ldots, n-1$$

and

$$\Lambda^* = \left\{ w : \arg w \in \left[\frac{\pi}{2} - \frac{\pi}{n}, \frac{\pi}{2}\right] \right\} \ .$$

The main aim of the paper is to find the Koebe set and the covering set for the class $\mathcal{F}^{(n)}$ when n is an odd positive integer. Similar problems in related classes were discussed, for instance, in [1], [2], [5] and in the papers of the authors [3], [4].

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At the beginning let us consider the general properties of the Koebe sets and the covering sets for $\mathcal{F}^{(n)}$.

In [4] we proved that

Theorem 1. *The sets $K_{\mathcal{F}^{(n)}}$ and $L_{\mathcal{F}^{(n)}}$, for $n \in \mathbb{N}$, are symmetric with respect to both axes of the coordinate system.*

Theorem 2. *The sets $K_{\mathcal{F}^{(n)}}$ and $L_{\mathcal{F}^{(n)}}$, for $n \in \mathbb{N}$, are n-fold symmetric.*

To prove both the theorems it is enough to consider functions

$$g(z) = \overline{f(\overline{z})} \tag{1}$$

and

$$h(z) = -f(-z) \ . \tag{2}$$

Obviously,

$$f \in \mathcal{F}^{(n)} \Leftrightarrow g, h \in \mathcal{F}^{(n)} \ . \tag{3}$$

Moreover, if $D = f(\Delta)$ then $g(\Delta) = \overline{D}$, $h(\Delta) = -D$.

Taking (3) into account it is clear that the coordinate axes are the lines of symmetry for both the sets $K_{\mathcal{F}^{(n)}}$ and $L_{\mathcal{F}^{(n)}}$ for all $n \in \mathbb{N}$. Furthermore,

Lemma 1. *Each straight line $\varepsilon^{j/4} \cdot \{\zeta = t \ , \ t \in R\}$, $j = 0, 1, \ldots, 4n-1$ is the line of symmetry of $K_{\mathcal{F}^{(n)}}$ and $L_{\mathcal{F}^{(n)}}$ for every positive odd integer n .*

Proof.
Let n be a positive odd integer and let D be one of the two sets: $K_{\mathcal{F}^{(n)}}$ or $L_{\mathcal{F}^{(n)}}$.

Since D is symmetric with respect to the real axis and the positive real half-axis contains one side of the sector Λ_0, each rotation of the real axis about a multiple of $2\pi/n$ is the line of symmetry of D. Because of the equality

$$\{\zeta = t \ , \ t \in R\} \cdot \varepsilon^{1/2} = \{\zeta = t \ , \ t \in R\} \cdot \varepsilon^{(n+1)/2} \ ,$$

our claim is true for all even j, $j = 0, 1, \ldots, 4n-1$.

Let $n = 4k+1$, $k \geq 1$. The bisector of $\Lambda_{(n-1)/4}$ divides this sector into two subsectors: $\{w : \arg w \in [\pi/2 - \pi/2n, \pi/2 + \pi/2n]\}$ and $\{w : \arg w \in [\pi/2 + \pi/2n, \pi/2 + 3\pi/2n]\}$. Hence the imaginary axis is the bisector of the former. For this reason each rotation of the imaginary axis about a multiple of $2\pi/n$ is the line of symmetry of D. Moreover,

$$\{\zeta = it \ , \ t \in R\} \cdot \varepsilon^{1/2} = \{\zeta = it \ , \ t \in R\} \cdot \varepsilon^{(n+1)/2} \ .$$

Hence our claim is valid also for all odd j, $j = 0, 1, \ldots, 4n-1$.

If $n = 4k+3$, $k \geq 0$ then the bisector of $\Lambda_{(n-3)/4}$ divides this sector into two subsectors: $\{w : \arg w \in [\pi/2 - 3\pi/2n, \pi/2 - \pi/2n]\}$ and $\{w : \arg w \in [\pi/2 - \pi/2n, \pi/2 + \pi/2n]\}$. The imaginary axis is the bisector of the latter. Similar argument to the one for $n = 4k+1$ completes the proof for this choice of n. ■

Theorem 3. *The sets $K_{\mathcal{F}^{(n)}}$ and $L_{\mathcal{F}^{(n)}}$, for positive odd integers n, are $2n$-fold symmetric.*

Proof.
Let D be one of the two sets: $K_{\mathcal{F}^{(n)}}$ or $L_{\mathcal{F}^{(n)}}$. Let $w_0 = |w_0| e^{i\varphi_0}$ be an arbitrary point belonging to the boundary of D such that $\arg \varphi_0 \in [0, \pi/2n]$.

It is sufficient to apply Lemma 1. Firstly, the symmetric point to w_0 with respect to the straight line $\varepsilon^{1/4} \cdot \{\zeta = t\ ,\ t \in R\}$ is $w_1 = |w_0|e^{i(\pi/n-\varphi_0)}$. Secondly, the symmetric point to w_0 with respect to the real axis is $w_2 = |w_0|e^{-i\varphi_0}$. Both points w_1, w_2 also belong to the boundary of D. Consequently,

$$w_1 = \varepsilon^{1/2} \cdot w_2\ ,$$

which results in

$$D = \varepsilon^{1/2} \cdot D\ . \qquad \blacksquare$$

Remark 1. *On the basis of this lemma, we can observe that in order to find sets K and L, it is enough to determine their boundaries only in the sector of the measure $\pi/2n$.*

2. Extremal polygons and functions

Let n be a fixed positive odd integer, $n \geq 5$, and let K denote the Koebe set for $\mathcal{F}^{(n)}$.

We denote by $\mathcal{W}$, the family of n-fold symmetric polygons W convex in the direction of the real axis and which have $2n$ sides and interior angles $\pi + \pi/n$ and $\pi - 3\pi/n$ alternately.

Suppose that $w_* \in \partial K \cap \Lambda^*$, where ∂K stands for the boundary of K. According to Theorem 1, $-\overline{w_*} \in \partial K$. Consider the straight horizontal line containing a segment $I = \{-\lambda\overline{w_*} + (1-\lambda)w_* : \lambda \in (0,1)\}$. There are two possibilities: the intersection of this line with K is either I or the empty set. Assume now that $I \subset K$. We shall see that the second case holds only if $\operatorname{Re} w_* = 0$.

Since K is n-fold symmetric, all points $w_* \cdot \varepsilon^j$, $j = 0, 1, \dots, n-1$ belong to ∂K. On the one hand, $w_* \in \Lambda^*$ means that w_* has the greatest imaginary part among points $w_* \cdot \varepsilon^j$. On the other hand, $w_* \in \partial K$ means that there exists $f_* \in \mathcal{F}^{(n)}$ such that $w_* \in \partial f_*(\Delta)$.

From the convexity of f_* in the direction of the real axis, at least one of the two horizontal rays emanating from w_* is disjoint from $f_*(\Delta)$. Since $\{w_* - t : t \geq 0\} \cap K = I$, it is a ray $l = \{w_* + t : t \geq 0\}$ is disjoint from $f_*(\Delta)$. Taking into account the n-fold symmetry of f_*, all rays $l \cdot \varepsilon^j$, $j = 0, 1, \dots, n-1$ are disjoint from $f_*(\Delta)$.

Observe that the point $w_* \cdot \varepsilon^{(n+1)/2}$ has the lowest imaginary part among points $w_* \cdot \varepsilon^j$, $j = 0, 1, \dots, n-1$. Only if $\arg w_* = \pi/2$, this point is one of two points with the same imaginary part. The convexity in the direction of the real axis of f_* implies that one of two horizontal rays emanating from $w_* \cdot \varepsilon^{(n+1)/2}$ is also disjoint from $f_*(\Delta)$. If this ray is of the form $k_1 = \{w_* \cdot \varepsilon^{(n+1)/2} + t : t \geq 0\}$, then $(k_1 \cdot \varepsilon^j) \cap f_*(\Delta) = \emptyset$ and, conseqeuntly, $f_*(\Delta)$ is included in a polygon of the family $\mathcal{W}$. Indeed, the rays l and $k_1 \cdot \varepsilon^{(n-1)/2}$ form a sector with the vertex in w_* and the opening angle $\pi + \pi/n$. From the n-fold symmetry of f_* we obtain the polygon mentioned above. The conjugate angle to this opening angle is the vertex angle of the polygon at w_*. It is easy to check that the angle of the polygon between $l \cdot \varepsilon$ and $k_1 \cdot \varepsilon^{(n+1)/2}$ has the measure $\pi - 3\pi/n$.

But there is another possibility, i.e. $k_2 \cap f_*(\Delta) = \emptyset$, where $k_2 = \{w_* \cdot \varepsilon^{(n+1)/2} - t : t \geq 0\}$.

If $\arg w_* = \pi/2$ the set $\mathbb{C} \setminus \{k_2 \cdot \varepsilon^j, j = 0, 1, \dots, n-1\}$ consists of two parts: an unbounded part and a bounded one which is a regular n-gon, see Figure 1. A regular polygon is convex and it can be treated as the generalization of a set of the family $\mathcal{W}$. Every second side of this generalized polygon has the length 0.

If $\arg w_* \in [\pi/2 - \pi/n, \pi/2)$ then the set $\mathbb{C} \setminus \{l \cdot \varepsilon^j, k_2 \cdot \varepsilon^j, j = 0, 1, \dots, n-1\}$ is not bounded and it is not convex in the direction of the real axis. Since $w_* \cdot \varepsilon \in \partial f_*(\Delta)$, one of two horizontal rays emanating from this point is also disjoint from $f_*(\Delta)$. If $m_1 = \{w_* \cdot \varepsilon + t : t \geq 0\}$ has no common points with $f_*(\Delta)$ then $w_* \notin \partial f_*(\Delta)$, because $\operatorname{Im}(w_*\varepsilon) < \operatorname{Im} w_*$ and $\operatorname{Re}(w_*\varepsilon) < \operatorname{Re} w_*$, a contradiction. For this reason $m_2 \cap f_*(\Delta) = \emptyset$, where $m_2 = \{w_* \cdot \varepsilon - t : t \geq 0\}$. Hence $(m_2 \cdot \varepsilon^j) \cap f_*(\Delta) = \emptyset$.

In this way we obtain $3n$ rays emanating from n points: $w_* \cdot \varepsilon^j$, $j = 0, 1, \ldots, n-1$. Let us take three rays starting from w_*. These rays are: $l = \{w_* + t : t \geq 0\}$, $k_2\varepsilon^{-(n+1)/2} = \{w_* - t \cdot \varepsilon^{-(n+1)/2} : t \geq 0\}$ and $m_2\varepsilon^{-1} = \{w_* - t \cdot \varepsilon^{-1} : t \geq 0\}$. The angles between them and the positive real half-axis are equal to: 0, $-\pi/n$, $\pi - 2\pi/n$. It means that l lies in the sector with the vertex in w_* and with the sides $k_2\varepsilon^{-(n+1)/2}$ and $m_2\varepsilon^{-1}$. The opening angle of this sector is equal to $\pi - 3\pi/n$.

Consequently, $f_*(\Delta)$ is included in a polygon generated by $k_2\varepsilon^j$ and $m_2\varepsilon^j$, $j = 0, 1, \ldots, n-1$. This polygon belongs to the family $\mathcal{W}$. Two examples of members of $\mathcal{W}$ are shown in Figure 2.

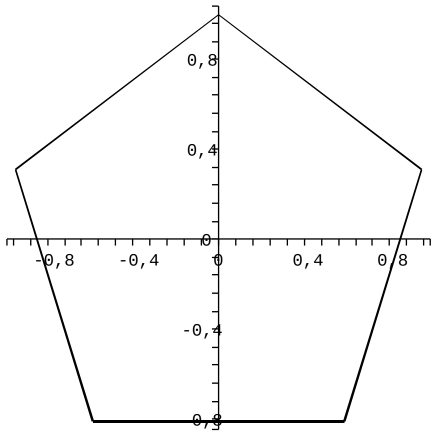

FIGURE 1. Extremal n-gon for $n = 5$.

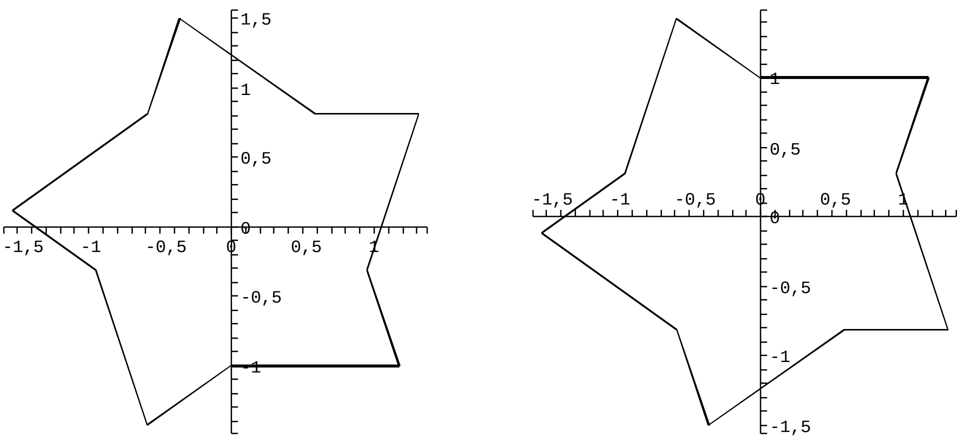

FIGURE 2. Extremal polygons for $n = 5$.

From now on, we assume that all members of $\mathcal{W}$ are open sets.

Let

(4) $$f_\alpha(z)=\int_0^z (1-\zeta^n e^{-in\alpha})^{\frac{1}{n}}(1+\zeta^n e^{-in\alpha/3})^{-\frac{3}{n}}d\zeta \quad , \quad \alpha\in\left[-\frac{3\pi}{n},\frac{3\pi}{n}\right] ,$$

(5) $$g_\alpha(z)=\int_0^z (1-\zeta^n e^{-in\alpha})^{\frac{1}{n}}(1-\zeta^n e^{-i(n\alpha/3+5\pi/3)})^{-\frac{3}{n}}d\zeta \quad , \quad \alpha\in\left[-\frac{2\pi}{n},\frac{4\pi}{n}\right] .$$

We choose the principal branch of n-th root. Since the exponential function is periodic, in the above definitions we restrict the range of variability of α to the intervals of length $6\pi/n$. The choice of these intervals depends on the properties of f_α and g_α. Some additional information will be given in Remark 2.

The definition of the family $\mathcal{W}$ may be extended for $n=3$. In this case the sets belonging to $\mathcal{W}$ may be treated as generalized polygons. The measure of the angles is equal to $4\pi/3$ and 0 alternately. These sets have the shape of an unbounded three-pointed star, see Figure 3. Moreover, for $n=3$ the functions g_α map Δ onto these generalized polygons.

Lemma 2. *All functions f_α, $\alpha\in\left[-\frac{3\pi}{n},\frac{3\pi}{n}\right]$, belong to $\mathcal{F}^{(n)}$ for $n=4k+1$, $k\geq 1$.*

Lemma 3. *All functions g_α, $\alpha\in\left[-\frac{2\pi}{n},\frac{4\pi}{n}\right]$, belong to $\mathcal{F}^{(n)}$ for $n=4k+3$, $k\geq 0$.*

Proof of Lemma 2.

At the beginning we shall show that the functions f_α, $\alpha\in[-3\pi/n,3\pi/n]$ are univalent. Observe that

(6) $$f'_\alpha(z)=p(z)\cdot\frac{h(z)}{z} ,$$

where

$$p(z)=\left(\frac{1-z^n e^{-ia}}{1+z^n e^{-ib}}\right)^{1/n}$$

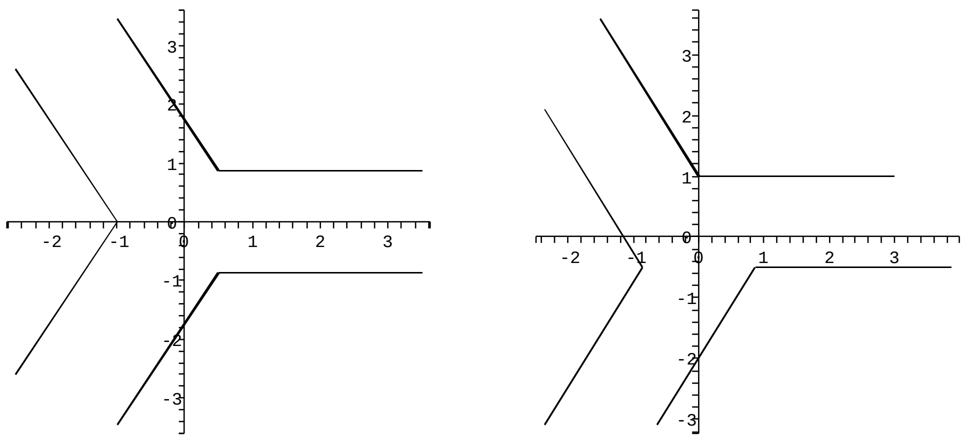

FIGURE 3. Extremal polygons for $n=3$.

and

$$h(z)=\frac{z}{(1+z^n e^{-ib})^{2/n}}$$

with $a=n\alpha$, $b=n\alpha/3$.

A Möbius function $p_1(z)=(1-ze^{-ia})/(1+ze^{-ib})$, $a,b\in\mathbb{R}$ satisfies the condition $\operatorname{Re} e^{i\beta}p_1(z)>0$ with some $\beta\in\mathbb{R}$; hence, for all $n\in\mathbb{N}$ the inequality $\operatorname{Re} e^{i\beta}p(z)=\operatorname{Re} e^{i\beta}p_1(z^n)^{1/n}>0$ holds with the same β. A function $h_1(z)=z/(1+ze^{-ib})^2$, $b\in\mathbb{R}$ is starlike, so is $h(z)=\sqrt[n]{h_1(z^n)}$. Combining these two facts with (6) we conclude that

$$\operatorname{Re} e^{i\beta}p(z)=\operatorname{Re} e^{i\beta}\frac{zf'_\alpha(z)}{h(z)}>0\ ,$$

which means that f_α is close-to-convex and consequently univalent.

Next, we claim that each polygon $f_\alpha(\Delta)$ is a set which is convex in the direction of the real axis. It is sufficient to discuss the argument of the tangent line to $\partial f_\alpha(\Delta)$. Observe that

$$\text{(7)}\quad \arg\left(\frac{\partial}{\partial\varphi}\left(f_\alpha(e^{i\varphi})\right)\right)=\arg\left(f'_\alpha(e^{i\varphi})ie^{i\varphi}\right)=$$
$$\frac{1}{n}\arg\left(1+e^{i(\pi+n\varphi-n\alpha)}\right)-\frac{3}{n}\arg\left(1+e^{i(n\varphi-n\alpha/3)}\right)+\frac{\pi}{2}+\varphi\ .$$

Let $\varphi\in[\alpha/3-\pi/n,\alpha)$. Then $\pi+n\varphi-n\alpha$ as well as $n\varphi-n\alpha/3$ are in $[-\pi,\pi)$ and from (7) we get

$$\text{(8)}\quad \arg\left(\frac{\partial}{\partial\varphi}\left(f_\alpha(e^{i\varphi})\right)\right)=\frac{1}{2n}(\pi+n\varphi-n\alpha)-\frac{3}{2n}\left(n\varphi-n\frac{\alpha}{3}\right)+\frac{\pi}{2}+\varphi=\frac{\pi}{2}+\frac{\pi}{2n}\ .$$

The above means that the tangent for φ in $(\alpha/3-\pi/n,\alpha)$ has the constant argument $\pi/2+\pi/2n$. Since $f_\alpha(\Delta)$ is a polygon with angles measuring $\pi+\pi/n$ and $\pi-3\pi/n$ alternately, the argument of the tangent line takes values $\pi/2+\pi/2n+2j\pi/n$ and $\pi/2-\pi/2n+2j\pi/n$, $j=1,2,\ldots,n$ alternately. What is more, putting $j=k$ in $\pi/2+\pi/2n+2j\pi/n$, we obtain the argument equal to π and putting $j=3k+1$ in $\pi/2-\pi/2n+2j\pi/n$, we obtain the argument equal to 2π. Hence two of the sides of $f_\alpha(\Delta)$ are horizontal; consequently $f_\alpha\in\mathcal{F}^{(n)}$. ■
The proof of Lemma 3 is similar.

Remark 2. *A polygon $f_\alpha(\Delta)$ has vertices in points $f_\alpha(e^{i\alpha})\cdot\varepsilon^j$ and $f_\alpha(e^{i(\alpha/3+\pi/n)})\cdot\varepsilon^j$, $j=0,1,\ldots,n-1$. These vertices correspond to angles measuring $\pi+\pi/n$ and $\pi-3\pi/n$ respectively. It is worth pointing out some particular cases of polygons belonging to $\mathcal{W}$. For $\alpha=-3\pi/2n$ and $\alpha=3\pi/2n$ they become regular n-gons and for $\alpha=-3\pi/n$, $\alpha=0$ and $\alpha=3\pi/n$ these sets are n-pointed stars symmetric with respect to the real axes. The functions $f_{-3\pi/n}$, f_0 and $f_{3\pi/n}$ have real coefficients. In all other cases coefficients are nonreal. These particular functions are as follows:*

$$f_{-3\pi/2n}(z)=\int_0^z(1+i\zeta^n)^{-\frac{2}{n}}d\zeta\quad,\quad f_{3\pi/2n}(z)=\int_0^z(1-i\zeta^n)^{-\frac{2}{n}}d\zeta\ ,$$

$$f_0(z)=\int_0^z(1-\zeta^n)^{\frac{1}{n}}(1+\zeta^n)^{-\frac{3}{n}}d\zeta\quad,\quad f_{-3\pi/n}(z)=f_{3\pi/n}(z)=\int_0^z(1+\zeta^n)^{\frac{1}{n}}(1-\zeta^n)^{-\frac{3}{n}}d\zeta\ .$$

Likewise, for $\alpha=-\pi/2n$ and $\alpha=5\pi/2n$ the sets $g_\alpha(\Delta)$ are regular n-gons, and for $\alpha=-2\pi/n$, $\alpha=\pi/n$ and $\alpha=4\pi/n$ these sets are n-pointed stars. These functions g_α which map Δ on n-pointed stars have real coefficients.

Let $n = 4k+1, k \geq 1$ be fixed. Let us denote by W_α a set $f_\alpha(\Delta)$ for a fixed $\alpha \in [-3\pi/n, 3\pi/n]$.

Observe that the following equalities hold:

$$f'_{-\alpha}(z) = \overline{f'_\alpha(\overline{z})} \quad \text{for} \quad \alpha \in \left[-\frac{3\pi}{n}, \frac{3\pi}{n}\right]$$

and

$$f'_{\frac{3\pi}{2n}-\gamma}(-z) = \overline{f'_{\frac{3\pi}{2n}+\gamma}(\overline{z})} \quad \text{for} \quad \gamma \in \left[0, \frac{3\pi}{2n}\right] \ ,$$

which means that

$$f_{-\alpha}(z) = \overline{f_\alpha(\overline{z})} \quad \text{for} \quad \alpha \in \left[-\frac{3\pi}{n}, \frac{3\pi}{n}\right]$$

and

$$-f_{\frac{3\pi}{2n}-\gamma}(-z) = \overline{f_{\frac{3\pi}{2n}+\gamma}(\overline{z})} \quad \text{for} \quad \gamma \in \left[0, \frac{3\pi}{2n}\right] \ .$$

Consequently,

$$W_{-\alpha} = \overline{W_\alpha} \quad \text{and} \quad -W_{\frac{3\pi}{2n}-\gamma} = \overline{W_{\frac{3\pi}{2n}+\gamma}} \ .$$

For this reason a polygon W as well as $\overline{W}$ and $-\overline{W}$ belong to the family $\mathcal{W}$. From the geometric construction of polygons in $\mathcal{W}$ it follows that the ratio of lengths of any two adjacent sides of a polygon varies from 0 to infinity as α is changing in $[-3\pi/n, 3\pi/n]$; in each case a polygon is convex in the direction of the real axis. Multiplying sets W_α, $\alpha \in [-3\pi/n, 3\pi/n]$ by $\lambda > 0$ we obtain all members of the set $\mathcal{W}$.

We have proved one part of the following lemma (the second one can be proved analogously)

Lemma 4.

1. $\mathcal{W} = \{\lambda \cdot f_\alpha(\Delta), \lambda > 0, \alpha \in [-3\pi/n, 3\pi/n]\}$ *for* $n = 4k+1, k \geq 1$,
2. $\mathcal{W} = \{\lambda \cdot g_\alpha(\Delta), \lambda > 0, \alpha \in [-2\pi/n, 4\pi/n]\}$ *for* $n = 4k+3, k \geq 0$.

3. Koebe sets for $\mathcal{F}^{(n)}$

Let us define

$$F(\alpha) \equiv f_\alpha(e^{i\alpha}) \quad , \quad \alpha \in \left[-\frac{3\pi}{n}, \frac{3\pi}{n}\right]$$

and

$$G(\alpha) \equiv g_\alpha(e^{i\alpha}) \quad , \quad \alpha \in \left[-\frac{2\pi}{n}, \frac{4\pi}{n}\right] \ .$$

From (4) and (5)

$$F(\alpha) = e^{i\alpha} \int_0^1 (1-t^n)^{\frac{1}{n}} (1 + t^n e^{2in\alpha/3})^{-\frac{3}{n}} dt$$

and

$$G(\alpha) = e^{i\alpha} \int_0^1 (1-t^n)^{\frac{1}{n}} (1 - t^n e^{i(2n\alpha/3 - 5\pi/3)})^{-\frac{3}{n}} dt \ .$$

It can be easily checked that

$$\arg F(\alpha) = \alpha \quad \text{for} \quad \alpha \in \left\{ -\frac{3\pi}{n}, -\frac{3\pi}{2n}, 0, \frac{3\pi}{2n}, \frac{3\pi}{n} \right\} , \tag{9}$$

$$\arg G(\alpha) = \alpha \quad \text{for} \quad \alpha \in \left\{ -\frac{2\pi}{n}, -\frac{\pi}{2n}, \frac{\pi}{n}, \frac{5\pi}{2n}, \frac{4\pi}{n} \right\} . \tag{10}$$

Theorem 4. *The Koebe set $K_{\mathcal{F}^{(n)}}$, for a fixed $n = 4k+1$, $k \in \mathbb{N}$, is a bounded and $2n$-fold symmetric domain such that*

$$\partial K_{\mathcal{F}^{(n)}} \cap \left\{ w : \arg w \in \left[-\frac{\pi}{2n}, \frac{\pi}{2n} \right] \right\} = F\left([-\alpha_F, \alpha_F] \right) , \tag{11}$$

where α_F is the only solution of the equation

$$\arg F(\alpha) = \frac{\pi}{2n} \tag{12}$$

in $[0, 3\pi/2n]$.

Proof.
Let K denote the Koebe set for $\mathcal{F}^{(n)}$.

Let us consider a polygon $V_\alpha = f_\alpha(\Delta)$ belonging to $\mathcal{W}$, such that one of its vertices, let say v_*, lies in Λ^* (its argument is in $[\pi/2 - \pi/n, \pi/2]$) and the interior angle at v_* has the measure $\pi(1 + 1/n)$. Suppose additionaly that w_* is a point of the boundary of K such that $\arg w_* = \arg v_*$ and $|w_*| < |v_*|$. We denote the quotient $w_*/v_* = |w_*|/|v_*|$ by λ. Hence $\lambda < 1$.

Since $w_* \in \partial K$, there exists $f_* \in \mathcal{F}^{(n)}$ such that

$$f_*(\Delta) \subset \lambda V_\alpha \subsetneq V_\alpha = f_\alpha(\Delta) . \tag{13}$$

Therefore, $f_* \prec f_\alpha$ and $1 = f_*'(0) \le f_\alpha'(0) = 1$. Consequently $f_* = f_\alpha$, which contradicts (13). It means that $v_* = w_*$, or in other words, w_* coincides with some vertex of $f_\alpha(\Delta)$. Hence w_* is equal to $f_\alpha(e^{i\alpha})$ rotated about a multiple of $2\pi/n$, namely about $2\pi/n \cdot (n-1)/4$. However, it is true only for those α, for which $w_* = F(\alpha) \cdot \varepsilon^{(n-1)/4}$ is in Λ^*.

Observe that for $\alpha \in [-3\pi/n, 3\pi/n]$ we have

$$F(-\alpha) = \overline{F(\alpha)} ,$$

that is,

$$\arg F(-\alpha) = -\arg F(\alpha) .$$

From this and (9),(12) it follows that

$$\left\{ F(\alpha) \cdot \varepsilon^{(n-1)/4} : \alpha \in [-\alpha_F, \alpha_F] \right\}$$

is the boundary of the Koebe set for $\mathcal{F}^{(n)}$ in Λ^*. Combining this with Theorem 3 the equality (11) follows.

Finally, we claim that α_F is the only solution of (12) in $[0, 3\pi/2n]$. On the contrary, assume that there exist two different numbers $\alpha_1, \alpha_2 \in [0, 3\pi/2n]$ such that

$$\arg F(\alpha_1) = \arg F(\alpha_2) ,$$

or equivalently,

$$\arg f_{\alpha_1}(e^{i\alpha_1}) = \arg f_{\alpha_2}(e^{i\alpha_2}) .$$

The sets $W_{\alpha_1} = f_{\alpha_1}(\Delta)$, $W_{\alpha_2} = f_{\alpha_2}(\Delta)$ are polygons of the family $\mathcal{W}$. This and the definition of $\mathcal{W}$ (or Lemma 4) result in

$$W_{\alpha_1} \subset W_{\alpha_2} \quad \text{or} \quad W_{\alpha_2} \subset W_{\alpha_1} \ .$$

The normalization of f_{α_1} and f_{α_2} leads to $W_{\alpha_1} = W_{\alpha_2}$. Hence $\alpha_1 = \alpha_2$, a contradiction. This means that (12) has only one solution in the set $[0, 3\pi/2n]$. ■

The above proof gives more. Namely, F is starlike for $\alpha \in [-3\pi/2n, 3\pi/2n]$. Moreover,

$$\arg F\left(\alpha + \frac{3\pi}{n}\right) = \arg F(\alpha) + \frac{3\pi}{n} \ .$$

It implies that F is starlike for $\alpha \in [-\pi, \pi]$.

Furthermore, it is not difficult to see that there do not exist two different points in the set $\partial K_{\mathcal{F}^{(n)}} \cap \Lambda^*$ with the same imaginary part. For contrary suppose that it is not the case, ie. there exist v_1 and v_2 such that $v_1 \neq v_2$, $v_1, v_2 \in \partial K_{\mathcal{F}^{(n)}} \cap \Lambda^*$ and $\operatorname{Im} v_1 = \operatorname{Im} v_2$.

With use of an argument similar to those in the proof of previous theorem we can see that there exist two polygons $V_1, V_2 \in \mathcal{W}$ with vertices v_1, v_2 respectively. The angles at these vertices have the same measure. Hence the sides of these polygons are pairwise parallel and if $\operatorname{Re} v_1 < \operatorname{Re} v_2$ then $V_1 \subset V_2$. This means that there exist $f_1, f_2 \in \mathcal{F}^{(n)}$ that $f_1(\Delta) = V_1$, $f_2(\Delta) = V_2$ and $f_1 \prec f_2$. But the normalization of f_1 and f_2 is the same, hence $f_1 = f_2$; a contradiction.

Theorem 5. *The Koebe set $K_{\mathcal{F}^{(n)}}$, for a fixed $n = 4k+3$, $k \geq 0$ is a bounded and $2n$-fold symmetric domain such that*

$$\partial K_{\mathcal{F}^{(n)}} \cap \left\{ w : \arg w \in \left[\frac{\pi}{2n}, \frac{3\pi}{2n}\right] \right\} = G\left(\left[\alpha_G, \frac{2\pi}{n} - \alpha_G\right]\right) \ , \tag{14}$$

where α_G is the only solution of the equation

$$\arg G\left(\alpha\right) = \frac{\pi}{2n} \tag{15}$$

in $[-\pi/2n, \pi/n]$.

Proof.
A consideration similar to the above shows that α_G is the only solution of the equation (15) in $[-\pi/2n, \pi/n]$.

Suppose that $w_* \in \partial K \cap \Lambda^*$. The analogous argument to this in the proof of Theorem 4 yields that K is contained in some polygon W of the family $\mathcal{W}$.

Let g_* be a function from $\mathcal{F}^{(n)}$ for which $w_* \in \partial g_*(\Delta)$. We have $g_*(\Delta) \subset W = g_\alpha(\Delta)$ for some $\alpha \in [-\pi/2n, 5\pi/n]$. For this reason $g_* \prec g_\alpha$, but taking into account the normalization of both functions we obtain $g_* = g_\alpha$. Hence $w_* = g_\alpha(e^{i\alpha}) \cdot \varepsilon^{(n-3)/4}$, but only if $w_* \in \Lambda^*$.

For $\alpha \in [-\pi/2n, \pi/n]$,

$$G\left(\frac{2\pi}{n} - \alpha\right) = \varepsilon \overline{G\left(\alpha\right)} \ ,$$

and so

$$\arg G\left(\frac{2\pi}{n} - \alpha\right) = \frac{2\pi}{n} - \arg G\left(\alpha\right) \ .$$

From this, (10) and (15), we conclude that

$$\left\{G(\alpha)\cdot\varepsilon^{(n-3)/4} : \alpha\in\left[\alpha_G, \frac{2\pi}{n}-\alpha_G\right]\right\}$$

is the boundary of the Koebe set for $\mathcal{F}^{(n)}$ in Λ^*. Theorem 3 concludes the proof of our theorem. ■

Now we can derive the Koebe constant for $\mathcal{F}^{(n)}$.

Theorem 6. *For a fixed positive odd integer n, $n\geq 3$ and for every function $f\in\mathcal{F}^{(n)}$ the disk Δ_{r_n}, where $r_n = B(1/n, 1/2n+1/2)/n\sqrt[n]{4}$, is included in $f(\Delta)$. The number r_n cannot be increased.*

The symbol B stands for the Beta and Δ_r, $r>0$ means $\Delta_r=\{\zeta\in\mathbb{C} : |\zeta|<r\}$.

Proof.
According to Theorems 4 and 5, the Koebe constant is equal to

$$\min\{|F(\alpha)| : \alpha\in[-\alpha_F,\alpha_F]\}\quad \text{for}\quad n=4k+1\ ,$$

or

$$\min\left\{|G(\alpha)| : \alpha\in\left[\alpha_G,\frac{2\pi}{n}-\alpha_G\right]\right\}\quad\text{for}\quad n=4k+3\ .$$

But

$$|F(\alpha)|^2\geq\left(\int_0^1(1-t^n)^{\frac{1}{n}}\,\mathrm{Re}(1+t^ne^{2in\alpha/3})^{-\frac{3}{n}}dt\right)^2$$

and the integrand in this expression is nonnegative; thus

$$|F(\alpha)|\geq\int_0^1(1-t^n)^{\frac{1}{n}}\,\mathrm{Re}\,q_F(\alpha,t)dt\quad ,\quad q_F(\alpha,t)=(1+t^ne^{2in\alpha/3})^{-\frac{3}{n}}.$$

Likewise,

$$|G(\alpha)|\geq\int_0^1(1-t^n)^{\frac{1}{n}}\,\mathrm{Re}\,q_G(\alpha,t)dt\quad ,\quad q_G(\alpha,t)=(1-t^ne^{i(2n\alpha/3-5\pi/3)})^{-\frac{3}{n}}.$$

It is easy to check that for $n\geq 3$ the functions $p(z)=(1\pm t^nz)^{-3/n}$ are convex in Δ and they have real coefficients. This means that

(16) $$\mathrm{Re}(1\pm t^nz)^{-\frac{3}{n}}\geq(1+t^n)^{-\frac{3}{n}}\ .$$

Applying (16) for both q_F and q_G, we get

$$|F(\alpha)|\geq q_0\quad\text{and}\quad|G(\alpha)|\geq q_0\ ,$$

where

(17) $$q_0=\int_0^1(1-t^n)^{\frac{1}{n}}(1+t^n)^{-\frac{3}{n}}dt\ .$$

This results in

$$\min\{|F(\alpha)| : \alpha\in[-\alpha_F,\alpha_F]\}=|F(0)|$$

and

$$\min\left\{|G(\alpha)| : \alpha\in\left[\alpha_G,\frac{2\pi}{n}-\alpha_G\right]\right\}=\left|G\left(\frac{\pi}{n}\right)\right|\ .$$

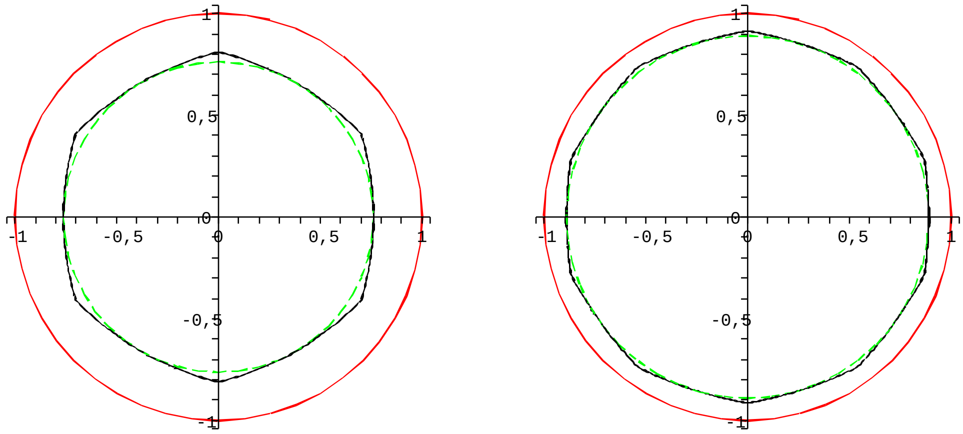

FIGURE 4. Koebe domain (solid line) and Koebe disk (dashed line) for $\mathcal{F}^{(3)}$ and $\mathcal{F}^{(5)}$.

Moreover, substituting $t^n = \tan^2(x/2)$ in (17) we get

$$q_0 = \frac{2}{n\sqrt[n]{4}} \int_0^{\pi/2} (\sin x)^{\frac{2}{n}-1} (\cos x)^{\frac{1}{n}}\, dx = \frac{1}{n\sqrt[n]{4}} B\left(\frac{1}{n}, \frac{1}{2n} + \frac{1}{2}\right) . \quad \blacksquare$$

Corollary 1. *For a fixed positive odd integer n, $n \geq 3$ the Koebe constant for $\mathcal{F}^{(n)}$ is equal to $r_n = B(1/n, 1/2n + 1/2)/n\sqrt[n]{4}$.*

The Koebe sets and the Koebe disks for $n = 3$ and $n = 5$ are shown in Figure 4.

Remark 3. *The results established in Theorem 4 and in Corollary 1 are actually valid also for $n = 1$. They were obtained by Złotkiewicz and Reade in* [6].

One can check that for $n = 1$ the function F takes the form

$$F(\alpha) = e^{i\alpha} \int_0^1 \frac{1-t}{(1 + te^{2i\alpha/3})^3} dt = \frac{e^{2i\alpha/3}}{4\cos(\alpha/3)} .$$

The equation (12) gives $\alpha_F = 3\pi/4$; thus the boundary of the Koebe set in the upper half-plane can be written as follows

$$u = \frac{\cos(2\alpha/3)}{4\cos(\alpha/3)} , \; v = \frac{1}{2}\sin(\alpha/3) , \; \alpha \in \left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right] .$$

This fact we can rewrite in a different way

$$K_{\mathcal{F}} = \{w \in \mathbb{C} : 8|w|\,(|w| + |\operatorname{Re} w|) < 1\} .$$

The extremal functions f_α given by (4) are of the form

$$f_\alpha(z) = \frac{z + Bz^2}{(1 + ze^{-i\alpha/3})^2} , \; B = i\sin(\alpha/3)e^{-2i\alpha/3}$$

and

$$g_\alpha(z) = -f_\alpha(-z) ,$$

where $\alpha \in [-3\pi/4, 3\pi/4]$. The image set $f_\alpha(\Delta)$ for a fixed $\alpha \in (-3\pi/4, 3\pi/4)$ coincides with the plane with a horizontal ray excluded. For $\alpha = -3\pi/4, 3\pi/4$, the sets $f_\alpha(\Delta)$ are half-planes.

Moreover, $r_1 = B(1,1)/4 = 1/4$.

4. Covering domains for $\mathcal{F}^{(n)}$

Theorem 7. *The covering set $L_{\mathcal{F}^{(n)}}$ for odd $n \geq 5$ is a bounded and $2n$-fold symmetric domain such that*

$$\partial L_{\mathcal{F}^{(n)}} \cap \left\{ w : \arg w \in \left[0, \frac{\pi}{2n}\right] \right\} = H\left(\left[0, \frac{\pi}{2n}\right]\right) . \tag{18}$$

In the proof of this theorem we need the following lemma.

Lemma 5. *Let $n \geq 5$ be a fixed odd integer. If $f \in \mathcal{F}^{(n)}$ and $w \in f(\Delta) \cap \Lambda^*$ then $f(\Delta)$ contains a polygon $W \in \mathcal{W}$ such that W has one of its vertices at w and the interior angle at w has the measure $\pi - 3\pi/n$.*

Proof.

Let $f \in \mathcal{F}^{(n)}$ and $w \in f(\Delta) \cap \Lambda^*$, i.e. $\arg w \in [\pi/2 - \pi/n, \pi/2]$. Because of the n-fold symmetry of f every point $w \cdot \varepsilon^j$, $j = 0, 1, \ldots, n-1$ belongs to $f(\Delta)$.

It can be easily checked that

$$\max\left\{\operatorname{Im}\left(w \cdot \varepsilon^j\right), j = 0, 1, \ldots, n-1\right\} = \operatorname{Im}(w)$$

and

$$\min\left\{\operatorname{Im}\left(w \cdot \varepsilon^j\right), j = 0, 1, \ldots, n-1\right\} = \operatorname{Im}\left(w \cdot \varepsilon^{2k+1}\right) .$$

Let $w_1 = w \cdot \varepsilon$ and $w_2 = w \cdot \varepsilon^{2k}$. The point w_1 has the second biggest imaginary part among points $w, w \cdot \varepsilon, \ldots, w \cdot \varepsilon^{n-1}$. Likewise, w_2 has the second lowest imaginary part among those points.

Let, moreover, l_1 and l_2 stand for two horizontal rays emanating from w_1 and w_2: $l_1 = \{w_1 + t : t \geq 0\}$, $l_2 = \{w_2 + t : t \geq 0\}$ respectively.

From the inequality $\operatorname{Im} w_2 > \operatorname{Im} w \cdot \varepsilon^{2k+1}$ we conclude that the point $w \cdot \varepsilon^{2k+1}$ lies on the opposite side of the straight line which contains l_2 with respect to the origin. As a consequence, w_1 lies on the other side of the straight line including $l_2 \cdot \varepsilon^{-2k}$ with respect to the origin. Hence, two rays l_1 and $l_2 \cdot \varepsilon^{-2k}$ have a common point, let say w_0.

We shall show that w_0 also belongs to $f(\Delta)$. Suppose, contrary to our claim, that $w_0 \notin f(\Delta)$. The points w_0, w_1 lie on the ray l_1 and $w_1 \in f(\Delta)$. Therefore, taking into account the convexity in the direction of the real axis of f, a ray $m_1 = \{w_0 + t : t \geq 0\}$ is disjoint from $f(\Delta)$.

Since w_0, w belong to $l_2 \cdot \varepsilon^{-2k}$, the points $w_0 \cdot \varepsilon^{2k}, w \cdot \varepsilon^{2k}$ belong to l_2. Moreover, $w_0 \cdot \varepsilon^{2k} \notin f(\Delta)$ and $w_2 \in f(\Delta)$. Consequently, $m_2 = \{w_0 \cdot \varepsilon^{2k} + t : t \geq 0\}$ is disjoint from $f(\Delta)$, and, generally, $m_2 \varepsilon^j \cap f(\Delta) = \emptyset$, $j = 0, 1, \ldots, n-1$.

We have proved that the rays m_1 and $m_2 \varepsilon^{-2k}$ with the common vertex w_0 are disjoint from $f(\Delta)$. It means that the reflex sector with the vertex in w_0 and these two rays as the sides has no common points with $f(\Delta)$. But w_1 lies in this reflex sector; hence $w_1 \notin f(\Delta)$, a contradiction.

From the argument given above all points $w\varepsilon^j$, $w_0\varepsilon^j$, $j = 0, 1, \ldots, n-1$ belong to $f(\Delta)$. Applying n-fold symmetry and the convexity of f in the direction of the real axis we can see that a polygon W with succeeding vertices at points $w, w_0, w\varepsilon, w_0\varepsilon, \ldots, w\varepsilon^{n-1}, w_0\varepsilon^{n-1}$ is contained in $f(\Delta)$. It is easy to check that this polygon has the interior angles $\pi - 3\pi/n$ and $\pi + \pi/n$ alternately. For this reason W is in $\mathcal{W}$. ∎

According to Lemmas 2 and 3, every function in $\mathcal{F}^{(n)}$ mapping Δ onto a polygon of the family $\mathcal{W}$ has the form (4)-(5) with appropriately taken α. These functions may be written in the form

$$f_\beta(z) = \int_0^z (1+\zeta^n e^{-3in\beta})^{\frac{1}{n}}(1-\zeta^n e^{-in\beta})^{-\frac{3}{n}}d\zeta \quad , \quad \beta \in \left[0, \frac{2\pi}{n}\right] \ , \tag{19}$$

$$g_\beta(z) = \int_0^z (1+\zeta^n e^{-3in\beta})^{\frac{1}{n}}(1-\zeta^n e^{-in\beta})^{-\frac{3}{n}}d\zeta \quad , \quad \beta \in \left[\frac{\pi}{n}, \frac{3\pi}{n}\right] \ , \tag{20}$$

equivalent to (4)-(5).

In fact, the functions defined by (4) and (19) are connected by the relation $\beta = \alpha/3 + \pi/n$ and the functions in (5) and (20) are connected by $\beta = \alpha/3 + 5\pi/3n$.

Let us define

$$H(\beta) = f_\beta(e^{i\beta}) \quad \text{for} \quad \beta \in \left[0, \frac{2\pi}{n}\right] \ ,$$

and

$$H(\beta) = g_\beta(e^{i\beta}) \quad \text{for} \quad \beta \in \left[\frac{\pi}{n}, \frac{3\pi}{n}\right] \ .$$

Hence

$$H(\beta) \equiv e^{i\beta}\int_0^1 (1+t^n e^{-2in\beta})^{\frac{1}{n}}(1-t^n)^{-\frac{3}{n}}dt \ , \ \beta \in \mathbb{R} \ .$$

Observe that

$$\arg H(\beta) = \beta \quad \text{for} \quad \beta = \frac{\pi}{2n}\cdot j \ , \ j = 0, 1, \ldots, 4n-1 \ . \tag{21}$$

Furthermore,

$$H\left(\beta + \frac{\pi}{n}\right) = e^{i\frac{\pi}{n}}H(\beta) \ . \tag{22}$$

Now we can prove Theorem 7.

Proof of Theorem 7.

Let L denote the covering set for $\mathcal{F}^{(n)}$. We additionaly assume that $n = 4k+1, k \geq 1$. The proof for the case $n = 4k+3, k \geq 0$ is almost similar.

Let us consider a polygon $W_\beta = f_\beta(\Delta)$ belonging to $\mathcal{W}$, such that one of its vertices, let say w^*, lies in Λ^* and the interior angle at w^* has the measure $\pi(1-3/n)$. Suppose additionaly that v^* is a point of the boundary of L such that $\arg v^* = \arg w^*$ and $|v^*| > |w^*|$. We denote the quotient $v^*/w^* = |v^*|/|w^*|$ by μ. Hence $\mu > 1$.

Since $v^* \in \partial L$, there exists $f^* \in \mathcal{F}^{(n)}$ such that v^* is a boundary point of $f^*(\Delta)$. From Lemma 5

$$f^*(\Delta) \supset \mu W_\beta \supsetneq W_\beta = f_\beta(\Delta) \ . \tag{23}$$

Therefore, $f_\beta \prec f^*$ and $1 = f_\beta'(0) \leq {f^*}'(0) = 1$. Consequently $f_\beta = f^*$, which contradicts (23). It means that $w^* = v^*$, or in other words, v^* coincides with some vertex of $f_\beta(\Delta)$. Hence v^* is equal to $f_\beta(e^{i\beta})$ rotated about a multiple of $2\pi/n$, namely about $2\pi/n\cdot(n-1)/4$. It is enough to take such β that $v^* = H(\beta)\cdot\varepsilon^{(n-1)/4}$ is in Λ^*. From this we conclude that $\beta \in [0, \pi/2n]$. ■

Theorem 8. *For a fixed odd integer $n \geq 5$ and for every function $f \in \mathcal{F}^{(n)}$ the set $f(\Delta)$ is included in Δ_{R_n}, where $R_n = B(1/n, 1/2 - 3/2n)/n\sqrt[n]{4}$. The number R_n cannot be decreased.*

Proof.
We have

$$|H(\beta)| \leq \int_0^1 \left|(1 + t^n e^{-2in\beta})^{\frac{1}{n}}(1 - t^n)^{-\frac{3}{n}}\right| dt \leq \int_0^1 \frac{(1 + t^n)^{\frac{1}{n}}}{(1 - t^n)^{\frac{3}{n}}} dt = |H(0)| \ .$$

It can be shown that $H(0) = B(1/n, 1/2 - 3/2n)/n\sqrt[n]{4}$. ■

Corollary 2. *For a fixed odd integer $n \geq 5$ the covering constant for $\mathcal{F}^{(n)}$ is equal to $R_n = B(1/n, 1/2 - 3/2n)/n\sqrt[n]{4}$.*

The results presented above are valid for positive odd integers greater than or equal to 5. In the last part of this section we turn to the case $n = 3$.

As it was said in Section 2 (see also Figure 3) for $n = 3$ and $\beta \in [\pi/3, \pi] \setminus \{\pi/2, 5\pi/6\}$ the functions given by (20) map Δ onto the polygons with the interior angles $4\pi/3$ and 0 alternately, and the vertices in points $a \cdot \varepsilon^j$, $\infty \cdot a \cdot \varepsilon^j$, $j = 0, 1, 2$ alternately, where $a = g_\beta(e^{i\beta}) = H(\beta)$. Both sides adjacent to every vertex in infinity are parallel. Hence $g_\beta(\Delta)$ are star-shaped sets with three unbounded strips. The strips have the direction $\pi/3$, π, $5\pi/3$ if $\beta \in [\pi/3, \pi/2) \cup (5\pi/6, \pi]$ and 0, $2\pi/3$, $4\pi/3$ if $\beta \in (\pi/2, 5\pi/6)$. The thickness of the strips is changing as β varies in $\beta \in [\pi/3, \pi] \setminus \{\pi/2, 5\pi/6\}$, but when β tends to $\pi/2$ or $5\pi/6$ the thickness of the strips tends to 0.

For $\beta = \pi/2$ and $\beta = 5\pi/6$ the functions

$$g_{\frac{\pi}{2}}(z) = \int_0^z \frac{1}{(1 - i\zeta^3)^{2/3}} d\zeta$$

and

$$g_{\frac{5\pi}{6}}(z) = \int_0^z \frac{1}{(1 + i\zeta^3)^{2/3}} d\zeta$$

map Δ onto the equilateral triangles symmetric with respect to the imaginary axis. The first triangle has one of its vertices in the point ic, the second one - in the point $-ic$, where

$$c = \frac{B(\frac{1}{3}, \frac{1}{6})}{3\sqrt[3]{4}} = 1.76\ldots \ .$$

Let

$$\Omega_0 = \left\{ w : \operatorname{Re} w \geq 0, |\operatorname{Im} w| < \frac{1}{2}c \right\} \ .$$

Theorem 9. *The covering domain $L_{\mathcal{F}^{(3)}}$ is an unbounded and 6-fold symmetric domain*

$$L_{\mathcal{F}^{(3)}} = \bigcup_{j=0}^{5} e^{j\frac{\pi}{3}i} \cdot \Omega_0 \ .$$

Proof.
Let L denote the covering set for $\mathcal{F}^{(n)}$ and let L^* stand for $\bigcup_{j=0}^5 e^{j\pi i/3} \cdot \Omega_0$.

At the beginning we can see that L includes six-pointed star obtained as a union of $g_{\pi/2}(\Delta)$ and $g_{5\pi/6}(\Delta)$. We know that for $\beta \in (\pi/2, 5\pi/6)$ each set $g_\beta(\Delta)$ contains a part of a horizontal strip between two rays emanating from a/ε and a, where $a = H(\beta)$. From (21) it follows that

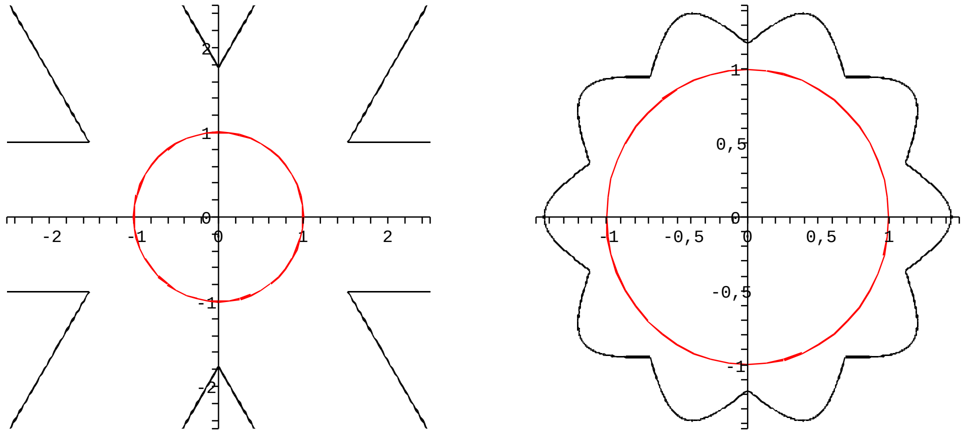

FIGURE 5. Covering domains for $\mathcal{F}^{(3)}$ and $\mathcal{F}^{(5)}$.

the arguments of these points vary continuously from $-\pi/6$ to $\pi/6$ for the point a/ε and from $\pi/2$ to $5\pi/6$ for the point a. This and the symmetry of L with respect to the imaginary axis result in $L^* \subset L$.

Now we shall prove that $L \subset L^*$. On the contrary, assume that $w_0 \notin L^*$ but $w_0 \in L$. It means that there exists a function $f_0 \in \mathcal{F}^{(3)}$ such that $w_0 \in f_0(\Delta)$. Without loss of generality we can assume that $\arg w_0 \in (0, \pi/6)$ because of Lemma 1 and Remark 1.

From the 3-fold symmetry of f_0 we know that $w_0\varepsilon, w_0\varepsilon^2 \in f_0(\Delta)$. Moreover,

$$\operatorname{Im} w_0 = |w_0| \sin \varphi_0 < |w_0| \sin\left(\varphi_0 + \frac{2\pi}{3}\right) = \operatorname{Im}(w_0\varepsilon) ,$$

because $\varphi_0 = \arg w_0 \in (0, \pi/6)$.

Observe that the point $w_1 = \{w_0 - t : t \geq 0\} \cap (\varepsilon \cdot \{w_0 - t : t \geq 0\})$ also belongs to $f_0(\Delta)$. If it were not the case, the points $w_1\varepsilon, w_1\varepsilon^2$ would not be in $f_0(\Delta)$ either. But $w_1, w_1\varepsilon^2 \in \{w_0 - t : t \geq 0\}$. Combining $w_1, w_1\varepsilon^2 \notin f_0(\Delta)$ with $w_0 \in f_0(\Delta)$ yields that the segment connecting w_1 and $w_1\varepsilon^2$ has no common points with $f_0(\Delta)$. From this and the 3-fold symmetry, all three segments connecting $w_1, w_1\varepsilon, w_1\varepsilon^2$ and, as a consequence, the equilateral triangle T with vertices in these points, would be disjoint with $f_0(\Delta)$, a contradiction. This means that $w_1, w_1\varepsilon, w_1\varepsilon^2 \in f_0(\Delta)$, which results in

$$T \subset f_0(\Delta) . \tag{24}$$

But

$$g_{\frac{5\pi}{6}}(\Delta) \subset T \quad \text{and} \quad g_{\frac{5\pi}{6}}(\Delta) \neq T . \tag{25}$$

From (24) and (25), $g_{5\pi/6}$ is subordinated to f, but $g_{5\pi/6}$ and f have the same normalization, a contradiction. It means that if $w_0 \in L$ then $w_0 \in L^*$, which completes the proof. ∎

The covering domains for $\mathcal{F}^{(3)}$ and $\mathcal{F}^{(5)}$ are shown in Figure 5.

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